Fundamentals of Structural Dynamics

1 Course description

Aim of the course is that students develop a "feeling for dynamic problems" and acquire the theoretical background and the tools to understand and to solve important problems relevant to the linear and, in part, to the nonlinear dynamic behaviour of structures, especially under seismic excitation.
The course will start with the analysis of **single-degree-of-freedom (SDoF) systems** by discussing: (i) Modelling, (ii) equations of motion, (iii) free vibrations with and without damping, (iv) harmonic, periodic and short excitations, (v) Fourier series, (vi) impacts, (vii) linear and nonlinear time history analysis, and (viii) elastic and inelastic response spectra.
Afterwards, **multi-degree-of-freedom (MDoF) systems** will be considered and the following topics will be discussed: (i) Equation of motion, (ii) free vibrations, (iii) modal analysis, (iv) damping, (v) Rayleigh's quotient, and (vi) seismic behaviour through response spectrum method and time history analysis.
To supplement the suggested reading, handouts with class notes and calculation spreadsheets with selected analysis cases to self-training purposes will be distributed.

Lecturer: Dr. Alessandro Dazio, UME School

2 Suggested reading

[Cho11] Chopra A., "Dynamics of Structures", Prentice Hall, Fourth Edition, 2011.

[CP03] Clough R., Penzien J., "Dynamics of Structures", Second Edition (revised), Computer and Structures Inc., 2003.

[Hum12] Humar J.L., "Dynamics of Structures". Third Edition. CRC Press, 2012.

3 Software

In the framework of the course the following software will be used by the lecturer to solve selected examples:

[Map10] Maplesoft: "Maple 14". User Manual. 2010

[Mic07] Microsoft: "Excel 2007". User Manual. 2007

[VN12] Visual Numerics: "PV Wave". User Manual. 2012

As an alternative to [VN12] and [Map10] it is recommended that students make use of the following software, or a previous version thereof, to deal with coursework:

[Mat12] MathWorks: "MATLAB 2012". User Manual. 2012

4 Schedule of classes

Date	Time	Topic
Day 1 Fri. April 19 2013	09:00 - 10:30	1. Introduction 2. **SDoF systems:** Equation of motion and modelling
	11:00 - 12:30	3. Free vibrations
	14:30 - 16:00	*Assignment 1*
	16:30 - 18:00	*Assignment 1*
Day 2 Sat. April 20 2013	9:00 - 10:30	4. Harmonic excitation
	11:00 - 12:30	5. Transfer functions
	14:30 - 16:00	6. Forced vibrations (Part 1)
	16:30 - 18:00	6. Forced vibrations (Part 2)
Day 3 Sun. April 21 2013	09:00 - 10:30	7. Seismic excitation (Part 1)
	11:00 - 12:30	7. Seismic excitation (Part 2)
	14:30 - 16:00	*Assignment 2*
	16:30 - 18:00	*Assignment 2*
Day 4 Mon. April 22 2013	9:00 - 10:30	8. **MDoF systems:** Equation of motion
	11:00 - 12:30	9. Free vibrations
	14:30 - 16:00	10. Damping 11. Forced vibrations
	16:30 - 18:00	11. Forced vibrations
Day 5 Tue. April 23 2013	09:00 - 10:30	12. Seismic excitation (Part 1)
	11:00 - 12:30	12. Seismic excitation (Part 2)
	14:30 - 16:00	*Assignment 3*
	16:30 - 18:00	*Assignment 3*

Table of Contents

1 Introduction

1.1 Goals of the course

- Presentation of the theoretical basis and of the relevant tools;
- General understanding of phenomena related to structural dynamics;
- Focus on earthquake engineering;
- Development of a "Dynamic Feeling";
- Detection of frequent dynamic problems and application of appropriate solutions.

1.2 Limitations of the course

- Only an introduction to the broadly developed field of structural dynamics (due to time constraints);
- Only deterministic excitation;
- No soil-dynamics and no dynamic soil-structure interaction will be treated (this is the topic of another course);
- Numerical methods of structural dynamics are treated only partially (No FE analysis. This is also the topic of another course);
- Recommendation of further readings to solve more advanced problems.

Course "Fundamentals of Structural Dynamics" *An-Najah 2013*

1.3 Topics of the course

1) Systems with one degree of freedom

- Modelling and equation of motion
- Free vibrations with and without damping
- Harmonic excitation

2) Forced oscillations

- Periodic excitation, Fourier series, short excitation
- Linear and nonlinear time history-analysis
- Elastic and inelastic response spectra

3) Systems with many degree of freedom

- Modelling and equation of motion
- Modal analysis, consideration of damping
- Forced oscillations,
- Seismic response through response spectrum method and time-history analysis

4) Continuous systems

- Generalised Systems

5) Measures against vibrations

- Criteria, frequency tuning, vibration limitation

1.4 References

Theory

[Bat96] Bathe KJ: “Finite Element Procedures”. Prentice Hall, Upper Saddle River, 1996.

[CF06] Christopoulos C, Filiatrault A: "Principles of Passive Supplemental Damping and Seismic Isolation". ISBN 88-7358-037-8. IUSSPress, 2006.

[Cho11] Chopra AK: “Dynamics of Structures”. Fourth Edition. Prentice Hall, 2011.

[CP03] Clough R, Penzien J: “Dynamics of Structures”. Second Edition (Revised). Computer and Structures, 2003. (http://www.csiberkeley.com)

[Den85] Den Hartog JP: “Mechanical Vibrations”. Reprint of the fourth edition (1956). Dover Publications, 1985.

[Hum12] Humar JL: “Dynamics of Structures”. Third Edition. CRC Press, 2012.

[Inm01] Inman D: “Engineering Vibration”. Prentice Hall, 2001.

[Prz85] Przemieniecki JS: “Theory of Matrix Structural Analysis”. Dover Publications, New York 1985.

[WTY90] Weawer W, Timoshenko SP, Young DH: “Vibration problems in Engineering”. Fifth Edition. John Wiley & Sons, 1990.

Practical cases (Vibration problems)

[Bac+97] Bachmann H et al.: “Vibration Problems in Structures”. Birkhäuser Verlag 1997.

Blank page

2 Single Degree of Freedom Systems

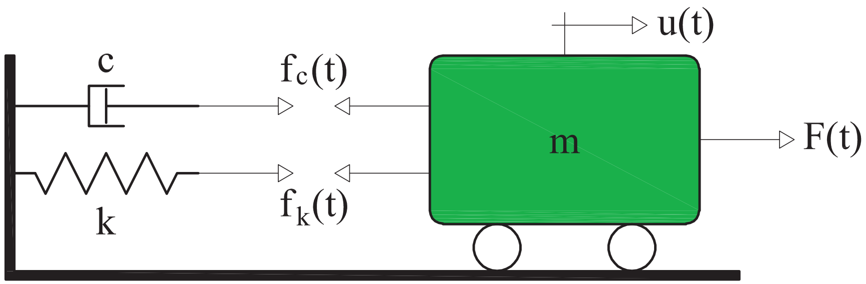
2.1 Formulation of the equation of motion

2.1.1 Direct formulation

1) Newton's second law (Action principle)

$$F = \frac{dI}{dt} = \frac{d}{dt}(m\dot{u}) = m\ddot{u} \qquad (I = \text{Impulse}) \qquad (2.1)$$

The force corresponds to the change of impulse over time.



$$-f_k(t) - f_c(t) + F(t) = m\ddot{u}(t) \qquad (2.2)$$

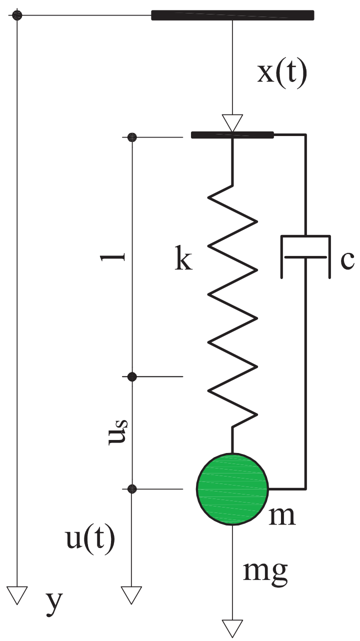
Introducing the spring force $f_k(t) = ku(t)$ and the damping force $f_c(t) = c\dot{u}(t)$ Equation (2.2) becomes:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F(t) \qquad (2.3)$$

2) D’Alembert principle

$$F + T = 0 \qquad (2.4)$$

The principle is based on the idea of a fictitious inertia force that is equal to the product of the mass times its acceleration, and acts in the opposite direction as the acceleration
The mass is at all times in equilibrium under the resultant force F and the inertia force $T = -m\ddot{u}$.



$$y = x(t) + l + u_s + u(t) \qquad (2.5)$$

$$\ddot{y} = \ddot{x} + \ddot{u} \qquad (2.6)$$

$$T = -m\ddot{y} = -m(\ddot{x} + \ddot{u}) \qquad (2.7)$$

$$\begin{aligned} F &= -k(u_s + u) - c\dot{u} + mg \qquad (2.8)\\ &= -\cancel{k}u_s - ku - c\dot{u} + m\cancel{g} \\ &= -ku - c\dot{u} \end{aligned}$$

$$F + T = 0 \qquad (2.9)$$

$$-c\dot{u} - ku - m\ddot{x} - m\ddot{u} = 0 \qquad (2.10)$$

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{x} \qquad (2.11)$$

- To derive the equation of motion, the dynamic equilibrium for each force component is formulated. To this purpose, forces, and possibly also moments shall be decomposed into their components according to the coordinate directions.

2.1.2 Principle of virtual work

$$\delta u \tag{2.12}$$

- Virtual displacement = imaginary infinitesimal displacement
- Should best be kinematically permissible, so that unknown reaction forces do not produce work

$$\delta A_i = \delta A_a \tag{2.13}$$

- Thereby, both inertia forces and damping forces must be considered

$$(f_m + f_c + f_k)\delta u = F(t)\delta u \tag{2.14}$$

2.1.3 Energy Formulation

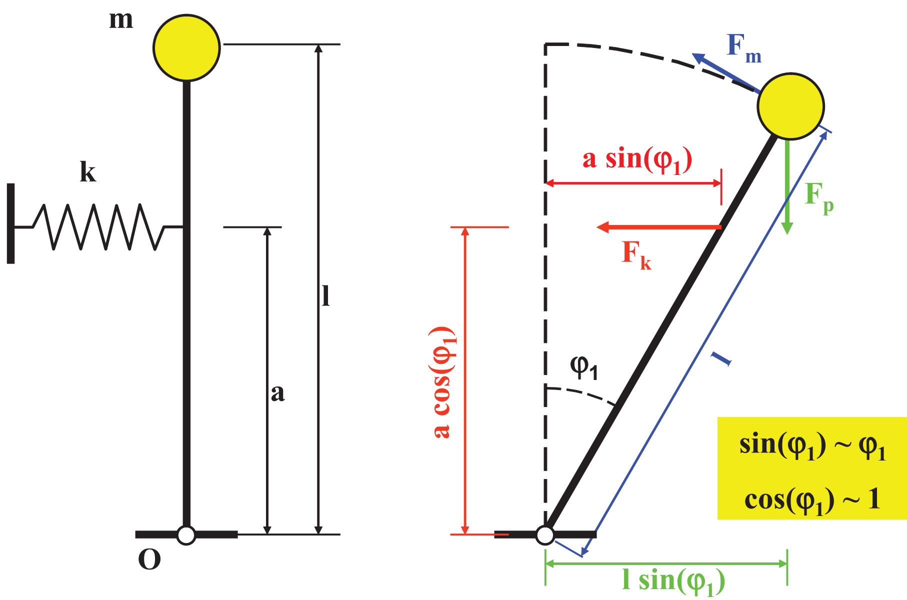
- **Kinetic energy T** (Work, that an external force needs to provide to move a mass)
- **Deformation energy U** (is determined from the work that an external force has to provide in order to generate a deformation)
- **Potential energy of the external forces V** (is determined with respect to the potential energy **at the position of equilibrium**)
- Conservation of energy theorem (Conservative systems)

$$E = T + U + V = T_o + U_o + V_o = \text{constant} \tag{2.15}$$

$$\frac{dE}{dt} = 0 \tag{2.16}$$

2.2 Example "Inverted Pendulum"

Direct Formulation



Spring force: $$F_k = a \cdot \sin(\varphi_1) \cdot k \approx a \cdot \varphi_1 \cdot k \tag{2.17}$$

Inertia force: $$F_m = \ddot{\varphi}_1 \cdot l \cdot m \tag{2.18}$$

External force: $$F_p = m \cdot g \tag{2.19}$$

Equilibrium

$$F_k \cdot a \cdot \cos(\varphi_1) + F_m \cdot l - F_p \cdot l \cdot \sin(\varphi_1) = 0 \tag{2.20}$$

$$m \cdot l^2 \cdot \ddot{\varphi}_1 + (a^2 \cdot k - m \cdot g \cdot l) \cdot \varphi_1 = 0 \qquad (2.21)$$

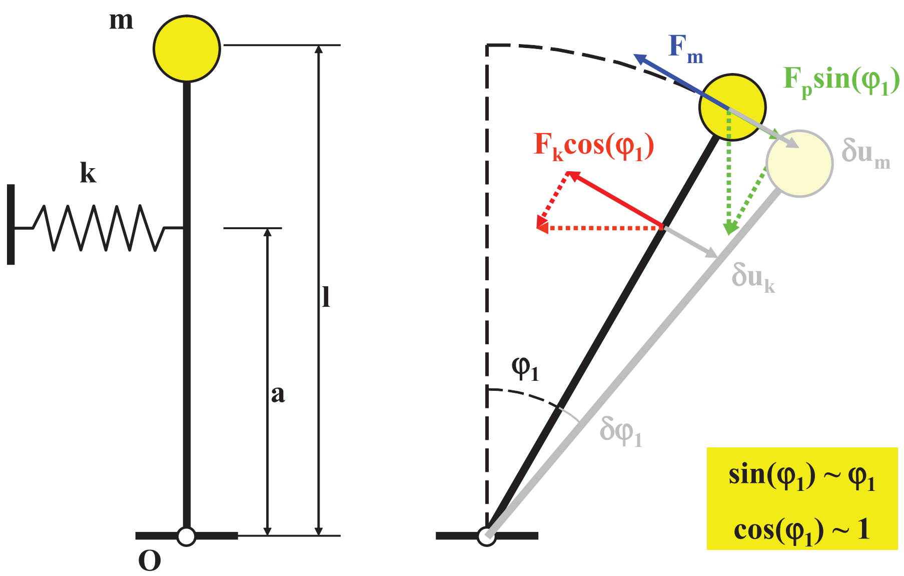
Circular frequency:

$$\omega = \sqrt{\frac{K_1}{M_1}} = \sqrt{\frac{a^2 \cdot k - m \cdot g \cdot l}{m \cdot l^2}} = \sqrt{\frac{a^2 \cdot k}{m \cdot l^2} - \frac{g}{l}} \qquad (2.22)$$

The system is stable if:

$$\omega > 0: \qquad a^2 \cdot k > m \cdot g \cdot l \qquad (2.23)$$

Principle of virtual work formulation



Spread force: see below.

Spring force: $F_k \cdot \cos(\varphi_1) \approx a \cdot \varphi_1 \cdot k$ (2.24)

Inertia force: $F_m = \ddot{\varphi}_1 \cdot l \cdot m$ (2.25)

External force: $F_p \cdot \sin(\varphi_1) \approx m \cdot g \cdot \varphi_1$ (2.26)

Virtual displacement:

$$\delta u_k = \delta\varphi_1 \cdot a, \qquad \delta u_m = \delta\varphi_1 \cdot l \qquad (2.27)$$

Principle of virtual work:

$$(F_k \cdot \cos(\varphi_1)) \cdot \delta u_k + (F_m - (F_p \cdot \sin(\varphi_1))) \cdot \delta u_m = 0 \qquad (2.28)$$

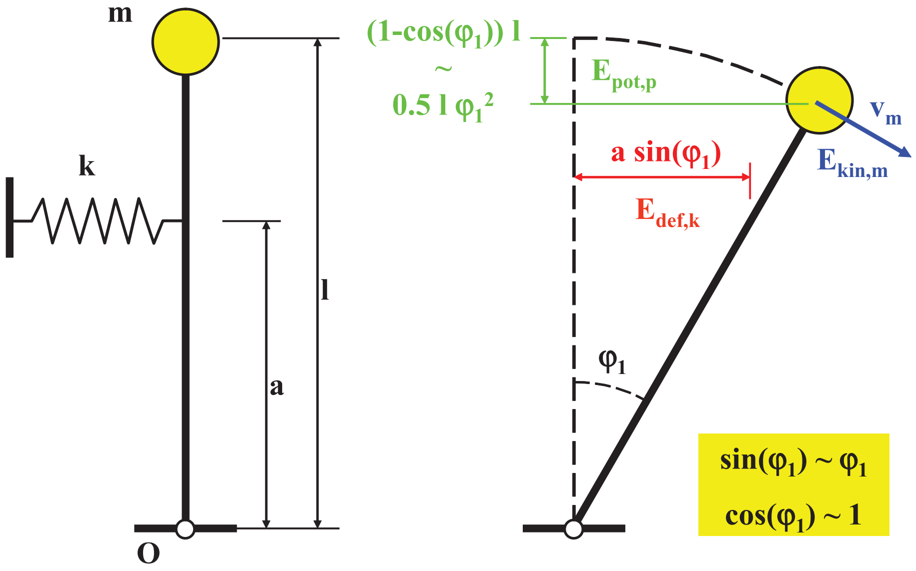
$$(a \cdot \varphi_1 \cdot k) \cdot \delta\varphi_1 \cdot a + (\ddot{\varphi}_1 \cdot l \cdot m - m \cdot g \cdot \varphi_1) \cdot \delta\varphi_1 \cdot l = 0 \qquad (2.29)$$

After cancelling out $\delta\varphi_1$ the following equation of motion is obtained:

$$m \cdot l^2 \cdot \ddot{\varphi}_1 + (a^2 \cdot k - m \cdot g \cdot l) \cdot \varphi_1 = 0 \qquad (2.30)$$

The equation of motion given by Equation (2.30) corresponds to Equation (2.21).

Energy Formulation



Spring: $E_{def,k} = \frac{1}{2} \cdot k \cdot [a \cdot \sin(\varphi_1)]^2 = \frac{1}{2} \cdot k \cdot (a \cdot \varphi_1)^2$ (2.31)

Mass: $E_{kin,m} = \frac{1}{2} \cdot m \cdot v_m^2 = \frac{1}{2} \cdot m \cdot (\dot{\varphi}_1 \cdot l)^2$ (2.32)

$E_{pot,p} = -(m \cdot g) \cdot (1 - \cos(\varphi_1)) \cdot l$ (2.33)

by means of a series development, $\cos(\varphi_1)$ can be expressed as:

$$\cos(\varphi_1) = 1 - \frac{\varphi_1^2}{2!} + \frac{\varphi_1^4}{4!} - \ldots + (-1)^k \cdot \frac{x^{2k}}{(2k)!} + \ldots \quad (2.34)$$

for small angles φ_1 we have:

$$\cos(\varphi_1) = 1 - \frac{\varphi_1^2}{2} \text{ and } \frac{\varphi_1^2}{2} = 1 - \cos(\varphi_1) \quad (2.35)$$

and Equation (2.33) becomes:

$$E_{pot,p} = -(m \cdot g \cdot 0.5 \cdot l \cdot \varphi_1^2) \quad (2.36)$$

Energy conservation:

$$E_{tot} = E_{def,k} + E_{kin,m} + E_{pot,p} = \text{constant} \quad (2.37)$$

$$E = \frac{1}{2}(m \cdot l^2) \cdot \dot{\varphi}_1^2 + \frac{1}{2}(k \cdot a^2 - m \cdot g \cdot l) \cdot \varphi_1^2 = \text{constant} \quad (2.38)$$

Derivative of the energy with respect to time:

$$\frac{dE}{dt} = 0 \qquad \text{Derivation rule: } (g \bullet f)' = (g' \bullet f) \cdot f' \quad (2.39)$$

$$(m \cdot l^2) \cdot \dot{\varphi}_1 \cdot \ddot{\varphi}_1 + (k \cdot a^2 - m \cdot g \cdot l) \cdot \varphi_1 \cdot \dot{\varphi}_1 = 0 \quad (2.40)$$

After cancelling out the velocity $\dot{\varphi}_1$:

$$m \cdot l^2 \cdot \ddot{\varphi}_1 + (a^2 \cdot k - m \cdot g \cdot l) \cdot \varphi_1 = 0 \quad (2.41)$$

The equation of motion given by Equation (2.41) corresponds to Equations (2.21) and (2.30).

Comparison of the energy maxima

$$KE = \frac{1}{2} \cdot m \cdot (\dot{\varphi}_{1,max} \cdot l)^2 \quad (2.42)$$

$$PE = \frac{1}{2} \cdot k \cdot (a \cdot \varphi_1)^2 - \frac{1}{2} \cdot g \cdot m \cdot l \cdot \varphi_1^2 \quad (2.43)$$

By equating KE and PE we obtain:

$$\dot{\varphi}_{1,max} = \left(\sqrt{\frac{a^2 \cdot k - m \cdot g \cdot l}{m \cdot l^2}}\right) \cdot \varphi_1 \quad (2.44)$$

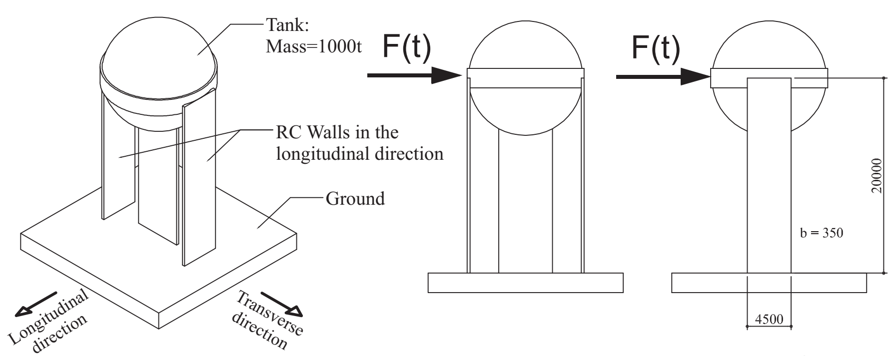
$$\dot{\varphi}_{1,max} = \omega \cdot \varphi_1 \quad (2.45)$$

- ω is independent of the initial angle φ_1
- the greater the deflection, the greater the maximum velocity.

2.3 Modelling

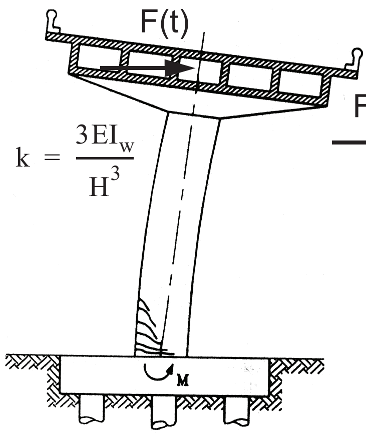
2.3.1 Structures with concentrated mass

Water tank

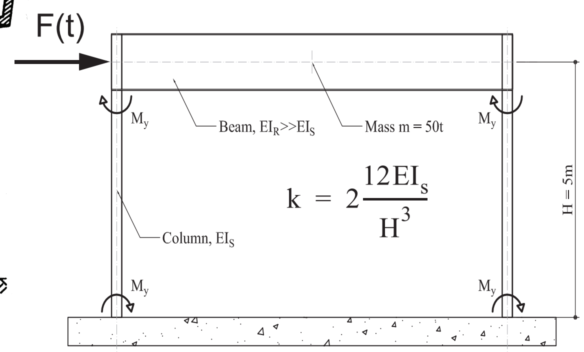


$k = \ldots$ $\qquad k = 2\frac{3EI_w}{H^3}$

Bridge in transverse direction

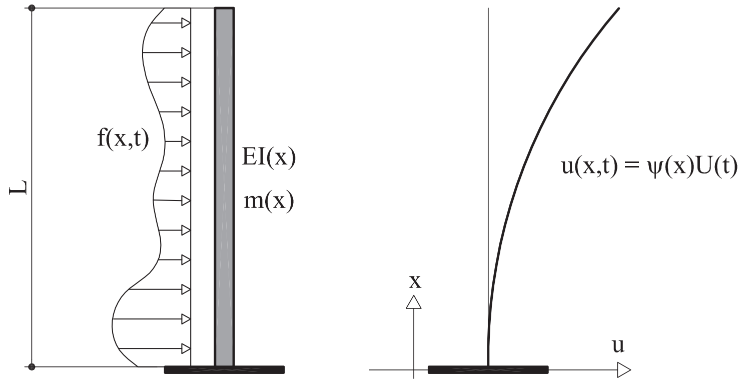


Frame with rigid beam



$$m\ddot{u} + ku = F(t) \quad (2.46)$$

2.3.2 Structures with distributed mass



Deformation: $u(x, t) = \psi(x)U(t)$ (2.47)

External forces: $t(x, t) = -m\ddot{u}(x, t)$

$f(x, t)$ (2.48)

- Principle of virtual work

$$\delta A_i = \delta A_a \quad (2.49)$$

$$\delta A_a = \int_0^L (t \cdot \delta u)dx + \int_0^L (f \cdot \delta u)dx \quad (2.50)$$

$$= -\int_0^L (m\ddot{u} \cdot \delta u)dx + \int_0^L (f \cdot \delta u)dx$$

$$\delta A_i = \int_0^L (M \cdot \delta\varphi)dx \text{ where:} \quad (2.51)$$

$$M = EIu'' \quad \text{and} \quad \delta\varphi = \delta[u''] \quad (2.52)$$

$$\delta A_i = \int_0^L (EIu'' \cdot \delta[u''])dx \quad (2.53)$$

- Transformations:

$$u'' = \psi''U \quad \text{and} \quad \ddot{u} = \psi\ddot{U} \quad (2.54)$$

- The virtual displacement is affine to the selected deformation:

$$\delta u = \psi\delta U \quad \text{and} \quad \delta[u''] = \psi''\delta U \quad (2.55)$$

- Using Equations (2.54) and (2.55), the work δA_a produced by the external forces is:

$$\delta A_a = -\int_0^L (m\psi\ddot{U} \cdot \psi\delta U)dx + \int_0^L (f \cdot \psi\delta U)dx \quad (2.56)$$

$$= \delta U\left[-\ddot{U}\int_0^L m\psi^2 dx + \int_0^L f\psi dx\right]$$

- Using Equations (2.54) and (2.55) the work δA_i produced by the internal forces is:

$$\delta A_i = \int_0^L (EI\psi''U \cdot \psi''\delta U)dx = \delta U\left[U\int_0^L (EI(\psi'')^2)dx\right] \quad (2.57)$$

- Equation (2.49) is valid for all virtual displacements, therefore:

$$U\int_0^L (EI(\psi'')^2)dx = -\ddot{U}\int_0^L m\psi^2 dx + \int_0^L f\psi dx \quad (2.58)$$

$$m^*\ddot{U} + k^*U = F^* \quad (2.59)$$

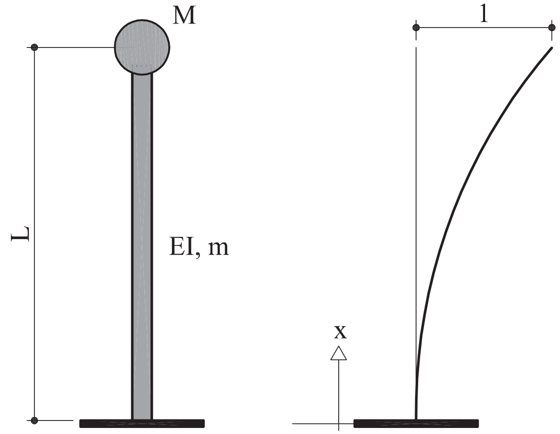
- Circular frequency

$$\omega_n^2 = \frac{k^*}{m^*} = \frac{\int_0^L (EI(\psi'')^2)dx}{\int_0^L m\psi^2 dx} \tag{2.60}$$

-> Rayleigh-Quotient

- Choosing the deformation figure
  - The accuracy of the modelling depends on the assumed deformation figure;
  - The best results are obtained when the deformation figure fulfills all boundary conditions;
  - The boundary conditions are automatically satisfied if the deformation figure corresponds to the deformed shape due to an external force;
  - A possible external force is the weight of the structure acting in the considered direction.
- Properties of the Rayleigh-Quotient
  - The estimated natural frequency is always larger than the exact one (**Minimization of the quotient!**);
  - Useful results can be obtained even if the assumed deformation figure is not very realistic.

- Example No. 1: Cantilever with distributed mass



$$\psi = 1 - \cos\left(\frac{\pi x}{2L}\right), \qquad \psi'' = \left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi x}{2L}\right) \tag{2.61}$$

$$m^* = \int_0^L m\left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)^2 dx + \psi^2(x = L)M \tag{2.62}$$

$$= \frac{1}{2}m\left(\frac{3\pi x - 8\sin\left(\frac{\pi x}{2L}\right)L + 2\cos\left(\frac{\pi x}{2L}\right)\sin\left(\frac{\pi x}{2L}\right)L}{\pi}\right)\Bigg|_0^L + M$$

$$= \frac{(3\pi - 8)}{2\pi}mL + M = 0.23mL + M$$

$$k^* = EI\left(\frac{\pi}{2L}\right)^4 \int_0^L \left(\cos\left(\frac{\pi x}{2L}\right)\right)^2 dx \tag{2.63}$$

$$= EI\left(\frac{\pi}{2L}\right)^4 \cdot \frac{1}{2}\left(\frac{\pi x + 2\cos\left(\frac{\pi x}{2L}\right)\sin\left(\frac{\pi x}{2L}\right)L}{\pi}\right)\Bigg|_0^L$$

$$= \frac{\pi^4}{32} \cdot \frac{EI}{L^3} = 3.04 \cdot \frac{EI}{L^3} \approx \frac{3EI}{L^3}$$

$$\omega = \sqrt{\frac{3EI}{(0.23mL + M)L^3}} \tag{2.64}$$

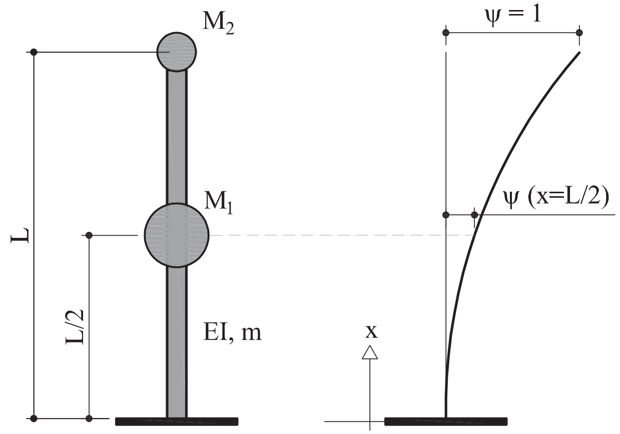
- Check of the boundary conditions of the deformation figure

$\psi(0) = 0$? -> $\psi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right)$: $\psi(0) = 0$ OK!

$\psi'(0) = 0$? -> $\psi'(x) = \frac{\pi}{2L}\sin\left(\frac{\pi x}{2L}\right)$: $\psi'(0) = 0$ OK!

$\psi''(L) = 0$? -> $\psi''(x) = \left(\frac{\pi}{2L}\right)^2\cos\left(\frac{\pi x}{2L}\right)$: $\psi''(L) = 0$ OK!

- Example No. 2: Cantilever with distributed mass



$$\psi = 1 - \cos\left(\frac{\pi x}{2L}\right), \qquad \psi'' = \left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi x}{2L}\right) \tag{2.65}$$

- Calculation of the mass m^*

$$m^* = \int_0^L m\left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)^2 dx + \psi^2\left(x = \frac{L}{2}\right)M_1 + \psi^2(x = L)M_2 \tag{2.66}$$

$$m^* = \frac{(3\pi - 8)}{2\pi}mL + \left(1 - \cos\left(\frac{\pi}{4}\right)\right)^2 \cdot M_1 + 1^2 \cdot M_2 \tag{2.67}$$

$$m^* = \frac{(3\pi - 8)}{2\pi}mL + \left(\frac{3 - 2\sqrt{2}}{2}\right) \cdot M_1 + M_2 \tag{2.68}$$

$$m^* = 0.23mL + 0.086M_1 + M_2 \tag{2.69}$$

- Calculation of the stiffness k^*

$$k^* = EI\left(\frac{\pi}{2L}\right)^4 \int_0^L \left(\cos\left(\frac{\pi x}{2L}\right)\right)^2 dx \tag{2.70}$$

$$k^* = \frac{\pi^4}{32} \cdot \frac{EI}{L^3} = 3.04 \cdot \frac{EI}{L^3} \approx \frac{3EI}{L^3} \tag{2.71}$$

- Calculation of the circular frequency ω

$$\omega = \sqrt{\frac{3.04EI}{(0.23mL + 0.086M_1 + M_2)L^3}} \tag{2.72}$$

Special case: $m = 0$ **and** $M_1 = M_2 = M$

$$\omega = \sqrt{\frac{3.04EI}{(1.086M)L^3}} = 1.673\sqrt{\frac{EI}{ML^3}} \tag{2.73}$$

The exact first natural circular frequency of a two-mass oscillator with constant stiffness and mass is:

$$\omega = \sqrt{\frac{3.007EI}{(1.102M)L^3}} = 1.652\sqrt{\frac{EI}{ML^3}} \tag{2.74}$$

As a numerical example, the first natural frequency of a $L = 10m$ tall steel shape HEB360 (bending about the strong axis) featuring two masses $M_1 = M_2 = 10t$ is calculated.

$$EI = 200000 \cdot 431.9\times10^6 = 8.638\times10^{13} Nmm^2 \tag{2.75}$$

$$EI = 8.638\times10^4 kNm^2 \tag{2.76}$$

By means of Equation (2.73) we obtain:

$$\omega = 1.673\sqrt{\frac{EI}{ML^3}} = 1.673\sqrt{\frac{8.638\times10^4}{10 \cdot 10^3}} = 4.9170 \tag{2.77}$$

$$f = \frac{1}{2\pi} \cdot \omega = \frac{4.9170}{2\pi} = 0.783Hz \tag{2.78}$$

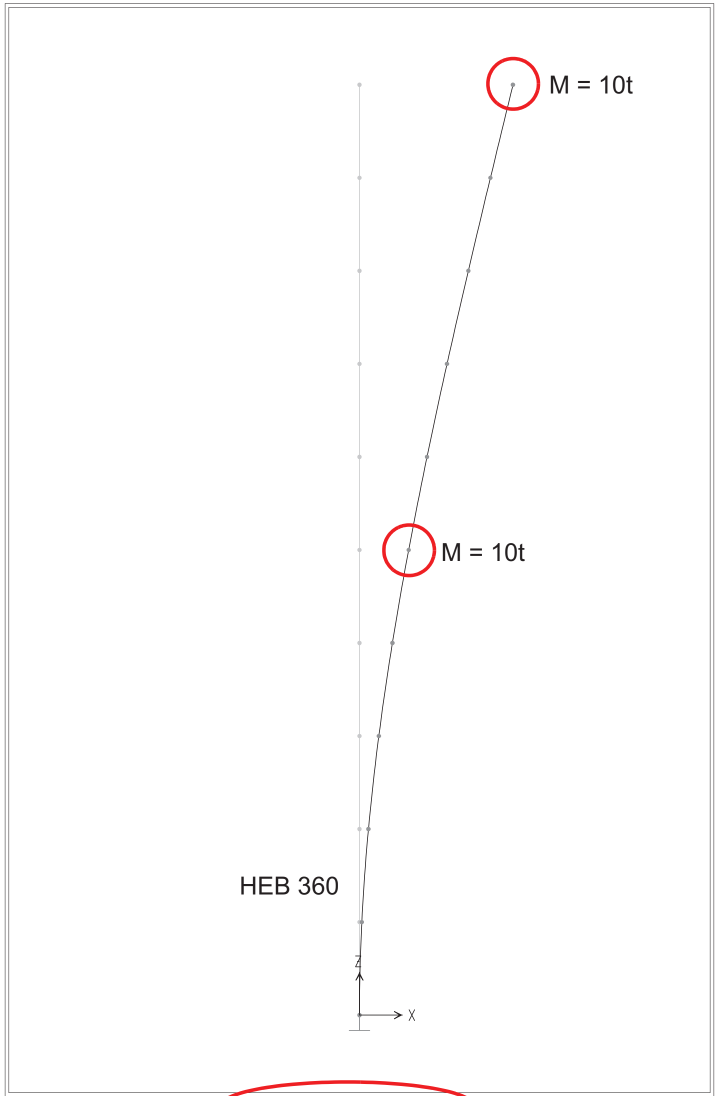
From Equation (2.74):

$$f = \frac{1.652}{2\pi}\sqrt{\frac{EI}{ML^3}} = \frac{1.652}{2\pi}\sqrt{\frac{8.638\times10^4}{10 \cdot 10^3}} = 0.773Hz \tag{2.79}$$

The first natural frequency of such a dynamic system can be calculated using a finite element program (e.g. SAP 2000), and it is equal to:

$$T = 1.2946s,\ f = 0.772Hz \tag{2.80}$$

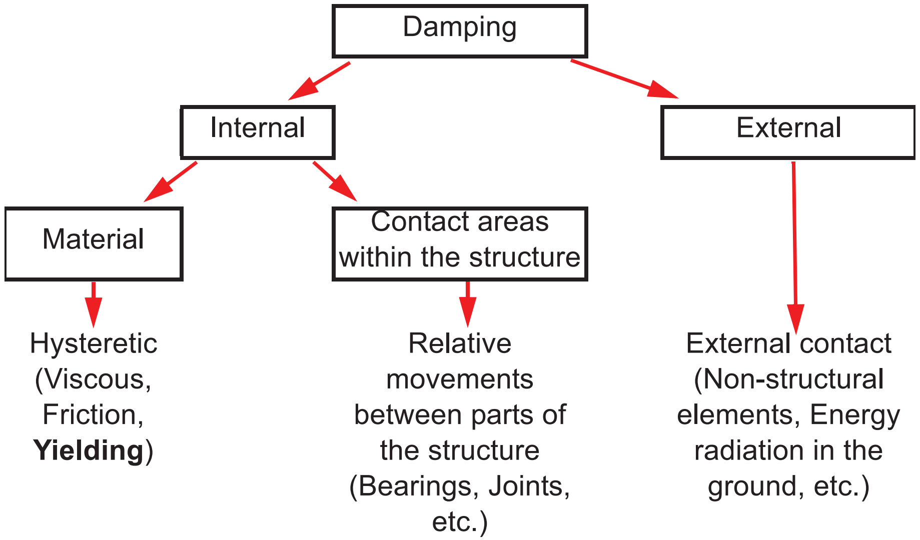
Equations (2.78), (2.79) and (2.80) are in very good accordance. The representation of the first mode shape and corresponding natural frequency obtained by means of a finite element program is shown in the next figure.



SAP2000 v8 - File:HEB_360 - Mode 1 Period 1.2946 seconds - KN-m Units

2.3.3 Damping

- Types of damping

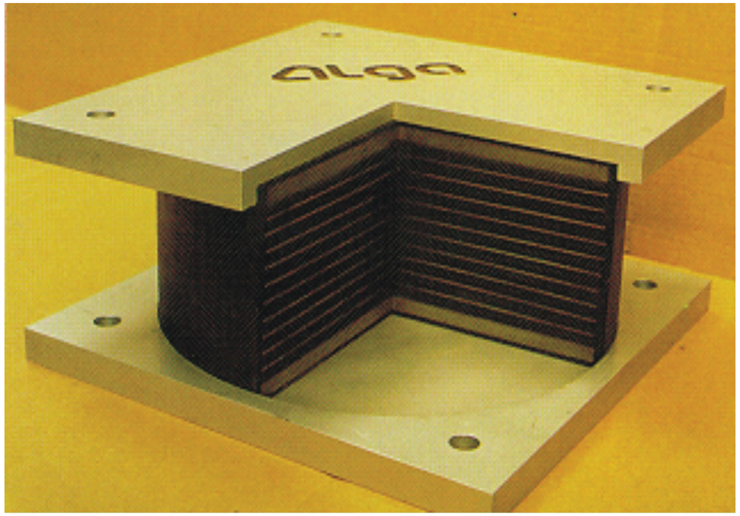


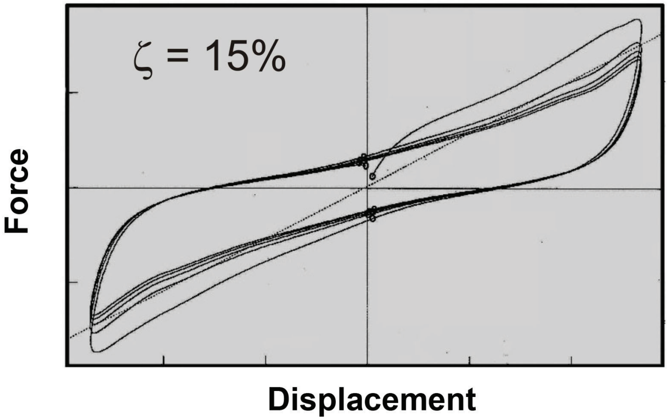
- Typical values of damping in structures

Material	Damping ζ
Reinforced concrete (uncraked)	0.007 - 0.010
Reinforced concrete (craked	0.010 - 0.040
Reinforced concrete (PT)	0.004 - 0.007
Reinforced concrete (partially PT)	0.008 - 0.012
Composite components	0.002 - 0.003
Steel	0.001 - 0.002

Table C.1 from [Bac+97]

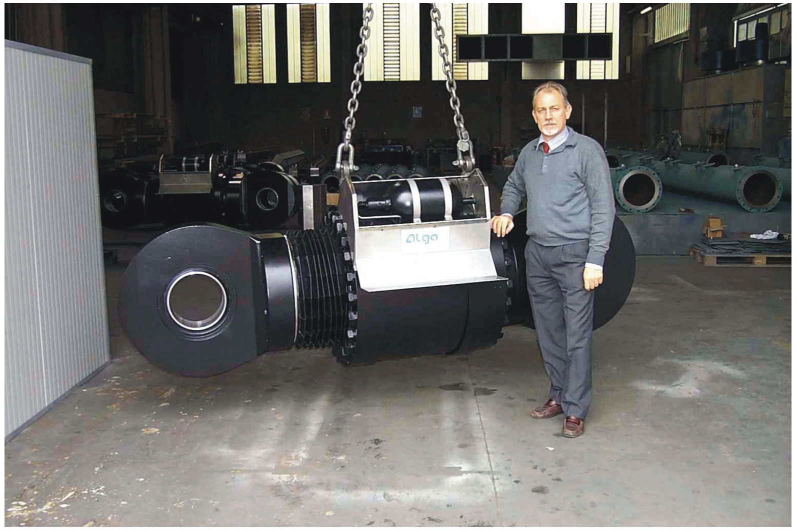
- Bearings

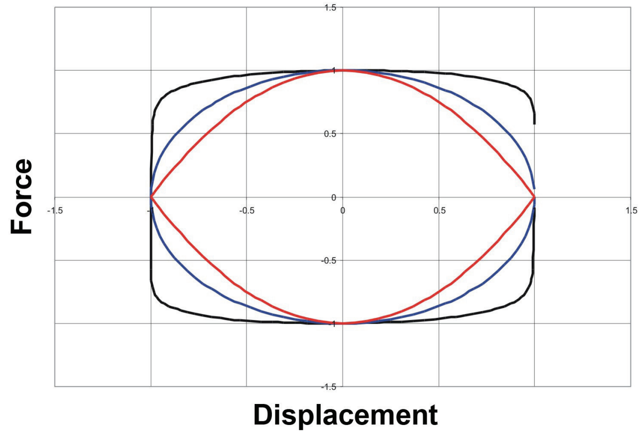




Source: A. Marioni: “Innovative Anti-seismic Devices for Bridges”. [SIA03]

- Dissipators





Source: A. Marioni: “Innovative Anti-seismic Devices for Bridges”. [SIA03]

3 Free Vibrations

"A structure undergoes free vibrations when it is brought out of its static equilibrium, and can then oscillate without any external dynamic excitation"

3.1 Undamped free vibrations

$$m\ddot{u}(t) + ku(t) = 0 \quad (3.1)$$

3.1.1 Formulation 1: Amplitude and phase angle

- Ansatz:

$$u(t) = A\cos(\omega_n t - \phi) \quad (3.2)$$

$$\ddot{u}(t) = -A\omega_n^2\cos(\omega_n t - \phi) \quad (3.3)$$

By substituting Equations (3.2) and (3.3) in (3.1):

$$A(-\omega_n^2 m + k)\cos(\omega_n t - \phi) = 0 \quad (3.4)$$

$$-\omega_n^2 m + k = 0 \quad (3.5)$$

$\omega_n = \sqrt{k/m}$ **"Natural circular frequency"** (3.6)

- Relationships

$\omega_n = \sqrt{k/m}$ [rad/s]: Angular velocity (3.7)

$f_n = \dfrac{\omega_n}{2\pi}$ [1/s], [Hz]: Number of revolutions per time (3.8)

$T_n = \dfrac{2\pi}{\omega_n}$ [s]: Time required per revolution (3.9)

- Transformation of the equation of motion

$$\ddot{u}(t) + \omega_n^2 u(t) = 0 \quad (3.10)$$

- Determination of the unknowns A and ϕ:

  The static equilibrium is disturbed by the initial displacement $u(0) = u_0$ and the initial velocity $\dot{u}(0) = v_0$:

$$A = \sqrt{u_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}\ ,\ \tan\phi = \frac{v_0}{u_0\omega_n} \quad (3.11)$$

- Visualization of the solution by means of the Excel file given on the web page of the course (SD_FV_viscous.xlsx)

3.1.2 Formulation 2: Trigonometric functions

$$m\ddot{u}(t) + ku(t) = 0 \tag{3.12}$$

- Ansatz:

$$u(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{3.13}$$

$$\ddot{u}(t) = -A_1 \omega_n^2 \cos(\omega_n t) - A_2 \omega_n^2 \sin(\omega_n t) \tag{3.14}$$

By substituting Equations (3.13) and (3.14) in (3.12):

$$A_1(-\omega_n^2 m + k)\cos(\omega_n t) + A_2(-\omega_n^2 m + k)\sin(\omega_n t) = 0 \tag{3.15}$$

$$-\omega_n^2 m + k = 0 \tag{3.16}$$

$$\omega_n = \sqrt{k/m} \quad \text{"Natural circular frequency"} \tag{3.17}$$

- Determination of the unknowns A_1 and A_2:

The static equilibrium is disturbed by the initial displacement $u(0) = u_0$ and the initial velocity $\dot{u}(0) = v_0$:

$$A_1 = u_0 \;,\; A_2 = \frac{v_0}{\omega_n} \tag{3.18}$$

3.1.3 Formulation 3: Exponential Functions

$$m\ddot{u}(t) + ku(t) = 0 \tag{3.19}$$

- Ansatz:

$$u(t) = e^{\lambda t} \tag{3.20}$$

$$\ddot{u}(t) = \lambda^2 e^{\lambda t} \tag{3.21}$$

By substituting Equations (3.20) and (3.21) in (3.19):

$$m\lambda^2 + k = 0 \tag{3.22}$$

$$\lambda^2 = -\frac{k}{m} \tag{3.23}$$

$$\lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_n \tag{3.24}$$

The complete solution of the ODE is:

$$u(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \tag{3.25}$$

and by means of Euler's formulas

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \;,\; \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \tag{3.26}$$

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) \;,\; e^{-i\alpha} = \cos(\alpha) - i\sin(\alpha) \tag{3.27}$$

Equation (3.25) can be transformed as follows:

$$u(t) = (C_1 + C_2)\cos(\omega_n t) + i(C_1 - C_2)\sin(\omega_n t) \qquad (3.28)$$

$$u(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \qquad (3.29)$$

Equation (3.29) corresponds to (3.13)!

3.2 Damped free vibrations

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0 \qquad (3.30)$$

- In reality vibrations subside
- Damping exists
- It is virtually impossible to model damping exactly
- From the mathematical point of view viscous damping is easy to treat

Damping constant: $c \left[N \cdot \frac{s}{m}\right]$ (3.31)

3.2.1 Formulation 3: Exponential Functions

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0 \qquad (3.32)$$

- Ansatz:

$$u(t) = e^{\lambda t}, \; \dot{u}(t) = \lambda e^{\lambda t}, \; \ddot{u}(t) = \lambda^2 e^{\lambda t} \qquad (3.33)$$

By substituting Equations (3.33) in (3.32):

$$(\lambda^2 m + \lambda c + k)e^{\lambda t} = 0 \qquad (3.34)$$

$$\lambda^2 m + \lambda c + k = 0 \qquad (3.35)$$

$$\lambda = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \qquad (3.36)$$

- Critical damping when: $c^2 - 4km = 0$

$$c_{cr} = 2\sqrt{km} = 2\omega_n m \qquad (3.37)$$

- Damping ratio

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2\omega_n m} \qquad (3.38)$$

- Transformation of the equation of motion

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0 \qquad (3.39)$$

$$\ddot{u}(t) + \frac{c}{m}\dot{u}(t) + \frac{k}{m}u(t) = 0 \qquad (3.40)$$

$$\ddot{u}(t) + 2\zeta\omega_n\dot{u}(t) + \omega_n^2 u(t) = 0 \qquad (3.41)$$

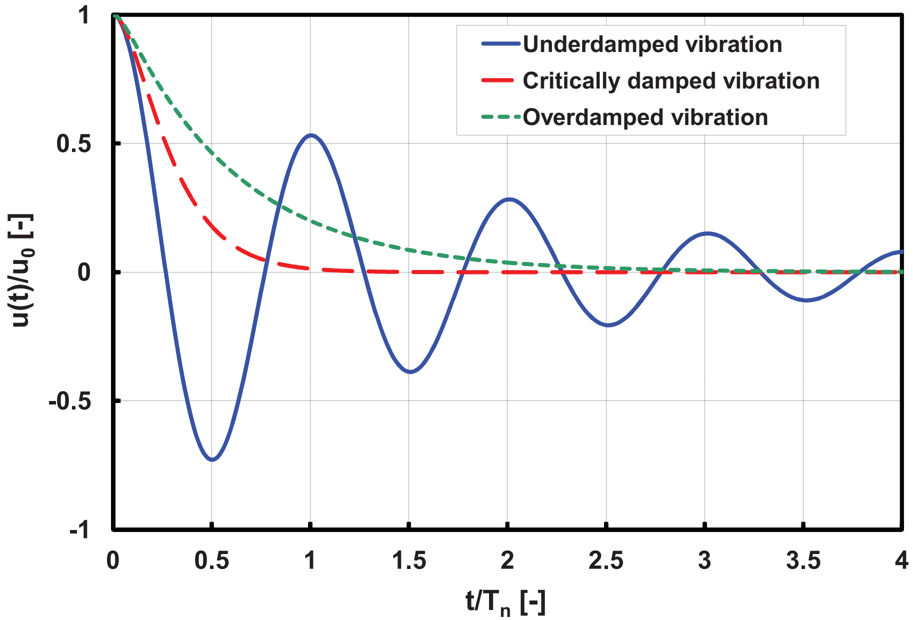
- Types of vibrations:

$\zeta = \frac{c}{c_{cr}} < 1$: Underdamped free vibrations

$\zeta = \frac{c}{c_{cr}} = 1$: Critically damped free vibrations

$\zeta = \frac{c}{c_{cr}} > 1$: Overdamped free vibrations

- Types of vibrations



Underdamped free vibrations $\zeta < 1$

By substituting:

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2\omega_n m} \text{ and } \omega_n^2 = \frac{k}{m} \quad (3.42)$$

in:

$$\lambda = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad (3.43)$$

it is obtained:

$$\lambda = -\zeta\omega_n \pm \sqrt{\omega_n^2\zeta^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \quad (3.44)$$

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} \quad (3.45)$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \quad \text{"damped circular frequency"} \quad (3.46)$$

$$\lambda = -\zeta\omega_n \pm i\omega_d \quad (3.47)$$

The complete solution of the ODE is:

$$u(t) = C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\omega_d)t} \quad (3.48)$$

$$u(t) = e^{-\zeta\omega_n t}(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}) \quad (3.49)$$

$$u(t) = e^{-\zeta\omega_n t}(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)) \quad (3.50)$$

The determination of the unknowns A_1 and A_2 is carried out as usual by means of the initial conditions for displacement ($u(0) = u_0$) and velocity ($\dot{u}(0) = v_0$) obtaining:

$$A_1 = u_0 \ , \ A_2 = \frac{v_0 + \zeta\omega_n u_0}{\omega_d} \quad (3.51)$$

3.2.2 Formulation 1: Amplitude and phase angle

Equation (3.50) can be rewritten as "the amplitude and phase angle":

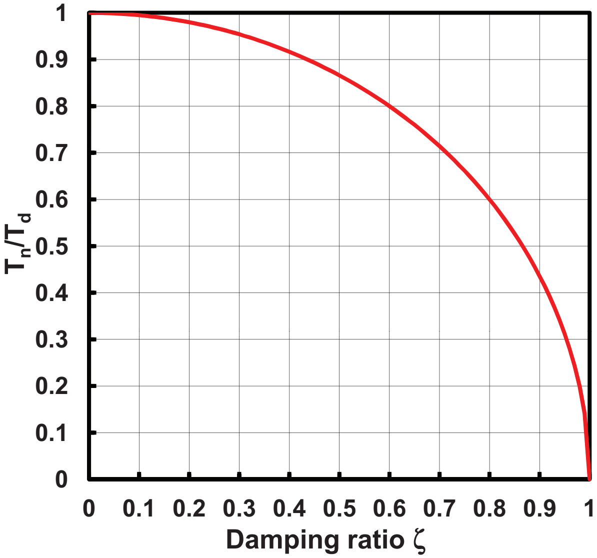
$$u(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \quad (3.52)$$

with

$$A = \sqrt{u_0^2 + \left(\frac{v_0 + \zeta\omega_n u_0}{\omega_d}\right)^2} \ , \ \tan\phi = \frac{v_0 + \zeta\omega_n u_0}{\omega_d u_0} \quad (3.53)$$

The motion is a sinusoidal vibration with circular frequency ω_d and decreasing amplitude $Ae^{-\zeta\omega_n t}$

- Notes
  - The period of the damped vibration is longer, i.e. the vibration is slower



$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

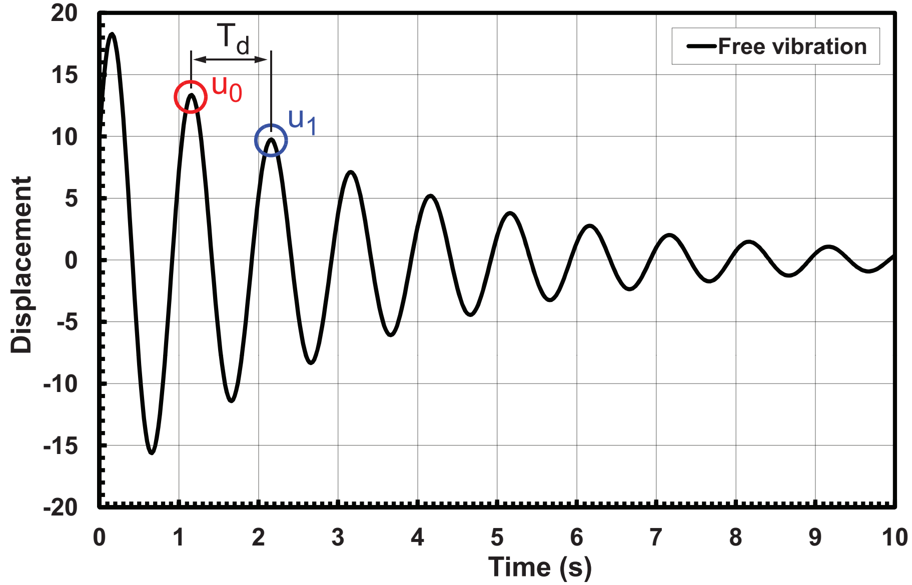
$$T_d = \frac{T_n}{\sqrt{1-\zeta^2}}$$

  - The envelope of the vibration is represented by the following equation:

$$u(t) = Ae^{-\zeta\omega_n t} \text{ with } A = \sqrt{u_0^2 + \left(\frac{v_0 + \zeta\omega_n u_0}{\omega_d}\right)^2} \quad (3.54)$$

  - Visualization of the solution by means of the Excel file given on the web page of the course (SD_FV_viscous.xlsx)

3.3 The logarithmic decrement



- Amplitude of two consecutive cycles

$$\frac{u_0}{u_1} = \frac{Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi)}{Ae^{-\zeta\omega_n(t+T_d)}\cos(\omega_d(t+T_d)-\phi)} \quad (3.55)$$

with

$$e^{-\zeta\omega_n(t+T_d)} = e^{-\zeta\omega_n t}e^{-\zeta\omega_n T_d} \quad (3.56)$$

$$\cos(\omega_d(t+T_d)-\phi) = \cos(\omega_d t + \omega_d T_d - \phi) = \cos(\omega_d t - \phi) \quad (3.57)$$

we obtain:

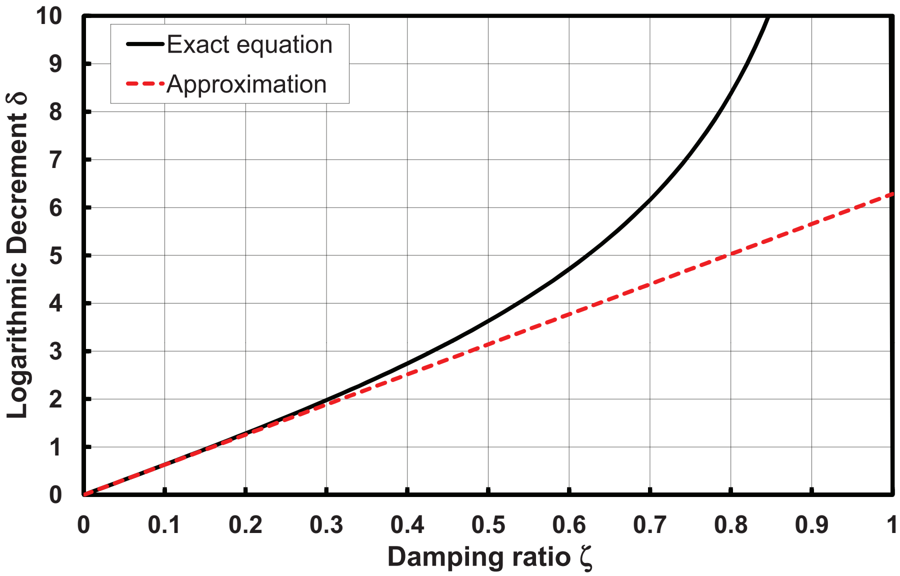
$$\frac{u_0}{u_1} = \frac{1}{e^{-\zeta\omega_n T_d}} = e^{\zeta\omega_n T_d} \quad (3.58)$$

- Logarithmic decrement δ

$$\delta = \ln\left(\frac{u_0}{u_1}\right) = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \cong 2\pi\zeta \text{ (if } \zeta \text{ small)} \quad (3.59)$$

The damping ratio becomes:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \cong \frac{\delta}{2\pi} \text{ (if } \zeta \text{ small)} \quad (3.60)$$



- Evaluation over several cycles

$$\frac{u_0}{u_N} = \frac{u_0}{u_1}\cdot\frac{u_1}{u_2}\cdot\ldots\cdot\frac{u_{N-1}}{u_N} = (e^{\zeta\omega_n T_d})^N = e^{N\zeta\omega_n T_d} \quad (3.61)$$

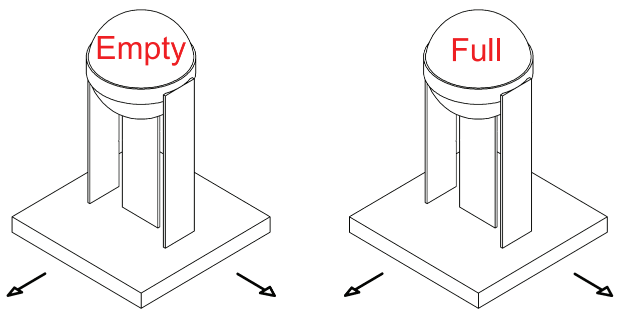
$$\delta = \frac{1}{N}\ln\left(\frac{u_0}{u_N}\right) \quad (3.62)$$

- Halving of the amplitude

$$\zeta = \frac{\frac{1}{N}\ln\left(\frac{u_0}{u_N}\right)}{2\pi} = \frac{\frac{1}{N}\ln(2)}{2\pi} = \frac{1}{9N} \cong \frac{1}{10N} \quad (3.63)$$

Useful formula for quick evaluation

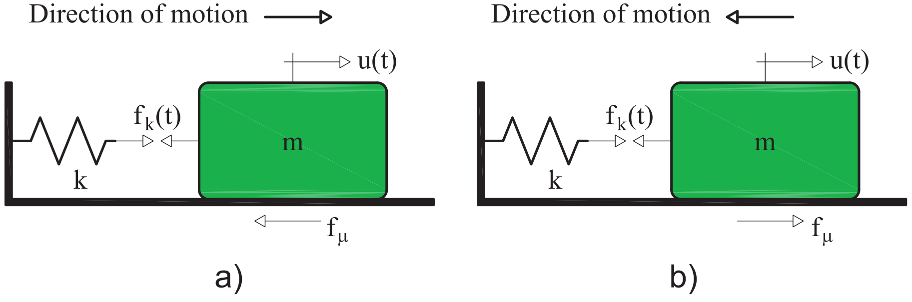
- Watch out: damping ratio vs. damping constant



m_1, k_1, c_1 $m_2 > m_1, k_1, c_1$

$$\zeta_1 = \frac{c_1}{2\sqrt{k_1 m_1}} \qquad \zeta_2 = \frac{c_1}{2\sqrt{k_1 m_2}} < \zeta_1$$

3.4 Friction damping



$$-f_k(t) - f_\mu = m\ddot{u}(t) \qquad -f_k(t) + f_\mu = m\ddot{u}(t)$$

$$m\ddot{u}(t) + ku(t) = -f_\mu \qquad m\ddot{u}(t) + ku(t) = f_\mu$$

- Solution of b)

$$u(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) + u_\mu \quad \text{with } u_\mu = \frac{f_\mu}{k} \quad (3.64)$$

$$\dot{u}(t) = -\omega_n A_1 \sin(\omega_n t) + \omega_n A_2 \cos(\omega_n t) \quad (3.65)$$

by means of the initial conditions $u(0) = u_0$, $\dot{u}(0) = v_0$ we obtain the constants:

$$A_1 = u_0 - u_\mu \ , \ A_2 = v_0/\omega_n$$

- Solution of a): Similar, with $-u_\mu$ instead of $+u_\mu$

- Free vibrations

  It is a nonlinear problem!

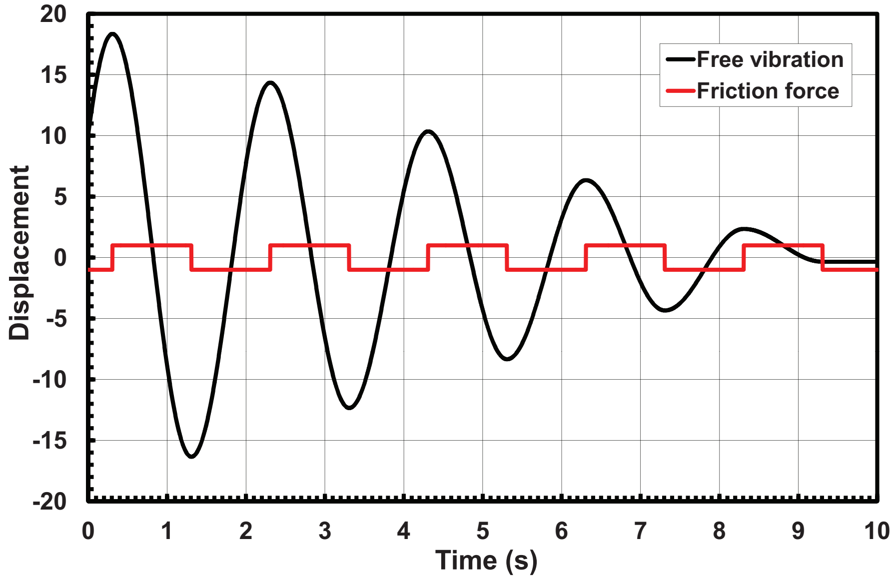


Figure: f=0.5 Hz , u_0=10 , v_0 = 50, u_f = 1

- Calculation example:
  - **Step 1:**

    Initial conditions $u(0) = u_0$, $\dot{u}(0) = 0$

$$A_1 = u_0 - u_\mu \ , \ A_2 = 0 \quad (3.66)$$

$$u(t) = [u_0 - u_\mu]\cos(\omega_n t) + u_\mu \qquad 0 \le t < \frac{\pi}{\omega_n} \quad (3.67)$$

End displacement: $u\left(\frac{\pi}{\omega_n}\right) = [u_0 - u_\mu](-1) + u_\mu = -u_0 + 2u_\mu$

- **Step 2:**

  Initial conditions $u(0) = -u_0 + 2u_\mu \ , \ \dot{u}(0) = 0$

$$A_1 = u(0) + u_\mu = -u_0 + 2u_\mu + u_\mu = -u_0 + 3u_\mu \ , \ A_2 = 0 \quad (3.68)$$

$$u(t) = [-u_0 + 3u_\mu]\cos(\omega_n t) - u_\mu \qquad 0 \le t < \frac{\pi}{\omega_n} \quad (3.69)$$

  End displacement: $u\left(\frac{\pi}{\omega_n}\right) = [-u_0 + 3u_\mu](-1) - u_\mu = u_0 - 4u_\mu$

- **Step 3:**

  Initial conditions ....

- **Important note:**
  The change between case a) and case b) occurs at velocity reversals. In order to avoid the build-up of inaccuracies, the displacement at velocity reversal should be identified with adequate precision (iterate!)
- Visualization of the solution by means of the Excel file given on the web page of the course (SD_FV_friction.xlsx)
- Characteristics of friction damping
  - Linear decrease in amplitude by $4u_\mu$ at each cycle
  - The period of the damped and of the undamped oscillator is the same:

$$T_n = \frac{2\pi}{\omega_n}$$

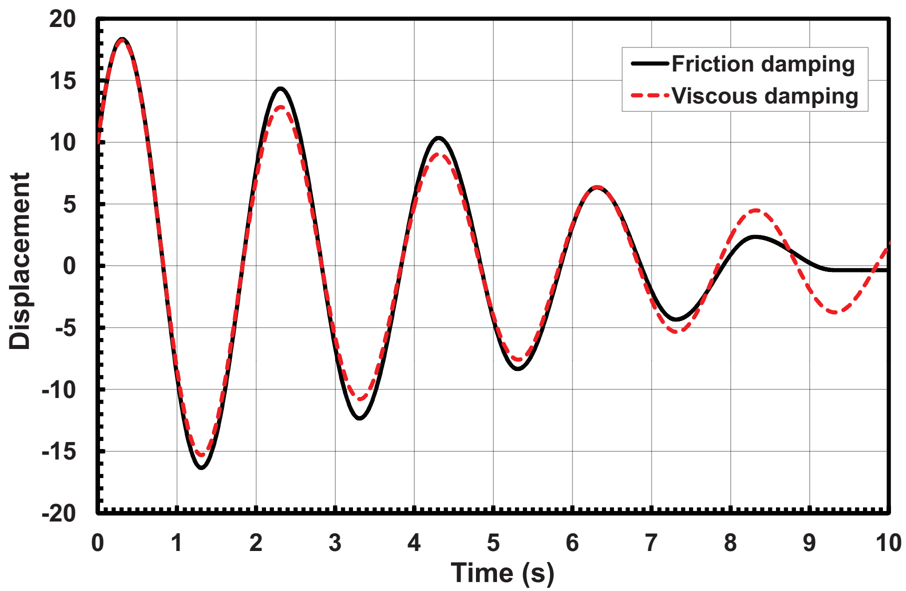
- Comparison Viscous damping vs. Friction damping

  Free vibration: f=0.5 Hz , u_0=10 , v_0 = 50, u_f = 1

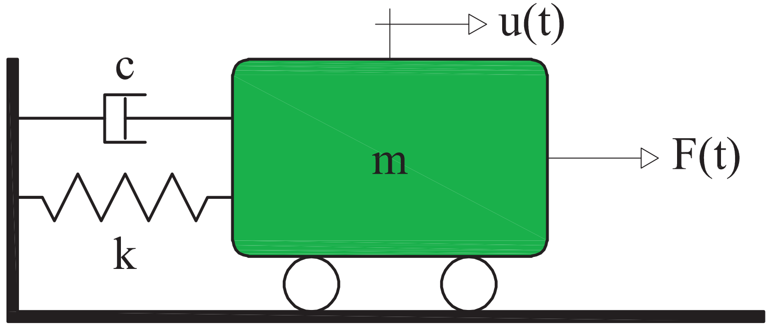
  Logarithmic decrement:

	U_0	U_N	N	δ	ζ [%]
1	18.35	14.35	1	0.245	3.91
2	18.35	10.35	2	0.286	4.56
3	18.35	6.35	3	0.354	5.63
4	18.35	2.35	4	0.514	8.18
Average					**5.57**

  Comparison:



4 Response to Harmonic Excitation



An harmonic excitation can be described either by means of a sine function (Equation 4.1) or by means of a cosine function (Equation 4.2):

$$m\ddot{u} + c\dot{u} + ku = F_o \sin(\omega t) \quad (4.1)$$

$$m\ddot{u} + c\dot{u} + ku = F_o \cos(\omega t) \quad (4.2)$$

Here we consider Equation (4.2) which after transformation becomes:

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = f_o \cos(\omega t) \quad (4.3)$$

where: ω_n: **Circular frequency of the SDoF system**

ω: **Circular frequency of the excitation**

$f_o = F_o/m = (F_o/k) \cdot \omega_n^2$

Course "Fundamentals of Structural Dynamics" *An-Najah 2013*

Linear inhomogeneous differential equation

- Particular solution: u_p

$$\ddot{u}_p + 2\zeta\omega_n\dot{u}_p + \omega_n^2 u_p = f(t) \quad (4.4)$$

- Solution of the homogeneous ODE: u_h

$$\ddot{u}_h + 2\zeta\omega_n\dot{u}_h + \omega_n^2 u_h = 0 \quad (4.5)$$

- Complete solution: $u = u_p + Cu_h$

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = f_o \cos(\omega t) \quad (4.6)$$

- Initial conditions

$$u(0) = u_0 \quad , \quad \dot{u}(0) = v_0 \quad (4.7)$$

4.1 Undamped harmonic vibrations

$$\ddot{u} + \omega_n^2 u = f_o \cos(\omega t) \quad (4.8)$$

- Ansatz for particular solution

$$u_p = A_o \cos(\omega t) \quad (4.9)$$

$$\ddot{u}_p = -A_o \omega^2 \cos(\omega t) \quad (4.10)$$

By substituting (4.9) and (4.10) in (4.8):

$$-A_o \omega^2 \cos(\omega t) + A_o \omega_n^2 \cos(\omega t) = f_o \cos(\omega t) \quad (4.11)$$

$$A_o(-\omega^2 + \omega_n^2) = f_o \quad (4.12)$$

$$A_o = \frac{f_o}{\omega_n^2 - \omega^2} = \frac{F_o}{k} \cdot \frac{1}{1-(\omega/\omega_n)^2} \quad (4.13)$$

$$u_p = \frac{f_o}{\omega_n^2 - \omega^2} \cos(\omega t) \quad (4.14)$$

- Ansatz for the solution of the homogeneous ODE (see section on free vibrations)

$$u_h = B_1 \cos(\omega_n t) + B_2 \sin(\omega_n t) \quad (4.15)$$

- Complete solution of the ODE:

$$u = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) + \frac{f_o}{\omega_n^2 - \omega^2} \cos(\omega t) \quad (4.16)$$

By means of the initial conditions given in Equation (4.7), the constants A_1 and A_2 can be calculated as follows:

$$A_1 = u_0 - \frac{f_o}{\omega_n^2 - \omega^2} \quad , \quad A_2 = \frac{v_0}{\omega_n} \quad (4.17)$$

- Denominations:
  - Homogeneous part of the solution: **"transient"**
  - Particular part of the solution: **"steady-state"**
- Visualization of the solution by means of the Excel file given on the web page of the course (SD_HE_cosine_viscous.xlsx)

- Harmonic vibration with sine excitation

$$u = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) + \frac{f_o}{\omega_n^2 - \omega^2} \sin(\omega t)$$

By means of the initial conditions given in Equation (3.7), the constants A_1 and A_2 can be calculated as follows:

$$A_1 = u_0 \quad , \quad A_2 = \frac{v_0}{\omega_n} - \frac{f_o(\omega/\omega_n)}{\omega_n^2 - \omega^2}$$

4.1.1 Interpretation as a beat

$$\ddot{u} + \omega_n^2 u = f_o \cos(\omega t) \text{ with } u(0) = \dot{u}(0) = 0 \tag{4.18}$$

The solution is:

$$u(t) = \frac{f_o}{\omega_n^2 - \omega^2} \cdot [\cos(\omega t) - \cos(\omega_n t)] \tag{4.19}$$

and using the trigonometric identity

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha - \beta}{2}t\right)\sin\left(\frac{\alpha + \beta}{2}t\right) \tag{4.20}$$

one gets the equation

$$u(t) = \frac{2f_o}{\omega^2 - \omega_n^2} \cdot \sin\left(\frac{\omega - \omega_n}{2}t\right)\sin\left(\frac{\omega + \omega_n}{2}t\right) \tag{4.21}$$

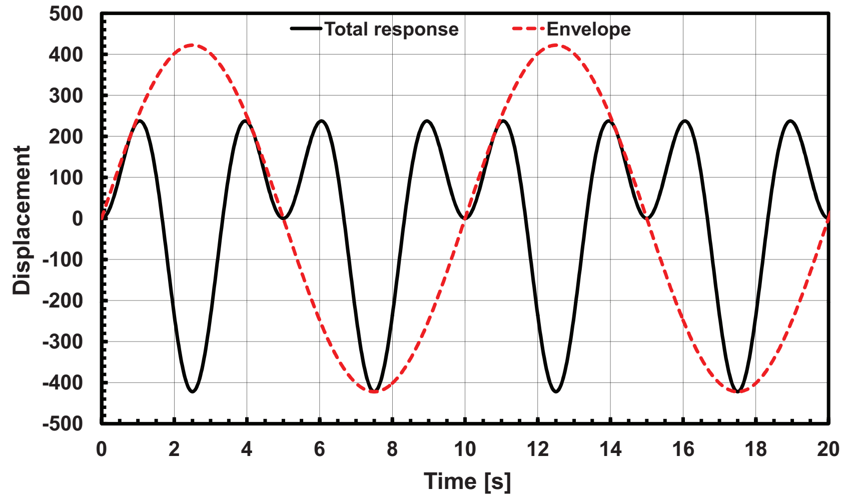
that describes a beat with:

Fundamental vibration: $$f_G = \frac{f + f_n}{2} \tag{4.22}$$

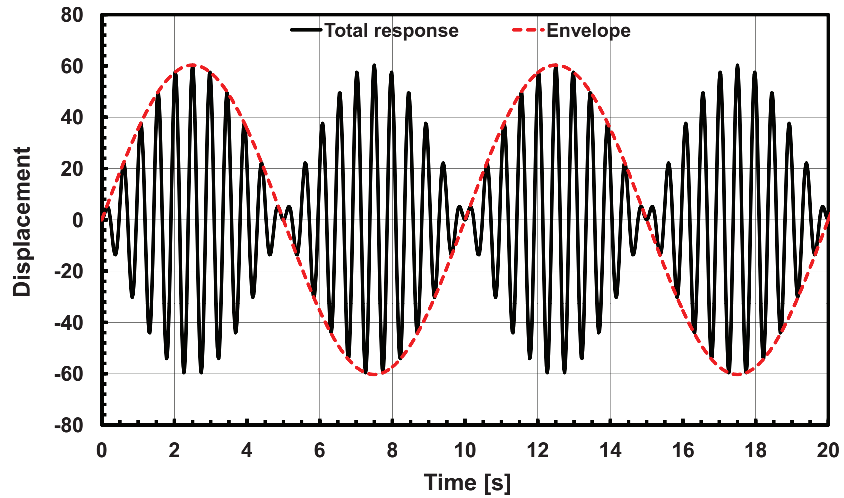
Envelope: $$f_U = \frac{f - f_n}{2} \tag{4.23}$$

A beat is always present, but is only evident when the natural frequency of the SDoF system and the excitation frequency are close (see figures on the next page)

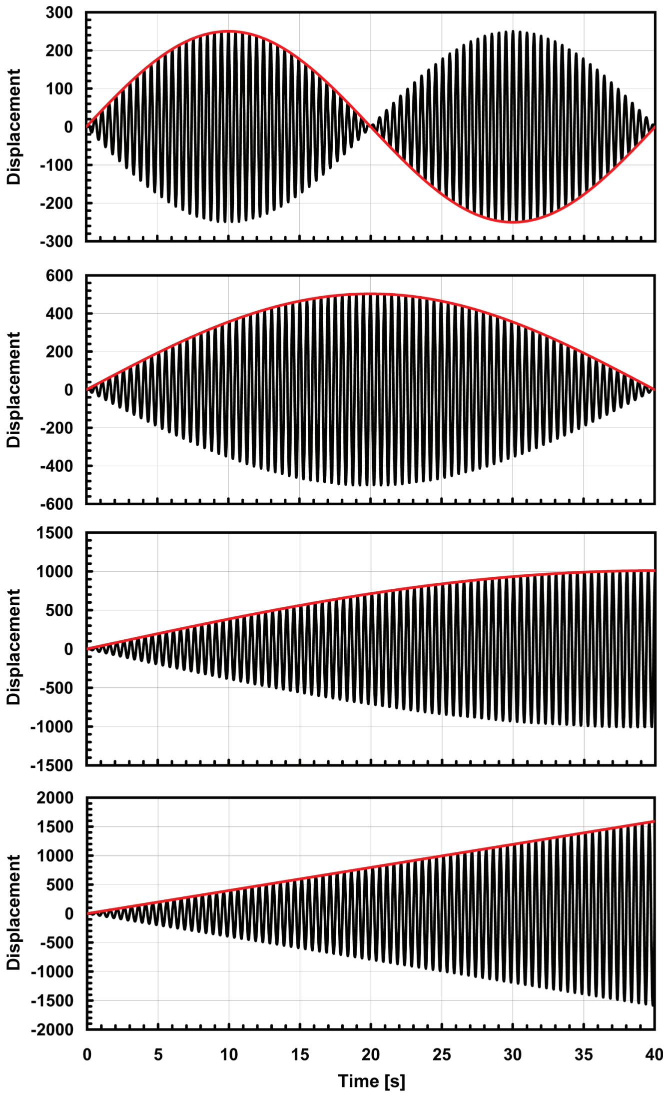
- Case 1: Natural frequency SDoF 0.2 Hz, excitation frequency 0.4 Hz



- Case 2: Natural frequency SDoF 2.0 Hz, excitation frequency 2.2 Hz



- Transition to $f = f_n$



$$\frac{f}{f_n} = \frac{2.0500}{2.0000}$$

$$\frac{f}{f_n} = \frac{2.0250}{2.0000}$$

$$\frac{f}{f_n} = \frac{2.0125}{2.0000}$$

$$\frac{f}{f_n} = \frac{2.0000}{2.0000}$$

Resonance!

4.1.2 Resonant excitation ($\omega = \omega_n$)

$$\ddot{u} + \omega_n^2 u = f_o \cos(\omega_n t) \quad (4.24)$$

- Ansatz for the particular solution

$$u_p = A_o t \sin(\omega_n t) \quad (4.25)$$

$$\dot{u}_p = A_o \sin(\omega_n t) + A_o \omega_n t \cos(\omega_n t) \quad (4.26)$$

$$\ddot{u}_p = 2A_o \omega_n \cos(\omega_n t) - A_o \omega_n^2 t \sin(\omega_n t) \quad (4.27)$$

By substituting Equations (4.25) and (4.27) in (4.24):

$$2A_o \omega_n \cos(\omega_n t) - \cancel{A_o \omega_n^2 t \sin(\omega_n t)} + \cancel{A_o \omega_n^2 t \sin(\omega_n t)} = f_o \cos(\omega_n t) \quad (4.28)$$

$$2A_o \omega_n = f_o \quad (4.29)$$

$$A_o = \frac{f_o}{2\omega_n} = \frac{F_o}{k} \cdot \frac{\omega_n}{2} \quad (4.30)$$

$$u_p = \frac{f_o}{2\omega_n} t \sin(\omega_n t) \quad (4.31)$$

- Ansatz for the solution of the homogeneous ODE (see section on free vibrations)

$$u_h = B_1 \cos(\omega_n t) + B_2 \sin(\omega_n t) \quad (4.32)$$

- Complete solution of the ODE:

$$u = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) + \frac{f_o}{2\omega_n} t \sin(\omega_n t) \tag{4.33}$$

By means of the initial conditions given in Equation (4.7), the constants A_1 and A_2 can be calculated as follows:

$$A_1 = u_0 \quad , \quad A_2 = \frac{v_0}{\omega_n} \tag{4.34}$$

- Special case $u_0 = v_0 = 0$
(The homogeneous part of the solution falls away)

$$u = \frac{f_o}{2\omega_n} t \sin(\omega_n t) \tag{4.35}$$

Is a sinusoidal vibration with amplitude:

$$A = \frac{f_o}{2\omega_n} t \tag{4.36}$$

  - The amplitude grows linearly with time (see last picture of interpretation “beat”);
  - We have $A \to \infty$ when $t \to \infty$, i.e. after infinite time the amplitude of the vibration is infinite as well.

- Visualization of the solution by means of the Excel file given on the web page of the course (SD_HE_cosine_viscous.xlsx)

4.2 Damped harmonic vibration

$$\ddot{u} + 2\zeta\omega_n \dot{u} + \omega_n^2 u = f_o \cos(\omega t) \tag{4.37}$$

- Ansatz for particular solution

$$u_p = A_3 \cos(\omega t) + A_4 \sin(\omega t) \tag{4.38}$$

$$\dot{u}_p = -A_3 \omega \sin(\omega t) + A_4 \omega \cos(\omega t) \tag{4.39}$$

$$\ddot{u}_p = -A_3 \omega^2 \cos(\omega t) - A_4 \omega^2 \sin(\omega t) \tag{4.40}$$

By substitution Equations (4.38) to (4.40) in (4.37):

$$[(\omega_n^2 - \omega^2)A_3 + 2\zeta\omega_n\omega A_4]\cos(\omega t) + [-2\zeta\omega_n\omega A_3 + (\omega_n^2 - \omega^2)A_4]\sin(\omega t) = f_o \cos(\omega t) \tag{4.41}$$

Equation (4.41) shall be true for all times t and for all constants A_3 and A_4, therefore Equations (4.42) and (4.43) can be written as follows:

$$(\omega_n^2 - \omega^2)A_3 + 2\zeta\omega_n\omega A_4 = f_o \tag{4.42}$$

$$-2\zeta\omega_n\omega A_3 + (\omega_n^2 - \omega^2)A_4 = 0 \tag{4.43}$$

The solution of the system [(4.42), (4.43)] allows the calculations of the constants A_3 and A_4 as:

$$A_3 = f_o \frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} , \quad A_4 = f_o \frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \tag{4.44}$$

- Ansatz for the solution of the homogeneous ODE (see Section 3.2 on damped free vibrations)

$$u_h = e^{-\zeta\omega_n t}(B_1\cos(\omega_d t) + B_2\sin(\omega_d t)) \tag{4.45}$$

with:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad \text{"damped circular frequency"} \tag{4.46}$$

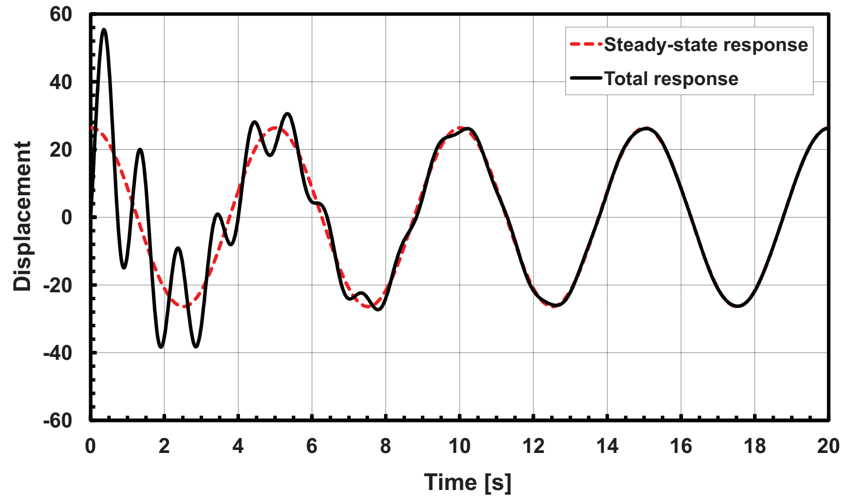
- Complete solution of the ODE:

$$u = e^{-\zeta\omega_n t}(A_1\cos(\omega_d t) + A_2\sin(\omega_d t)) + A_3\cos(\omega t) + A_4\sin(\omega t) \tag{4.47}$$

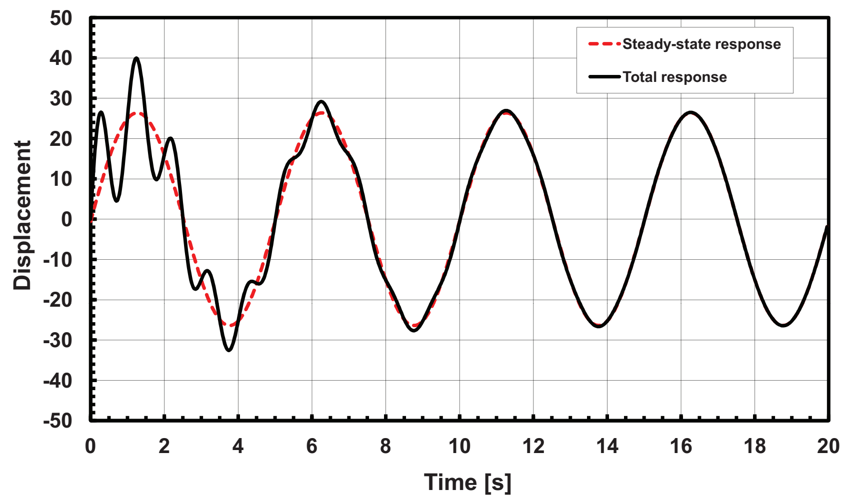
By means of the initial conditions of Equation (4.7), the constants A_1 and A_2 can be calculated. The calculation is laborious and should be best carried out with a mathematics program (e.g. Maple).

- Denominations:
  - Homogeneous part of the solution: **"transient"**
  - Particular part of the solution: **"steady-state"**
- Visualization of the solution by means of the Excel file given on the web page of the course (SD_HE_cosine_viscous.xlsx)

- Example 1: f_n = 1Hz, f = 0.2Hz, ζ = 5%, f_o = 1000, u_0 = 0, $v_0 = f_o/\omega_n$



- Example 2: Like 1 but with $F(t) = F_o\sin(\omega t)$ instead of $F_o\cos(\omega t)$



4.2.1 Resonant excitation ($\omega = \omega_n$)

By substituting $\omega = \omega_n$ in Equation (4.44) the constants A_3 and A_4 becomes:

$$A_3 = 0, \; A_4 = \frac{f_o}{2\zeta\omega_n^2} \tag{4.48}$$

i.e. if damping is present, the resonant excitation is not a special case any more, and the complete solution of the differential equation is:

$$u = e^{-\zeta\omega_n t}(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)) + \frac{f_o}{2\zeta\omega_n^2}\sin(\omega_n t) \tag{4.49}$$

- Special case $u_0 = v_0 = 0$

$$A_1 = 0 \quad , \quad A_2 = -\frac{f_o}{2\zeta\omega_n^2\sqrt{1-\zeta^2}} = -\frac{f_o}{2\zeta\omega_n\omega_d} \tag{4.50}$$

$$u = \frac{f_o}{2\zeta\omega_n^2}\left(\sin(\omega_n t) - \frac{\sin(\omega_d t)}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\right) \tag{4.51}$$

  - After a certain time, the homogeneous part of the solution subsides and what remains is a sinusoidal oscillation of the amplitude:

$$A = \frac{f_o}{2\zeta\omega_n^2} \tag{4.52}$$

- The amplitude is limited, i.e. the maximum displacement of the SDoF system is:

$$u_{max} = \frac{f_o}{2\zeta\omega_n^2} = \frac{F_o}{2\zeta k} = \frac{u_{st}}{2\zeta} \tag{4.53}$$

where $u_{st} = F_o/k$ is the static displacement.

- For small damping ratios ($\zeta \le 0.2$) $\omega_d \approx \omega_n$ and $\sqrt{1-\zeta^2} \approx 1$ hence Equations (4.51) becomes:

$$u = \frac{f_o}{2\zeta\omega_n^2}(1 - e^{-\zeta\omega_n t})\sin(\omega_n t) = u_{max}(1 - e^{-\zeta\omega_n t})\sin(\omega_n t) \tag{4.54}$$

It is a sinusoidal vibration with the amplitude:

$$A = u_{max}(1 - e^{-\zeta\omega_n t}) \tag{4.55}$$

and the magnitude of the amplitude at each maxima j is

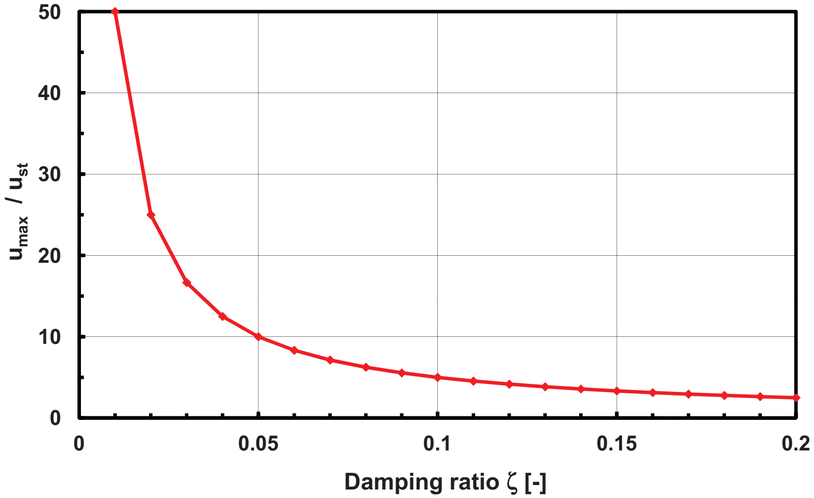
$$\frac{u_j}{u_{max}} = (1 - e^{-\zeta\omega_n t_j})\sin(\omega_n t_j) \tag{4.56}$$

Maxima occur when $\sin(\omega_n t) = -1$, d.h. when

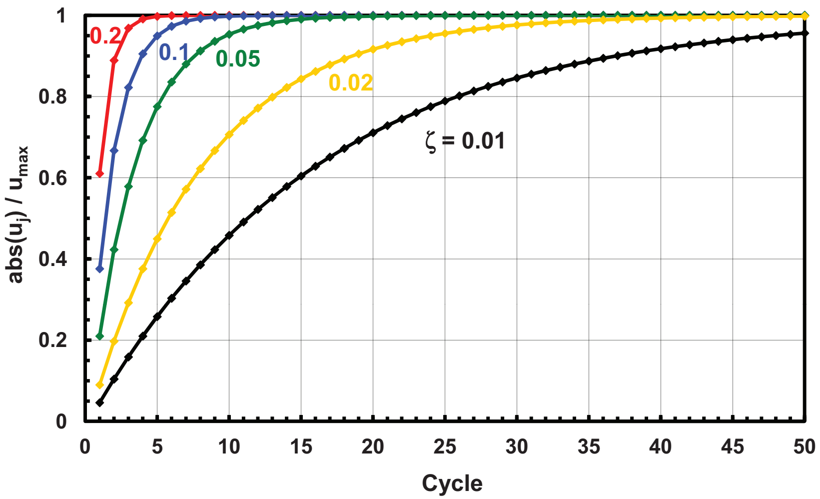
$$t_j = (4j-1)\cdot\frac{T_n}{4}, \; j = 1\ldots\infty \tag{4.57}$$

$$\frac{|u_j|}{u_{max}} = 1 - e^{-\zeta\omega_n(4j-1)\cdot\frac{T_n}{4}} = 1 - e^{-\zeta(4j-1)\cdot\frac{\pi}{2}} \tag{4.58}$$

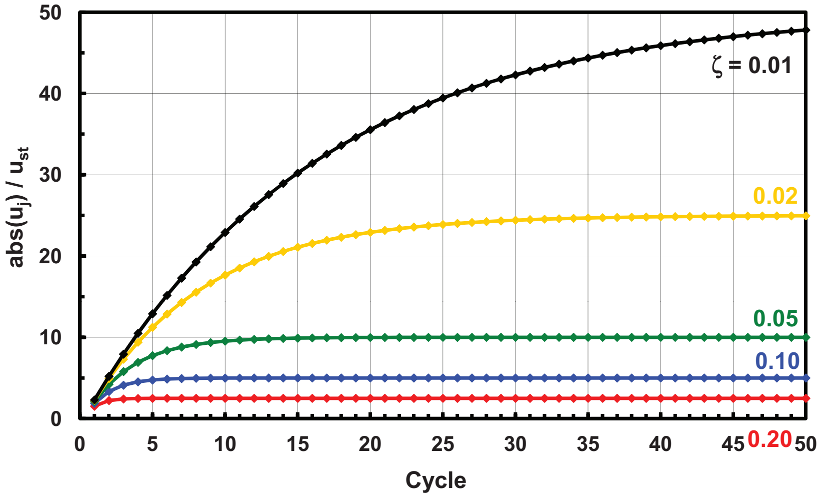
- Dynamic amplification



- Magnitude of the amplitude after each cycle: $f(u_{max})$



- Magnitude of the amplitude after each cycle: $f(u_{st})$



5 Transfer Functions

5.1 Force excitation

The steady-state displacement of a system due to harmonic excitation is (see Section 4.2 on harmonic excitation):

$$u_p = a_1 \cos(\omega t) + a_2 \sin(\omega t) \qquad (5.1)$$

with

$$a_1 = f_o \frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}, \quad a_2 = f_o \frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \qquad (5.2)$$

By means of the trigonometric identity

$$a\cos(\alpha) + b\sin(\alpha) = \sqrt{a^2 + b^2} \cdot \cos(\alpha - \phi) \text{ where } \tan\phi = \frac{b}{a} \qquad (5.3)$$

Equation (5.1) can be transformed as follows:

$$u_p = u_{max} \cos(\omega t - \phi) \qquad (5.4)$$

It is a cosine vibration with the maximum dynamic amplitude u_{max}:

$$u_{max} = \sqrt{{a_1}^2 + {a_2}^2} \qquad (5.5)$$

and the phase angle ϕ obtained from:

$$\tan\phi = \frac{a_2}{a_1} \qquad (5.6)$$

The maximum dynamic amplitude u_{max} given by Equation (5.5) can be transformed to:

$$u_{max} = \sqrt{\left(f_o \frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\right)^2 + \left(f_o \frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\right)^2} \qquad (5.7)$$

$$u_{max} = f_o \sqrt{\frac{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}{[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2]^2}} \qquad (5.8)$$

$$u_{max} = f_o \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \qquad (5.9)$$

$$u_{max} = \frac{f_o}{\omega_n^2} \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \qquad (5.10)$$

Introducing the maximum static amplitude $u_o = F_o/k = f_o/\omega_n^2$ the dynamic amplification factor $V(\omega)$ can be defined as:

$$V(\omega) = \frac{u_{max}}{u_o} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \qquad (5.11)$$

The maximum amplification factor $V(\omega)$ occurs when its derivative, given by Equation (5.12), is equal to zero.

$$\frac{dV}{d\omega} = \frac{2\omega\omega_n^2[\omega^2 - \omega_n^2(1-2\zeta^2)]}{[\omega^4 - 2(1-2\zeta^2)\omega^2\omega_n^2 + \omega_n^4]^{(3/2)}} \quad (5.12)$$

$$\frac{dV}{d\omega} = 0 \text{ when: } \omega = 0 \text{ , } \omega = \pm\omega_n\sqrt{1-2\zeta^2} \quad (5.13)$$

The maximum amplification factor $V(\omega)$ occurs when:

$$\omega = \omega_n\sqrt{1-2\zeta^2} \text{ for } \zeta < \frac{1}{\sqrt{2}} \approx 0.71 \quad (5.14)$$

and we have:

$$\omega = \omega_n: \qquad V = \frac{1}{2\zeta} \quad (5.15)$$

$$\omega = \omega_n\sqrt{1-2\zeta^2}: \qquad V = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad (5.16)$$

From Equation (5.6), the phase angle ϕ is:

$$\tan\phi = \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} = \frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2} \quad (5.17)$$

The phase angle has the following interesting property:

$$\frac{d\phi}{d(\omega/\omega_n)} = \frac{2\zeta[1+(\omega/\omega_n)^2]}{1-2(\omega/\omega_n)^2 + (\omega/\omega_n)^4 + 4\zeta^2(\omega/\omega_n)^2} \quad (5.18)$$

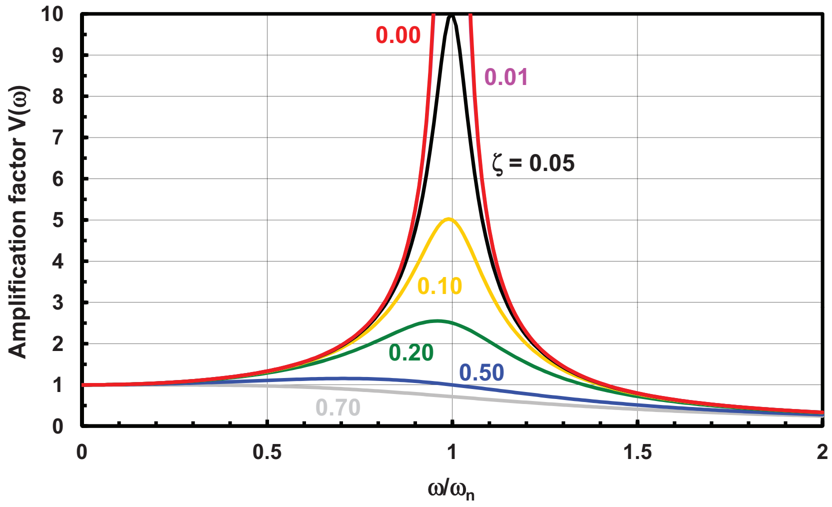
at $\omega/\omega_n = 1$ we have: $\frac{d\phi}{d(\omega/\omega_n)} = \frac{1}{\zeta}$ $\left(= \frac{1}{\zeta}\cdot\frac{180}{\pi}\right.$ when ϕ in deg) (5.19)

5.1.1 Comments on the amplification factor V

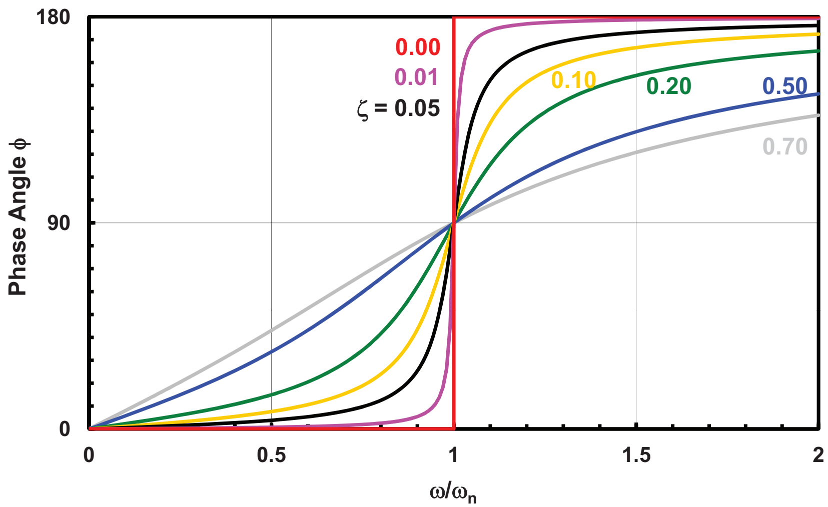
$$V(\omega) = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \quad (5.20)$$

- $\omega/\omega_n \ll 1$: Slow variation of the excitation (ζ not important)
  - $V(\omega) \approx 1$ therefore: $u_{max} \approx u_o$
  - $\phi \approx 0$: Motion and excitation force are in phase
- $\omega/\omega_n \gg 1$: Quick variation of the excitation (ζ not important)
  - $V(\omega) \approx \left(\frac{\omega_n}{\omega}\right)^2$
  - $u_{max} \approx u_o \cdot \left(\frac{\omega_n}{\omega}\right)^2 = F_o/(m\omega^2)$: Mass controls the behaviour
  - $\phi \approx 180$: Motion and excitation force are opposite
- $(\omega/\omega_n) \approx 1$: ($\zeta$ very important)
  - $V(\omega) \approx \frac{1}{2\zeta}$
  - $u_{max} \approx u_o/(2\zeta) = F_o/(c\omega_n)$: Damping controls the behaviour
  - $\phi \approx 90$: zero displacement when excitation force is maximum

- Amplification factor



- Phase angle



- Example:

An excitation produces the static displacement

$$u_{st} = \frac{F_o \cos(\omega t)}{k} \quad (5.21)$$

and its maximum is:

$$u_o = \frac{F_o}{k} \quad (5.22)$$

The steady-state dynamic response of the system is:

$$u_p = u_{max} \cos(\omega t - \phi) \quad (5.23)$$

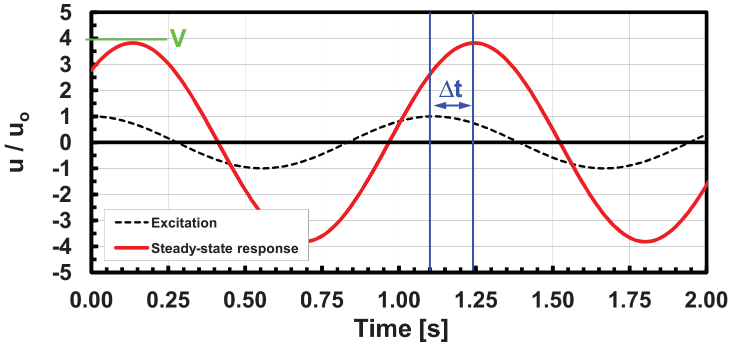
therefore:

$$\frac{u_{st}}{u_o} = \cos(\omega t) \quad , \quad \frac{u_p}{u_o} = V \cos(\omega t - \phi) \quad (5.24)$$

In the next plots the time histories of u_{st}/u_o and u_p/u_o are represented and compared.

The phase angle ϕ is always positive and because of the minus sign in Equation (5.24) it shows how much the response to the excitation lags behind.

Frequency of SDoF System f_n = 1Hz (ω_n = 6.28rad/s), Damping ζ = 0.1



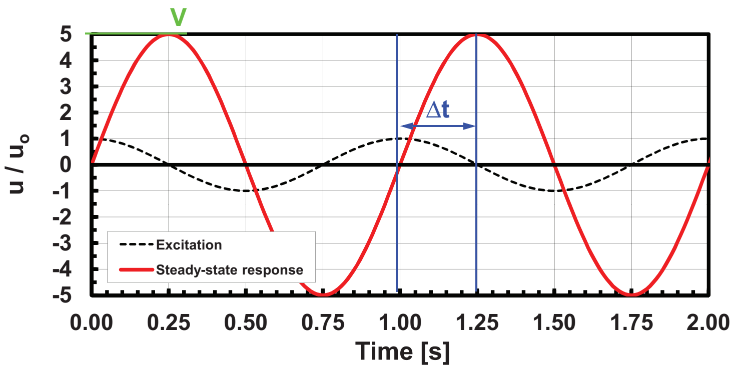
$\omega/\omega_n = 0.9$

$$\omega = 0.9 \cdot 6.28 = 5.65 \text{ rad/s}$$

$$V = 3.82$$

$$\phi = 43.45° = 0.76 \text{ rad}$$

$$\Delta t = \frac{\phi}{\omega} = \frac{0.76}{5.65} = 0.14 \text{ s}$$



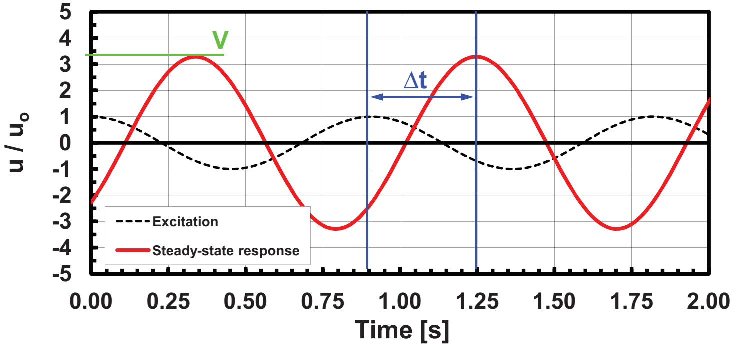
$\omega/\omega_n = 1.0$

$$\omega = 1.0 \cdot 6.28 = 6.28 \text{ rad/s}$$

$$V = 5.00$$

$$\phi = 90.00° = 1.57 \text{ rad}$$

$$\Delta t = \frac{\phi}{\omega} = \frac{1.57}{6.28} = 0.25 \text{ s}$$



$\omega/\omega_n = 1.1$

$$\omega = 1.1 \cdot 6.28 = 6.91 \text{ rad/s}$$

$$V = 3.28$$

$$\phi = 133.66° = 2.33 \text{ rad}$$

$$\Delta t = \frac{\phi}{\omega} = \frac{2.33}{6.91} = 0.34 \text{ s}$$

5.1.2 Steady-state displacement quantities

- Displacement: Corresponds to Equation (5.4)

$$\frac{u_p}{F_o/k} = V(\omega)\cos(\omega t - \phi) \tag{5.25}$$

- Velocity: Obtained by derivating Equation (5.25)

$$\frac{\dot{u}_p}{F_o/k} = -V(\omega)\omega\sin(\omega t - \phi) \tag{5.26}$$

$$\frac{\dot{u}_p}{(F_o/k)\omega_n} = -V(\omega)\frac{\omega}{\omega_n}\sin(\omega t - \phi) \tag{5.27}$$

$$\frac{\dot{u}_p}{F_o/\sqrt{km}} = -V_v(\omega)\sin(\omega t - \phi) \text{ with } V_v(\omega) = \frac{\omega}{\omega_n}V(\omega) \tag{5.28}$$

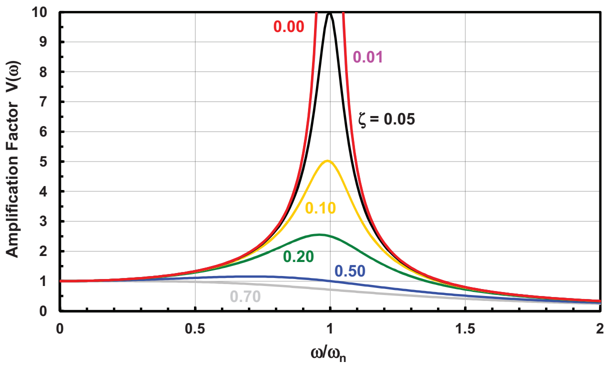
- Acceleration: Obtained by derivating Equation (5.26)

$$\frac{\ddot{u}_p}{F_o/k} = -V(\omega)\omega^2\cos(\omega t - \phi) \tag{5.29}$$

$$\frac{\ddot{u}_p}{(F_o/k)\omega_n^2} = -V(\omega)\frac{\omega^2}{\omega_n^2}\cos(\omega t - \phi) \tag{5.30}$$

$$\frac{\ddot{u}_p}{F_o/m} = -V_a(\omega)\cos(\omega t - \phi) \text{ with } V_a(\omega) = \frac{\omega^2}{\omega_n^2}V(\omega) \tag{5.31}$$

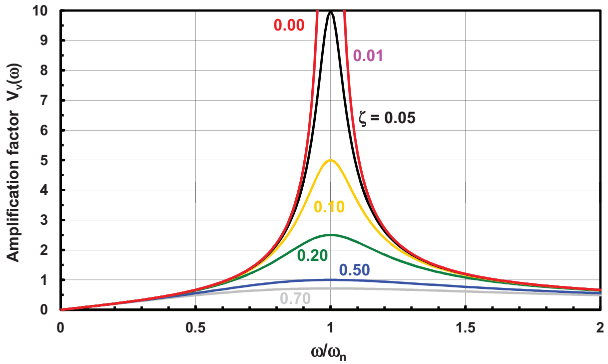
- Amplification factors



Resonant displacement

$$\omega = \omega_n\sqrt{1-2\zeta^2}$$

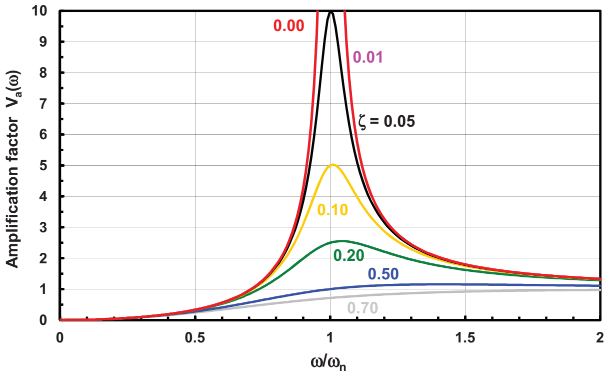
$$V = \frac{1}{2\zeta\sqrt{1-\zeta}}$$



Resonant velocity

$$\omega = \omega_n$$

$$V = \frac{1}{2\zeta}$$



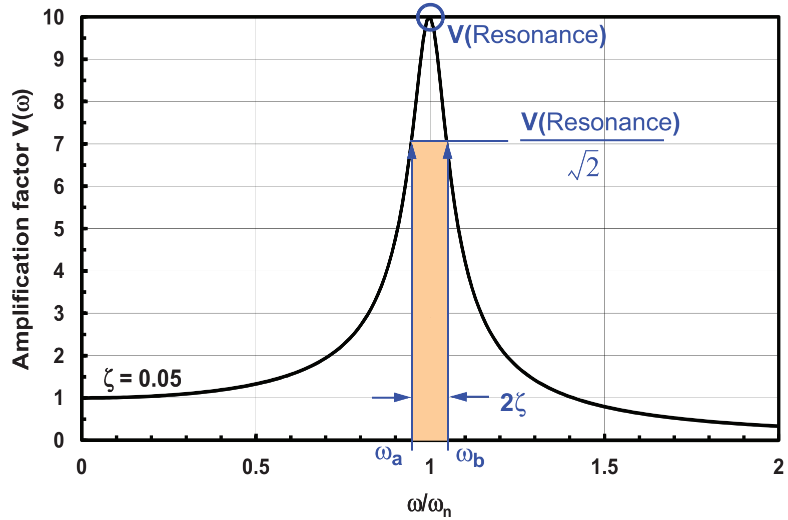
Resonant acceleration

$$\omega = \frac{\omega_n}{\sqrt{1-2\zeta^2}}$$

$$V = \frac{1}{2\zeta\sqrt{1-\zeta}}$$

5.1.3 Derivating properties of SDoF systems from harmonic vibrations

- Half-power bandwidth



Condition:

$$V(\omega) = \frac{V(\omega/\omega_n = \sqrt{1-2\zeta^2})}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{2\zeta\sqrt{1-\zeta}} \qquad (5.32)$$

$$\frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2\zeta(\omega/\omega_n)]^2}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{2\zeta\sqrt{1-\zeta}} \qquad (5.33)$$

$$\left(\frac{\omega}{\omega_n}\right)^4 - 2(1-2\zeta^2)\left(\frac{\omega}{\omega_n}\right)^2 + 1 - 8\zeta^2(1-\zeta^2) = 0 \qquad (5.34)$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = 1-2\zeta^2 \pm 2\zeta\sqrt{1-\zeta^2} \qquad (5.35)$$

For small damping, the terms featuring ζ^2 can be neglected:

$$\frac{\omega}{\omega_n} \approx \sqrt{1 \pm 2\zeta} \approx 1 \pm \zeta \qquad (5.36)$$

This yield the solution for the half-power bandwidth:

$$2\zeta = \frac{\omega_b - \omega_a}{\omega_n} \qquad (5.37)$$

- Remarks on the frequency response curve
  - The natural frequency of the system can be derived from the resonant response. However, it is sometimes problematic to build the whole frequency response curve because at resonance the system could be damaged. For this reason it is often better to determine the properties of a system based on vibration decay tests (see section on free vibration)
  - The natural frequency ω_n can be estimated by varying the Excitation until a $90°$ phase shift in the response occurs.
  - Damping can be calculated by means of Equation (5.15) as:

    $$\zeta = \frac{1}{2} \cdot \frac{u_o}{u_{max}}$$

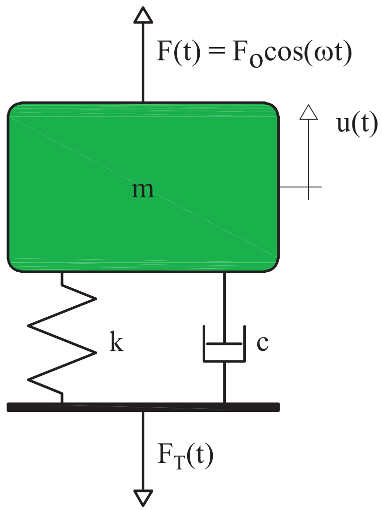
    However, it is sometimes difficult to determine the static deflection u_o, therefore, the definition of half-power bandwidth is used to estimate the damping.
  - Damping can be determined from the slope of the phase angle curve using Equation (5.19).

5.2 Force transmission (vibration isolation)

The mass-spring-damper system, shown here on the right, is excited by the harmonic force

$$F(t) = F_o \cos(\omega t)$$

What is the reaction force $F_T(t)$, which is introduced in the foundation?



The reaction force $F_T(t)$ results from the sum of the spring force F_s and the damper’s force F_c

$$F_T(t) = F_s(t) + F_c(t) = ku(t) + c\dot{u}(t) \qquad (5.38)$$

The steady-state deformation of the system due to harmonic excitation $F(t)$ is according to Equation (5.4):

$$u_p = u_{max}\cos(\omega t - \phi) \text{ with } u_{max} = u_o V(\omega) = \frac{F_o}{k} V(\omega) \qquad (5.39)$$

By substituting Equation (5.39) and its derivative into Equation (5.38) we obtain:

$$F_T(t) = \frac{F_o}{k} V(\omega)[k\cos(\omega t - \phi) - c\omega\sin(\omega t - \phi)] \qquad (5.40)$$

with the trigonometric identity from Equation (5.3):

$$F_T(t) = \frac{F_o}{k}V(\omega)[\sqrt{k^2 + c^2\omega^2}\cos(\omega t - \bar{\phi})] \tag{5.41}$$

and by substituting the identity $c = (2\zeta k)/\omega_n$:

$$F_T(t) = F_o V(\omega)\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\cos(\omega t - \bar{\phi}) \tag{5.42}$$

the maximum reaction force becomes:

$$\frac{F_{T,max}}{F_o} = TR(\omega) \tag{5.43}$$

where the quantity $TR(\omega)$ is called Transmissibility and it is equal to:

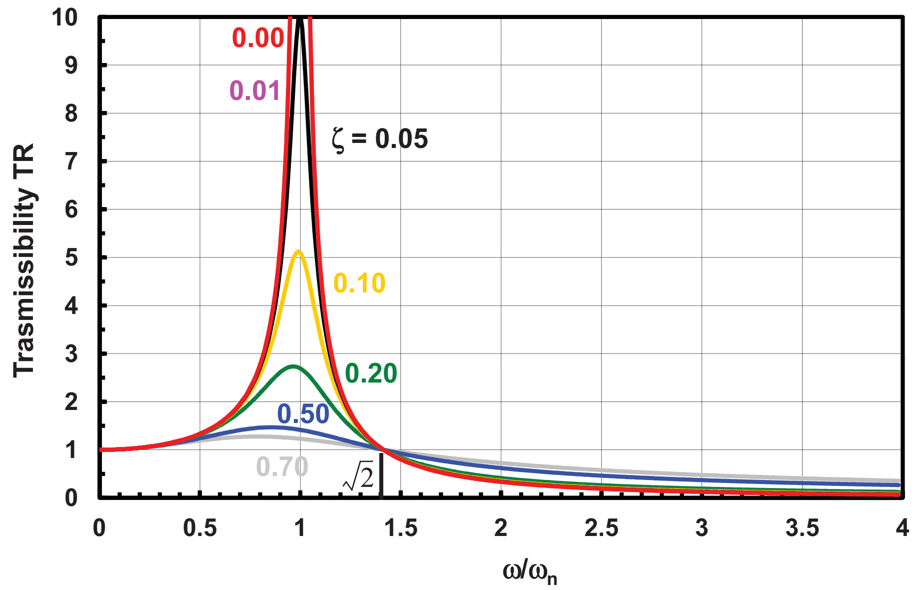
$$TR(\omega) = V(\omega)\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} \tag{5.44}$$

$$= \sqrt{\frac{1 + [2\zeta(\omega/\omega_n)]^2}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}$$

Special case:

$$TR\left(\frac{\omega}{\omega_n} = 1\right) = \frac{\sqrt{1 + 4\zeta^2}}{2\zeta} \tag{5.45}$$

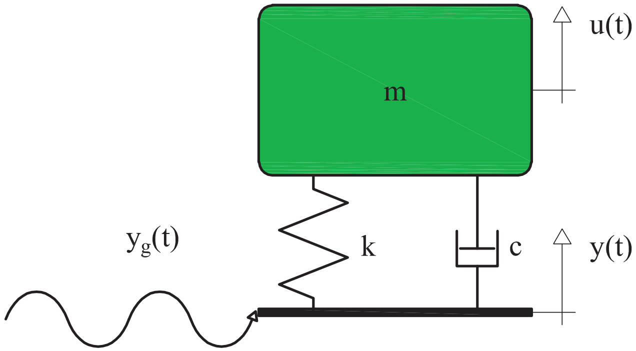
- Representation of the transmissibility TR



- When $\omega/\omega_n > \sqrt{2}$ then $TR < 1$: Vibration isolation
- When $\omega/\omega_n > \sqrt{2}$ damping has a stiffening effect
- High tuning (sub-critical excitation)
- Low tuning (super-critical excitation):
  Pay attention to the starting phase!

5.3 Base excitation (vibration isolation)

5.3.1 Displacement excitation



The mass-spring-damper system, shown here above is excited by the harmonic vertical ground displacement

$$y_g(t) = y_{go}\cos(\omega t) \qquad (5.46)$$

What is the absolute vertical displacement $u(t)$ of the system?

The differential equation of the system is:

$$m\ddot{u} + c(\dot{u} - \dot{y}) + k(u - y) = 0 \qquad (5.47)$$

after rearrangement:

$$m\ddot{u} + c\dot{u} + ku = ky + c\dot{y} \qquad (5.48)$$

The right hand side of the ODE (5.48) can be interpreted as an external excitation force $F(t) = ky + c\dot{y}$:

$$\begin{aligned} F(t) &= ky_{go}\cos(\omega t) - cy_{go}\omega\sin(\omega t) \qquad (5.49) \\ &= ky_{go}\left[\cos(\omega t) - 2\zeta\frac{\omega}{\omega_n}\sin(\omega t)\right] \\ &= ky_{go}\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\cos(\omega t + \phi) \end{aligned}$$

The external excitation force $F(t)$ is harmonic with amplitude:

$$F_o = ky_{go}\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} \qquad (5.50)$$

According to Equations (5.10) and (5.11) the maximum displacement of the system due to such a force is equal to:

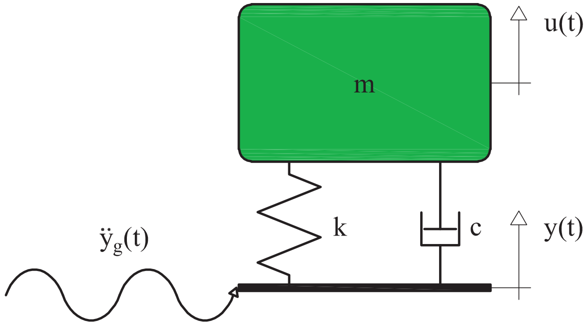
$$u_{max} = \frac{F_o}{k}V(\omega) = y_{go}\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}V(\omega) \qquad (5.51)$$

By substituting Equation (5.44) we obtain:

$$\frac{u_{max}}{y_{go}} = TR(\omega) \qquad (5.52)$$

where again $TR(\omega)$ is the transmissibility given by Equation (5.44).

5.3.2 Acceleration excitation



Pay attention:

This base excitation, like the excitation discussed in the previous Section 5.3.1, is an **harmonic excitation** and not an **arbitrary excitation** like e.g. an earthquake (see Section 7).

The mass-spring-damper system, shown above here, is excited by the harmonic vertical ground acceleration.

$$\ddot{y}_g(t) = \ddot{y}_{go}\cos(\omega t) \tag{5.53}$$

What is the absolute vertical acceleration $\ddot{u}(t)$ of the system?

The differential equation of the system is:

$$m\ddot{u} + c(\dot{u} - \dot{y}) + k(u - y) = 0 \tag{5.54}$$

after rearrangement:

$$m\ddot{u} + c(\dot{u} - \dot{y}) + k(u - y) - m\ddot{y} = -m\ddot{y} \tag{5.55}$$

$$m(\ddot{u} - \ddot{y}) + c(\dot{u} - \dot{y}) + k(u - y) = -m\ddot{y} \tag{5.56}$$

$$m\ddot{u}_{rel} + c\dot{u}_{rel} + ku_{rel} = -m\ddot{y}_g \tag{5.57}$$

$$m\ddot{u}_{rel} + c\dot{u}_{rel} + ku_{rel} = -m\ddot{y}_{go}\cos(\omega t) \tag{5.58}$$

The steady-state relative deformation u_{rel} of the system due to the harmonic ground acceleration $\ddot{y}_g$ is given by Equation (5.1):

$$u_{rel} = a_1\cos(\omega t) + a_2\sin(\omega t) \tag{5.59}$$

with the constants a_1 and a_2 according to Equation (5.2), and with:

$$f_o = \frac{F_o}{m} = \frac{-m\ddot{y}_{go}}{m} = -\ddot{y}_{go} \tag{5.60}$$

By double derivation of Equation (5.59), the relative acceleration $\ddot{u}_{rel}$ can be calculated as:

$$\ddot{u}_{rel} = -a_1\omega^2\cos(\omega t) - a_2\omega^2\sin(\omega t) \tag{5.61}$$

The desired absolute acceleration is:

$$\ddot{u} = \ddot{u}_{rel} + \ddot{y}_g = -a_1\omega^2\cos(\omega t) - a_2\omega^2\sin(\omega t) + \ddot{y}_{go}\cos(\omega t) \tag{5.62}$$

By substituting the constants a_1, a_2 and f_o given by Equations (5.2) and (5.60), and after a long but simple rearrangement, the equations for the maximum absolute vertical acceleration of the system is obtained as:

$$\frac{\ddot{u}_{max}}{\ddot{y}_{go}} = TR(\omega) \tag{5.63}$$

where again $TR(\omega)$ is the transmissibility given by Equation (5.44).

- Additional derivation:

The maximum relative displacement given by Equation (5.58) can be easily determined by means of Equations (5.10), (5.11) and (5.60) as:

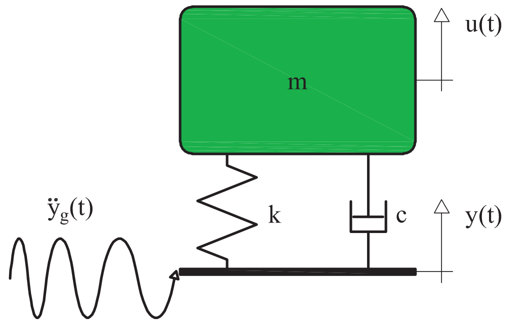
$$u_{rel,\,max} = \frac{f_o}{\omega_n^2}V(\omega) = \frac{|-\ddot{y}_{go}|}{\omega_n^2}V(\omega) \tag{5.64}$$

$$\frac{u_{rel,\,max}}{(\ddot{y}_{go}/\omega_n^2)} = V(\omega) \tag{5.65}$$

5.3.3 Example transmissibility by base excitation

Vertical base excitation:

$$\ddot{y}_g(t) = A_0 \cos(\omega_0 t)$$

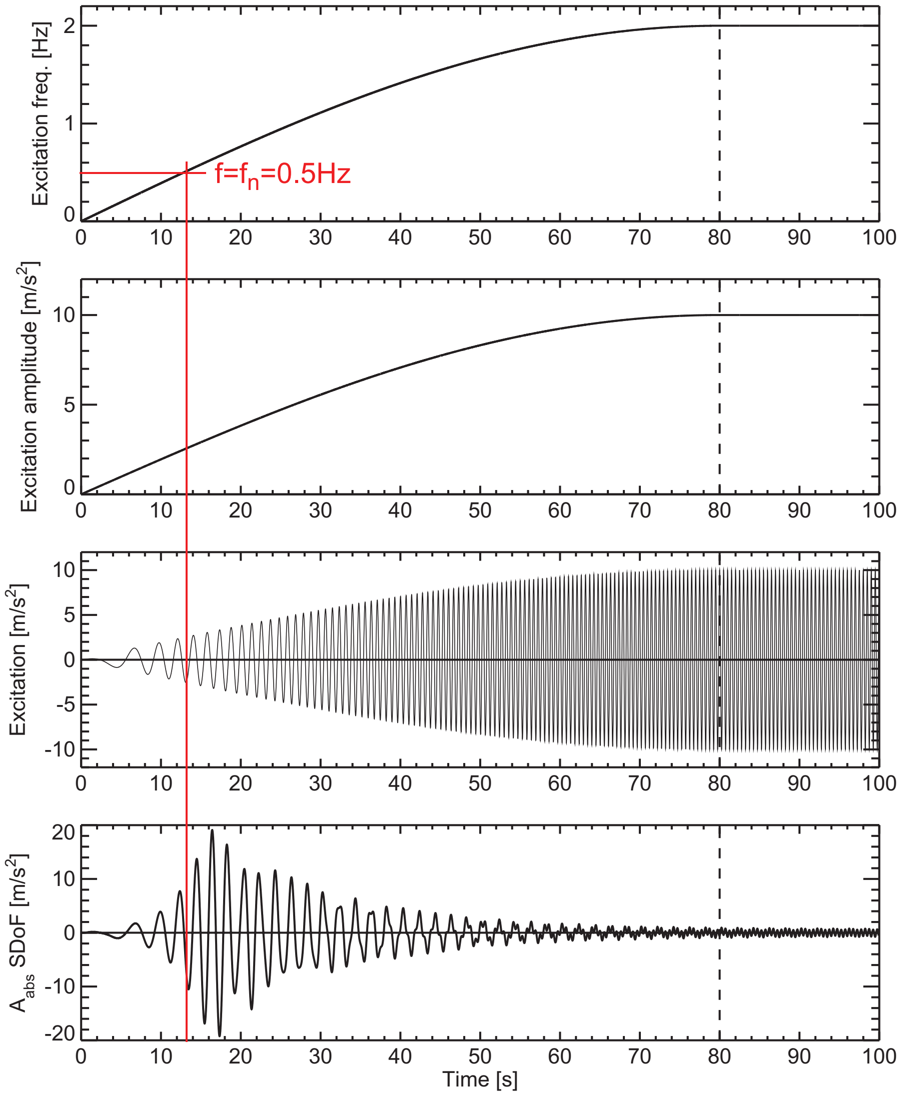


- Natural frequency SDoF system: $f_n = 0.5\text{Hz}$
- Excitation frequency: $f_0 = 2.0\text{Hz}$
- Excitation amplitude: $A_0 = 10\text{m/s}^2$

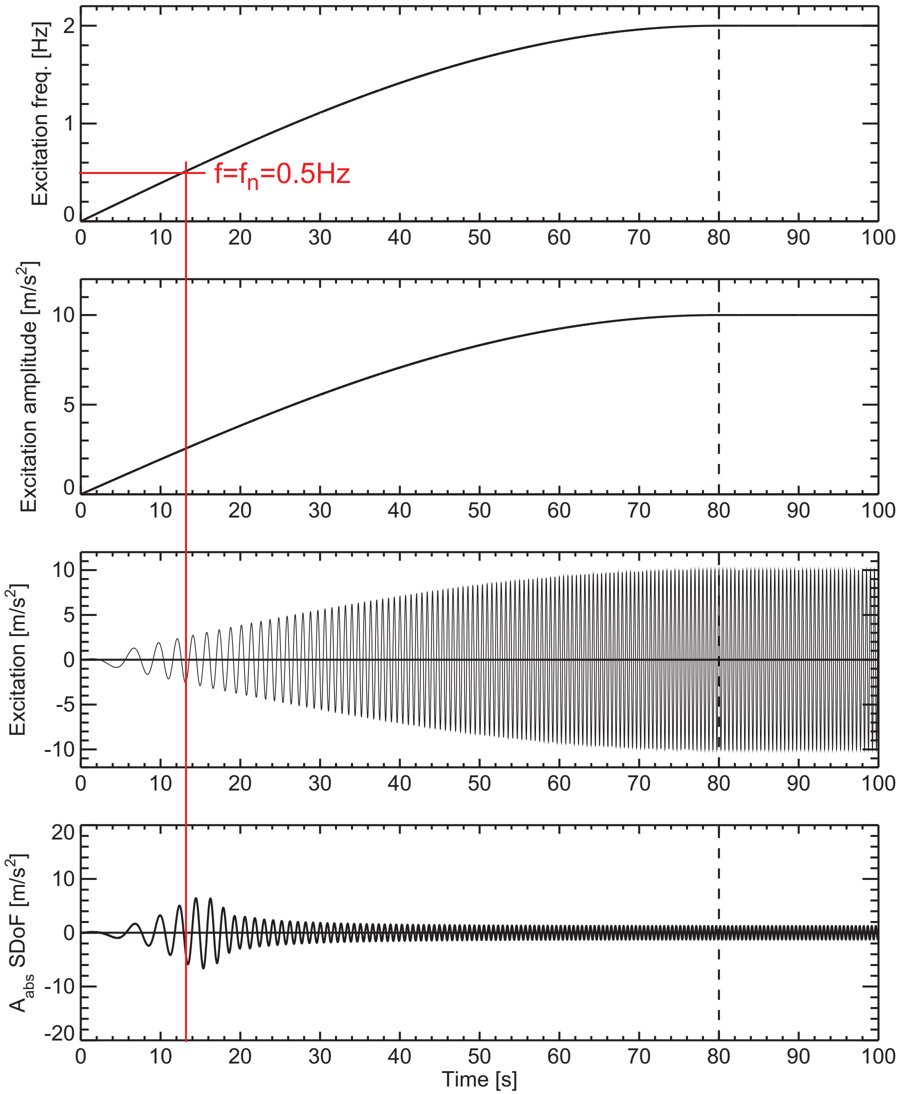
Sought is the maximum absolute acceleration $\ddot{u}_{max}$ of the SDoF system for $\zeta = 2\%$ and for $\zeta = 20\%$.

- The steady-state maximum absolute acceleration is:
  - $\zeta = 2\%$, $\omega_0/\omega_n = 4$: $TR = 0.068$ and $\ddot{u}_{max} = 0.68\text{m/s}^2$
  - $\zeta = 20\%$, $\omega_0/\omega_n = 4$: $TR = 0.125$ and $\ddot{u}_{max} = 1.25\text{m/s}^2$
- Is the steady-state maximum absolute acceleration really the maximum absolute acceleration or at start even larger absolute accelerations may result?
  - Assumptions: starting time $t_a = 80\text{s}$, sinusoidal start function for excitation frequency and excitation amplitude.
  - Numerical computation using Newmark’s Method (see Section 7)

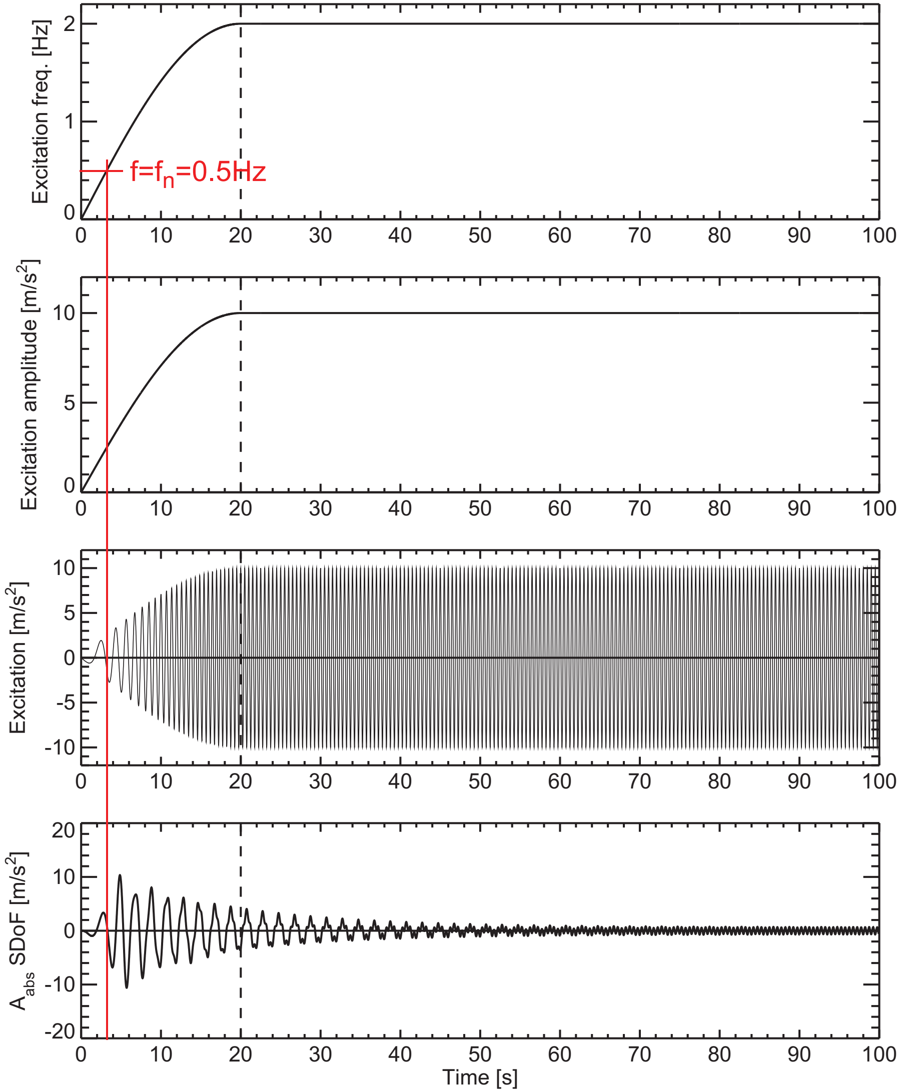
- Case 1: Initial situation with ζ = 2%



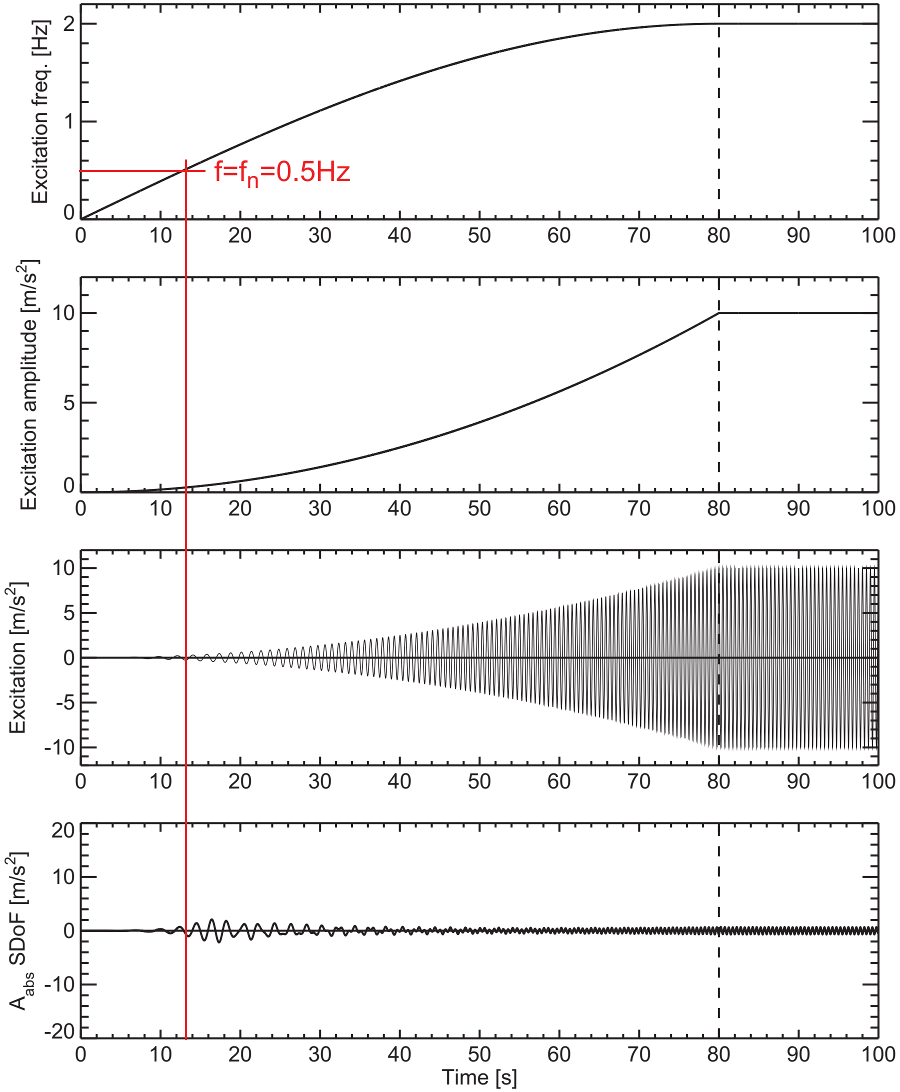
- Case 2: Increase of the damping rate from ζ = 2% to ζ = 20%



- Case 3: Reduction of starting time from t_a = 80s to t_a = 20s (ζ = 2%)



- Case 4: Change the start function for the amplitude (ζ = 2%)



- Notes

The excitation function in the starting phase has the form:

$$\ddot{y}_g(t) = A(t)\cos(\Omega(t)\cdot t) \quad (5.66)$$

The excitation angular frequency varies with time, and is:

$$\omega(t) = \frac{d}{dt}(\Omega(t)\cdot t) \quad (5.67)$$

- Linear variation of the excitation circular frequency

$$\Omega(t) = \frac{\omega_0}{2t_a}\cdot t \ : \qquad \omega(t) = \omega_0\cdot\frac{t}{t_a} \ (0 \le t \le t_a) \quad (5.68)$$

- Parabolic variation of the excitation circular frequency

$$\Omega(t) = \frac{\omega_0}{3t_a^2}\cdot t^2 \ : \qquad \omega(t) = \omega_0\cdot\left(\frac{t}{t_a}\right)^2 \ (0 \le t \le t_a) \quad (5.69)$$

- Sinusoidal variation of the excitation circular frequency

$$\Omega(t) = -\frac{2\omega_0 t_a}{\pi t}\cos\left(\frac{\pi}{2}\cdot\frac{t}{t_a}\right) : \ \omega(t) = \omega_0\cdot\sin\left(\frac{\pi}{2}\cdot\frac{t}{t_a}\right) \ (0 \le t \le t_a) \quad (5.70)$$

- Double-sinusoidal variation of the excitation circular frequency

$$\Omega(t) = \frac{\omega_0}{2}\left[1 - \frac{\sin\left(\pi\cdot\frac{t}{t_a}\right)t_a}{\pi t}\right] : \ \omega(t) = \omega_0\left[1-\cos\left(\pi\cdot\frac{t}{t_a}\right)\right] \quad (5.71)$$

- Visualization of the solution by means of the Excel file given on the web page of the course (SD_HE_Starting_Phase.xlsx)

5.4 Summary Transfer Functions

$$V(\omega) = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \quad (5.72)$$

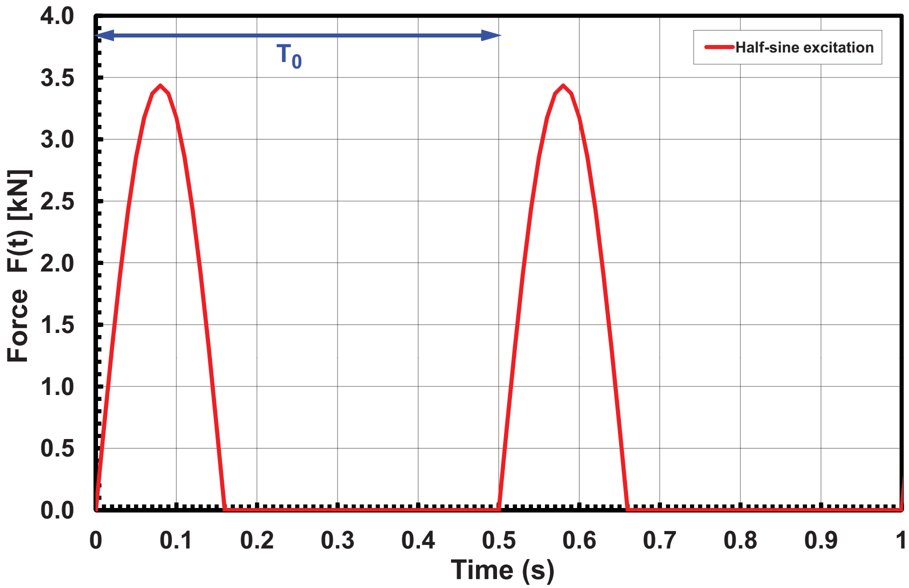
$$TR(\omega) = \sqrt{\frac{1+[2\zeta(\omega/\omega_n)]^2}{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \quad (5.73)$$

- Force excitation: $\frac{u_{max}}{u_o} = V(\omega)$
- Force transmission: $\frac{F_{T,max}}{F_o} = TR(\omega)$
- Displacement excitation: $\frac{u_{max}}{y_{go}} = TR(\omega)$
- Acceleration excitation $\frac{\ddot{u}_{max}}{\ddot{y}_{go}} = TR(\omega)$

  $\frac{u_{rel,\,max}}{(\ddot{y}_{go}/\omega_n^2)} = V(\omega)$
- For further cases check the literature.

6 Forced Vibrations

6.1 Periodic excitation



An excitation is periodic if:

$$F(t+nT_0) = F(t) \qquad \text{for} \qquad n = -\infty, \ldots, -1, 0, 1, \ldots, \infty \tag{6.1}$$

The function $F(t)$ can be represented as a sum of several harmonic functions in the form of a Fourier series, namely:

$$F(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \tag{6.2}$$

with the fundamental frequency

$$\omega_0 = \frac{2\pi}{T_0} \tag{6.3}$$

Taking into account the orthogonality relations:

$$\int_0^{T_0} \sin(n\omega_0 t)\sin(j\omega_0 t)dt = \begin{cases} 0 & \text{for } n \neq j \\ T_0/2 & \text{for } n = j \end{cases} \tag{6.4}$$

$$\int_0^{T_0} \cos(n\omega_0 t)\cos(j\omega_0 t)dt = \begin{cases} 0 & \text{for } n \neq j \\ T_0/2 & \text{for } n = j \end{cases} \tag{6.5}$$

$$\int_0^{T_0} \cos(n\omega_0 t)\sin(j\omega_0 t)dt = 0 \tag{6.6}$$

the Fourier coefficients a_n can be computed by multiplying Equation (6.2) by $\cos(j\omega_0 t)$ first, and then integrating it over the period T_0.

- $j = 0$

$$\int_0^{T_0} F(t)\cos(j\omega_0 t)dt = \int_0^{T_0} a_0 \cos(j\omega_0 t)dt + \sum_{n=1}^{\infty}\left[\int_0^{T_0} a_n \cos(n\omega_0 t)\cos(j\omega_0 t)dt + \int_0^{T_0} b_n \sin(n\omega_0 t)\cos(j\omega_0 t)dt\right] \tag{6.7}$$

$$\int_0^{T_0} F(t)dt = \int_0^{T_0} a_0 dt = a_0 T_0 \tag{6.8}$$

$$a_0 = \frac{1}{T_0} \cdot \int_0^{T_0} F(t)dt \tag{6.9}$$

- $j = n$

$$\int_0^{T_0} F(t)\cos(j\omega_0 t)dt = \int_0^{T_0} a_0\cos(j\omega_0 t)dt \quad (6.10)$$

$$+ \sum_{n=1}^{\infty}\left[\int_0^{T_0} a_n\cos(n\omega_0 t)\cos(j\omega_0 t)dt + \int_0^{T_0} b_n\sin(n\omega_0 t)\cos(j\omega_0 t)dt\right]$$

$$\int_0^{T_0} F(t)\cos(n\omega_0 t)dt = a_n \cdot \frac{T_0}{2} \quad (6.11)$$

$$a_n = \frac{2}{T_0}\cdot\int_0^{T_0} F(t)\cos(n\omega_0 t)dt \quad (6.12)$$

Similarly, the Fourier coefficients b_n can be computed by first multiplying Equation (6.2) by $\sin(j\omega_0 t)$ and then integrating it over the period T_0.

$$b_n = \frac{2}{T_0}\cdot\int_0^{T_0} F(t)\sin(n\omega_0 t)dt \quad (6.13)$$

- Notes
  - a_0 is the mean value of the function $F(t)$
  - The integrals can also be calculated over the interval $[-T_0/2, T_0/2]$
  - For $j = 0$ no b-coefficient exists

6.1.1 Steady state response due to periodic excitation

$$m\ddot{u} + c\dot{u} + ku = F(t) \quad (6.14)$$

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = \frac{F(t)}{m} \quad (6.15)$$

$$F(t) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)] \quad (6.16)$$

- Static Part (a_0)

$$u_0(t) = \frac{a_0}{k} \quad (6.17)$$

- Harmonic part "cosine" (see harmonic excitation)

$$u_n^{Cosine}(t) = \frac{a_n}{k}\cdot\frac{2\zeta\beta_n\sin(n\omega_0 t) + (1-\beta_n^2)\cos(n\omega_0 t)}{(1-\beta_n^2)^2 + (2\zeta\beta_n)^2}, \quad \beta_n = \frac{n\omega_0}{\omega_n} \quad (6.18)$$

- Harmonic part "sine" (similar as "cosine")

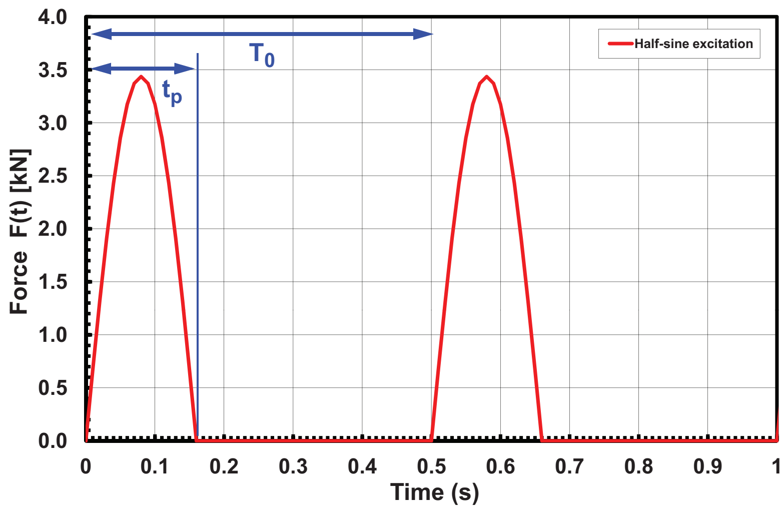
$$u_n^{Sine}(t) = \frac{b_n}{k}\cdot\frac{(1-\beta_n^2)\sin(n\omega_0 t) - 2\zeta\beta_n\cos(n\omega_0 t)}{(1-\beta_n^2)^2 + (2\zeta\beta_n)^2}, \quad \beta_n = \frac{n\omega_0}{\omega_n} \quad (6.19)$$

- The steady-state response $u(t)$ of a damped SDoF system under the periodic excitation force $F(t)$ is equal to the sum of the terms of the Fourier series.

$$u(t) = u_0(t) + \sum_{n=1}^{\infty} u_n^{Cosine}(t) + \sum_{n=1}^{\infty} u_n^{Sine}(t) \quad (6.20)$$

6.1.2 Half-sine

A series of half-sine functions is a good model for the force that is generated by a person jumping.



$$F(t) = \begin{cases} A\sin\left(\frac{\pi t}{t_p}\right) & \text{for } 0 \le t < t_p \\ 0 & \text{for } t_p \le t < T_0 \end{cases} \tag{6.21}$$

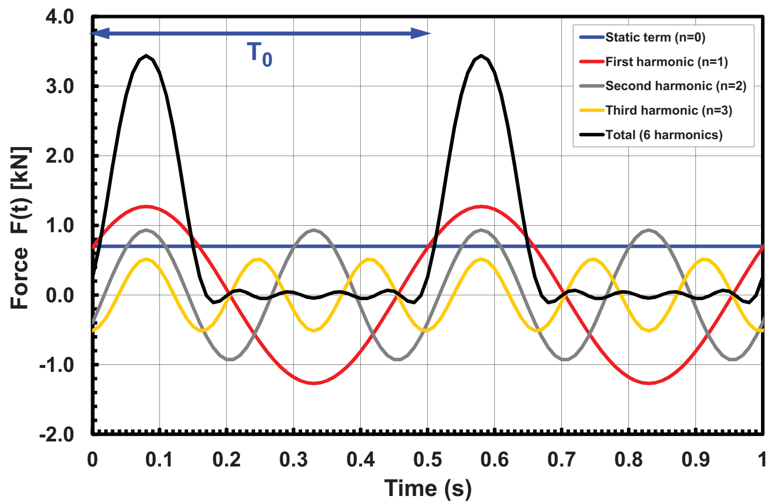
The Fourier coefficients can be calculated at the best using a mathematics program:

$$a_0 = \frac{A}{T_0} \cdot \int_0^{t_p} \sin\left(\frac{\pi t}{t_p}\right) dt = \frac{2A\tau}{\pi} \qquad \text{with} \qquad \tau = \frac{t_p}{T_0} \tag{6.22}$$

$$a_n = \frac{2A}{T_0} \cdot \int_0^{t_p} \sin\left(\frac{\pi t}{t_p}\right) \cos(n\omega_0 t) dt = \frac{4A\tau\cos(n\pi\tau)^2}{\pi(1-4n^2\tau^2)} \tag{6.23}$$

$$b_n = \frac{2A}{T_0} \cdot \int_0^{t_p} \sin\left(\frac{\pi t}{t_p}\right) \sin(n\omega_0 t) dt = \frac{4A\tau\sin(n\pi\tau)\cos(n\pi\tau)}{\pi(1-4n^2\tau^2)} \tag{6.24}$$

The approximation of the half-sine model for $T_0 = 0.5s$ and $t_p = 0.16s$ by means of 6 Fourier terms is as follows:

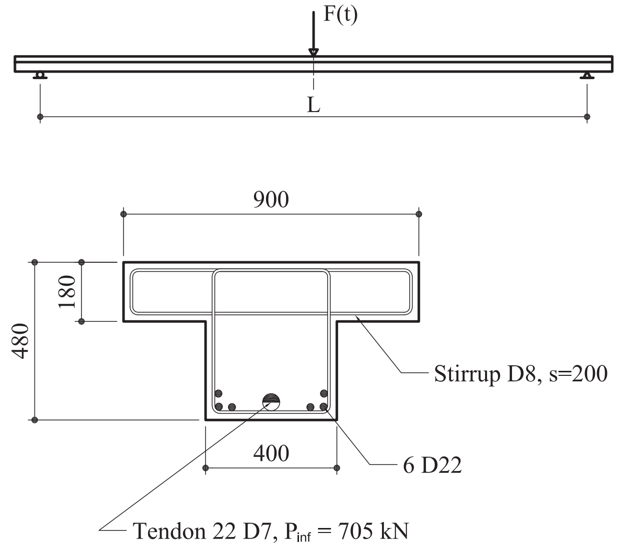


- Note

The static term $a_0 = 2A\tau/\pi = G$ corresponds to the weight of the person jumping.

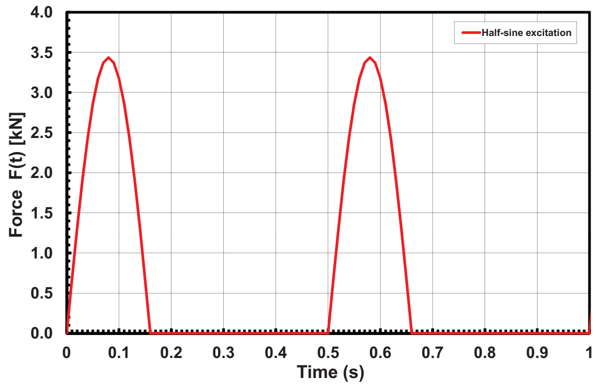
6.1.3 Example: “Jumping on a reinforced concrete beam”

- Beam



- Young’s Modulus: $E = 23500\text{MPa}$
- Density: $\rho = 20.6\text{kN/m}^3$
- Bending stiffness: $EI = 124741\text{kNm}^2$
- Damping rate $\zeta = 0.017$
- Modal mass $M_n = 0.5M_{tot}$
- Modal stiffness $K_n = \frac{\pi^4}{2} \cdot \frac{EI}{L^3}$

- Excitation (similar to page 186 of [Bac+97])



- Jumping frequency: $f_0 = 2\text{Hz}$
- Period: $T_0 = 0.5\text{s}$
- Contact time: $t_p = 0.16\text{s}$
- Person’s weight: $G = 0.70\text{kN}$
- Amplitude: $A = 3.44\text{kN}$

- Maximum deflections

Static: $u_{st} = \frac{G}{K_n}$

Dynamic: $u_{max} = \max(u(t))$ with $u(t)$ from Equation (6.20)

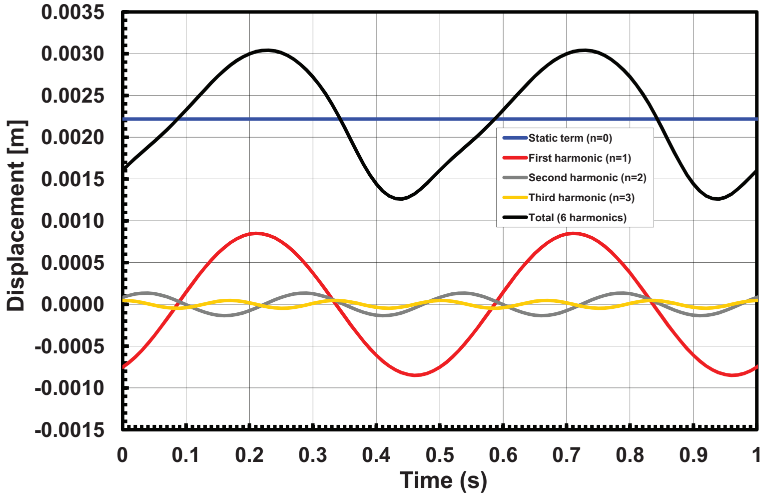
Ratio: $\bar{V} = \frac{u_{max}}{u_{st}}$

- Investigated cases

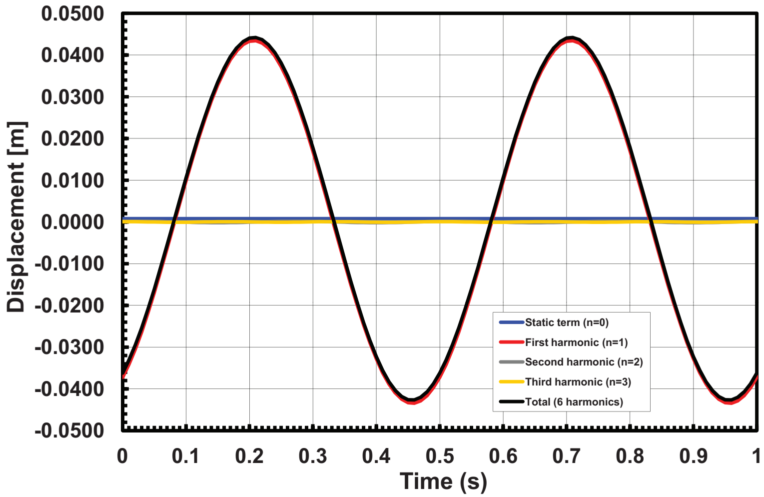
Length [m]	Frequency f_n [Hz]	u_{max} [m]	$\bar{V}$ [-]
26.80	1	0.003	1.37
19.00	2	0.044	55.94
15.50	3	0.002	3.62
13.42	4	0.012	41.61
12.01	5	0.001	4.20
10.96	6	0.004	25.02

- Notes
  - When the excitation frequency f_0 is twice as large as the natural frequency f_n of the beam, the magnification factor $\bar{V}$ is small.
  - Taking into account the higher harmonics can be important!

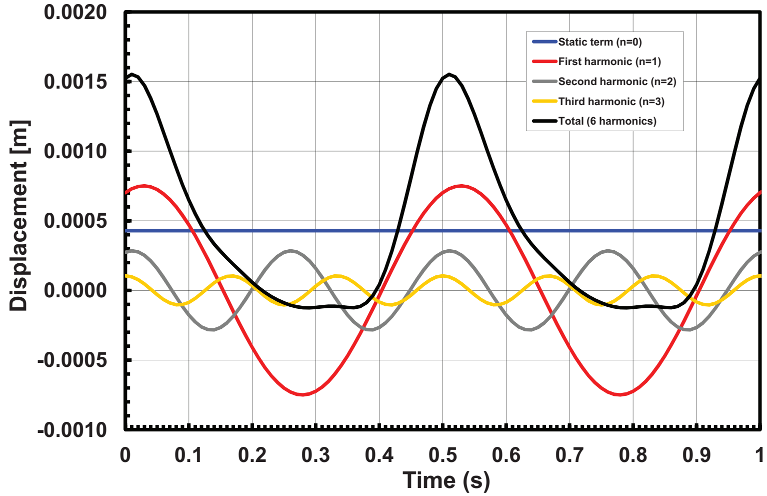
- Case1: f_0 = 2Hz, f_n = 1Hz



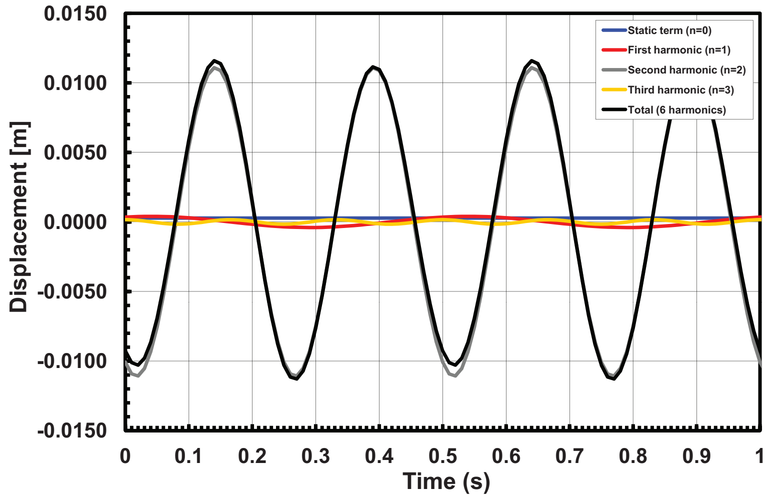
- Case 2: f_0 = 2Hz, f_n = 2Hz



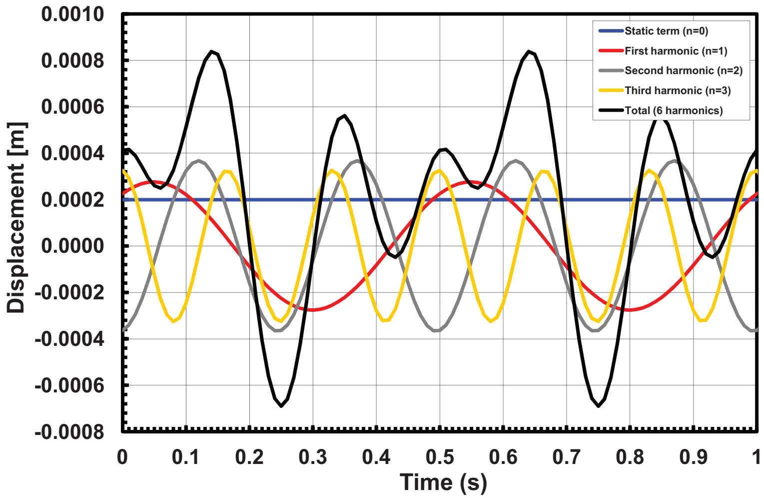
- Case 3: f_0 = 2Hz, f_n = 3Hz



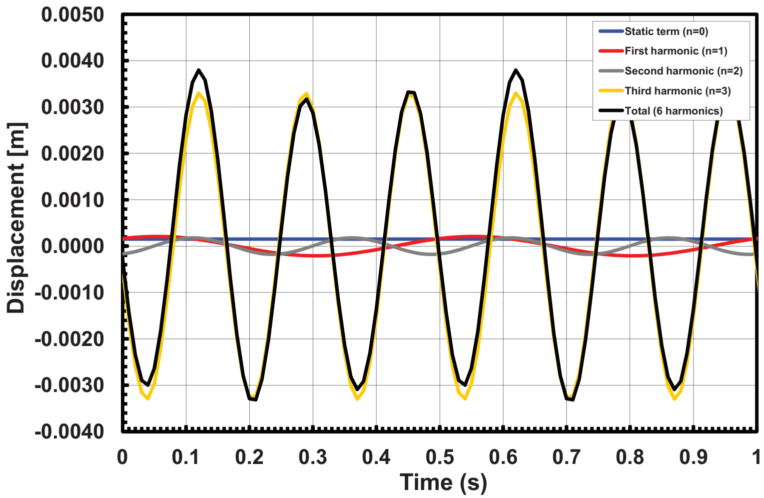
- Case 4: f_0 = 2Hz, f_n = 4Hz



- Case 5: f_0 = 2Hz, f_n = 5Hz



- Case 6: f_0 = 2Hz, f_n = 6Hz



6.2 Short excitation

6.2.1 Step force

The differential equation of an undamped SDoF System loaded with a force F_0 which is applied suddenly at the time $t = 0$ is:

$$m\ddot{u} + ku = F_0 \tag{6.25}$$

There is a homogeneous and a particular solution

$$u_h = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \text{ (see free vibrations)} \tag{6.26}$$

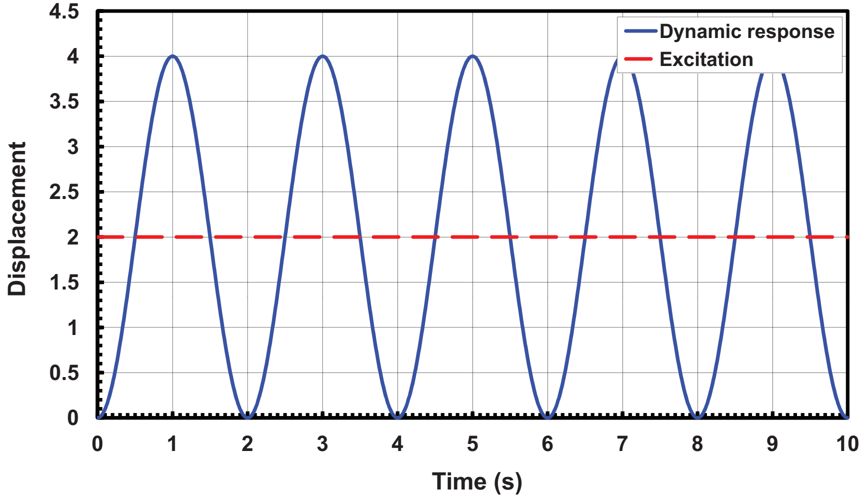
$$u_p = F_0/k \tag{6.27}$$

The overall solution $u(t) = u_h + u_p$ is completely defined by the initial conditions $u(0) = \dot{u}(0) = 0$ and it is:

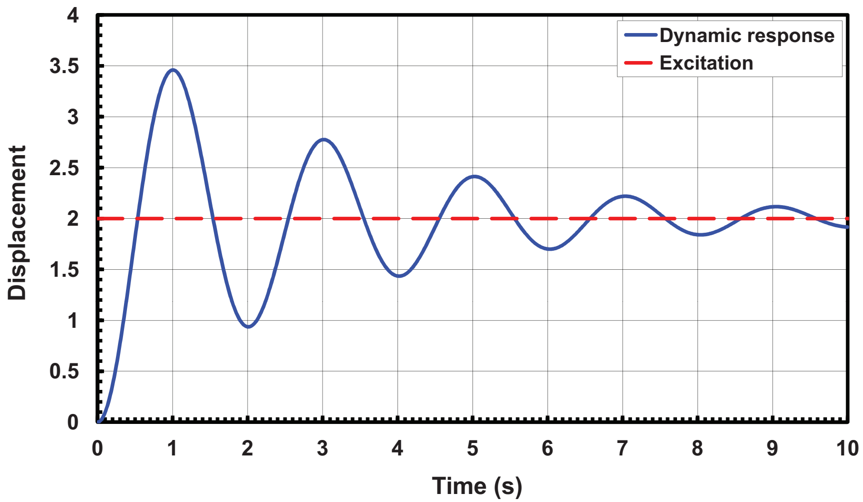
$$u(t) = \frac{F_0}{k}[1 - \cos(\omega_n t)] \tag{6.28}$$

- Notes
  - The damped case can be solved in the exact same way. On the web page of the course there is an Excel file to illustrate this excitation.
  - The maximum displacement of an undamped SDoF System under a step force is twice the static deflection $u_{st} = F_0/k$.
  - The deflection at the time $t = \infty$ of a damped SDoF System under a step force is equal to the static deflection $u_{st} = F_0/k$.

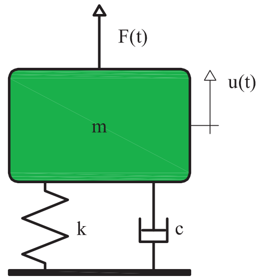
- Step force: T_n=2s, F_o/k=2, ζ=0

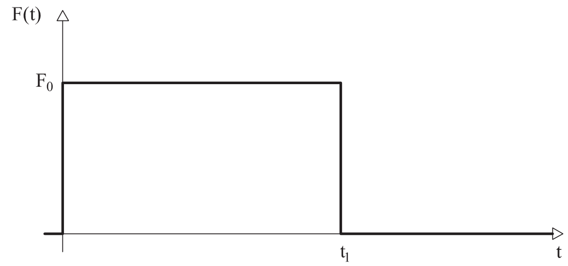


- Step force: T_n=2s, F_o/k=2, ζ=10%



6.2.2 Rectangular pulse force excitation





The differential equation of an undamped SDoF system under a rectangular pulse force excitation is:

$$\begin{cases} m\ddot{u} + ku = F_0 & \text{for } t \le t_1 \\ m\ddot{u} + ku = 0 & \text{for } t > t_1 \end{cases} \tag{6.29}$$

Up to time $t = t_1$ the solution of the ODE corresponds to Equation (6.28). From time $t = t_1$ onwards, it is a free vibration with the following initial conditions:

$$u(t_1) = \frac{F_0}{k}[1 - \cos(\omega_n t_1)] \tag{6.30}$$

$$\dot{u}(t_1) = \frac{F_0}{k}\omega_n \sin(\omega_n t_1) \tag{6.31}$$

The free vibration is described by the following equation:

$$u_h = A_1 \cos(\omega_n(t - t_1)) + A_2 \cos(\omega_n(t - t_1)) \tag{6.32}$$

and through the initial conditions (6.30) and (6.31), the constants A_1 and A_2 can be determined.

- Short duration of excitation (t_1/T_n is small)

The series expansion of sine and cosine is:

$$\cos(\omega_n t_1) = \cos\left(\frac{2\pi}{T_n}t_1\right) = 1 - \frac{(\omega_n t_1)^2}{2} + \dots \quad (6.33)$$

$$\sin(\omega_n t_1) = \sin\left(\frac{2\pi}{T_n}t_1\right) = \omega_n t_1 + \frac{(\omega_n t_1)^3}{6} + \dots \quad (6.34)$$

and for small t_1/T_n the expressions simplifies to:

$$\cos(\omega_n t_1) \cong 1 \quad , \quad \sin(\omega_n t_1) \cong \omega_n t_1 \quad (6.35)$$

By substituting Equation (6.35) in Equations (6.30) and (6.31) it follows that:

$$u(t_1) = 0 \quad , \quad \dot{u}(t_1) = \frac{F_0}{k}\omega_n^2 t_1 = \frac{F_0 t_1}{m} \quad (6.36)$$

Equation (6.36) shows, that a short excitation can be interpreted as a free vibration with initial velocity

$$v_0 = I/m \quad (6.37)$$

where I is the impulse generated by the force F_0 over the time t_1.

- Rectangular pulse force excitation: $I = F_0 t_1$
- Triangular pulse force excitation: $I = 0.5F_0 t_1$
- Arbitrary short excitation: $I = \int_0^{t_1} F(t)dt$

The equation of an undamped free vibration is:

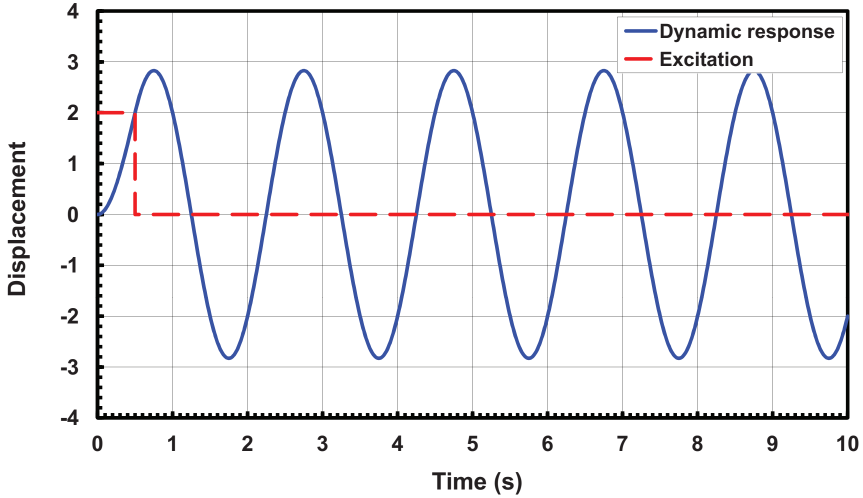
$$u(t) = A\cos(\omega_n t - \phi) \text{ with } A = \sqrt{u_0 + \left(\frac{v_0}{\omega_n}\right)^2} \text{ and } \tan\phi = \frac{v_0}{\omega_n u_0} \quad (6.38)$$

therefore, the maximum amplitude of a short excitation is:

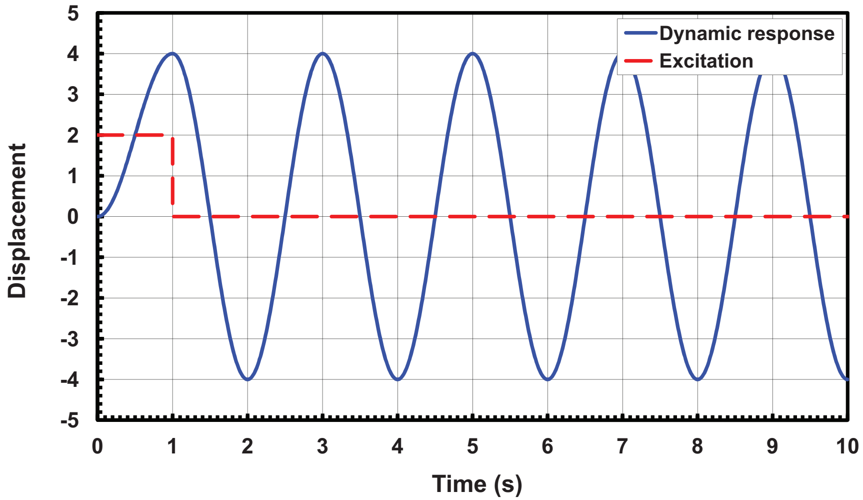
$$A = \frac{v_0}{\omega_n} \quad (6.39)$$

- Notes
  - The damped case can be solved in the exact same way. On the web page of the course there is an Excel file to illustrate this excitation.
  - Rectangular pulse force excitation: When $t_1 > T_n/2$, the maximum response of the SDoF system is equal to two times the static deflection $u_{st} = F_0/k$.
  - Rectangular pulse force excitation: When $t_1 > T_n/2$, for some t_1/T_n ratios (z.B.: 0.5, 1.5, ...) the maximum response of the SDoF system can even be as large as $4F_0/k$.
  - Rectangular pulse force excitation: try yourself using the provided Excel spreadsheet.
  - Short excitation: The shape of the excitation has virtually no effect on the maximum response of the SDoF system. Important is the impulse.
  - Short excitation: Equation (6.59) is exact only for $t_1/T_n \to 0$ and $\zeta = 0$. For all other cases, it is only an approximation, which overestimates the actual maximum deflection.

- Rectangular pulse: T_n=2s, t_1=0.5s (t_1/T_n=0.25), F_o/k=2, ζ=0%

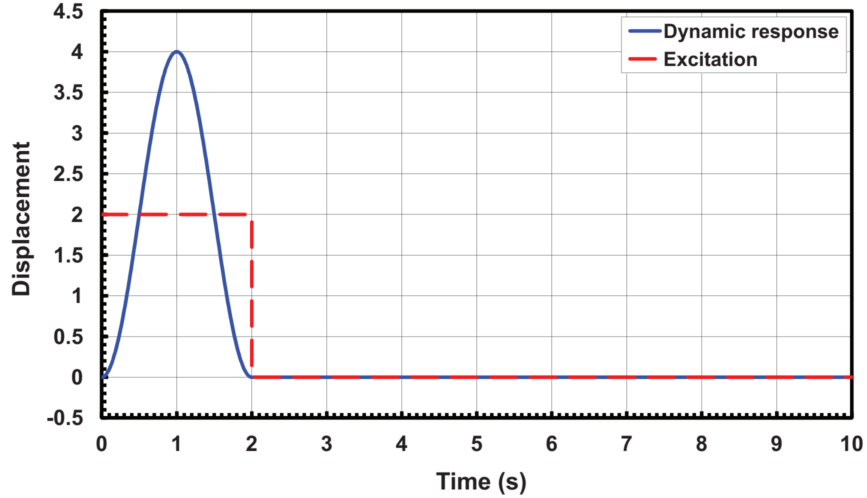


- Rectangular pulse: T_n=2s, t_1=1s (t_1/T_n=0.50), F_o/k=2, ζ=0%

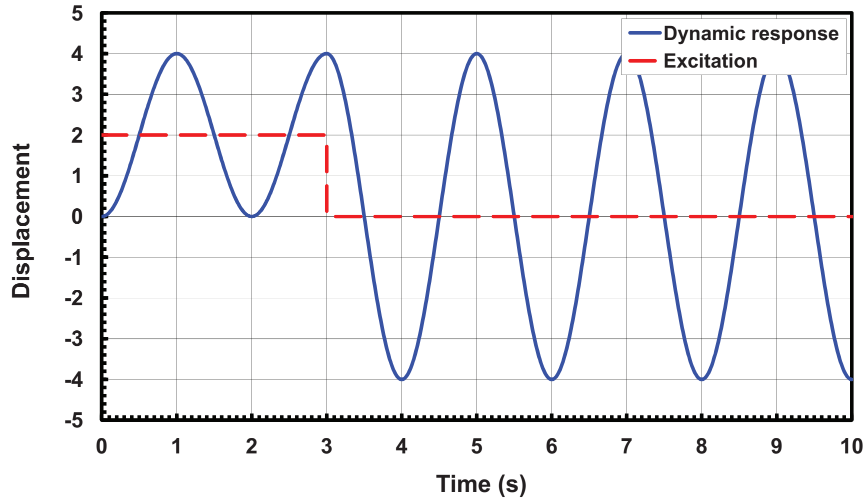


Course "Fundamentals of Structural Dynamics" *An-Najah 2013*

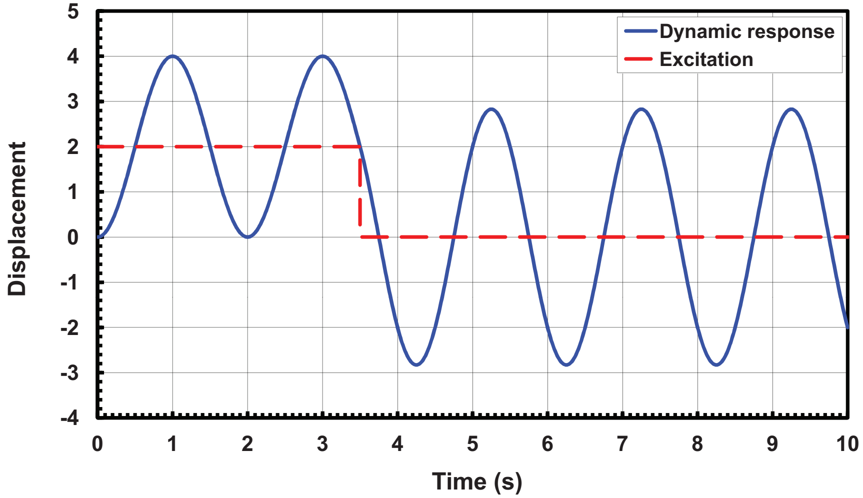
- Rectangular pulse: T_n=2s, t_1=2s (t_1/T_n=1.00), F_o/k=2, ζ=0%



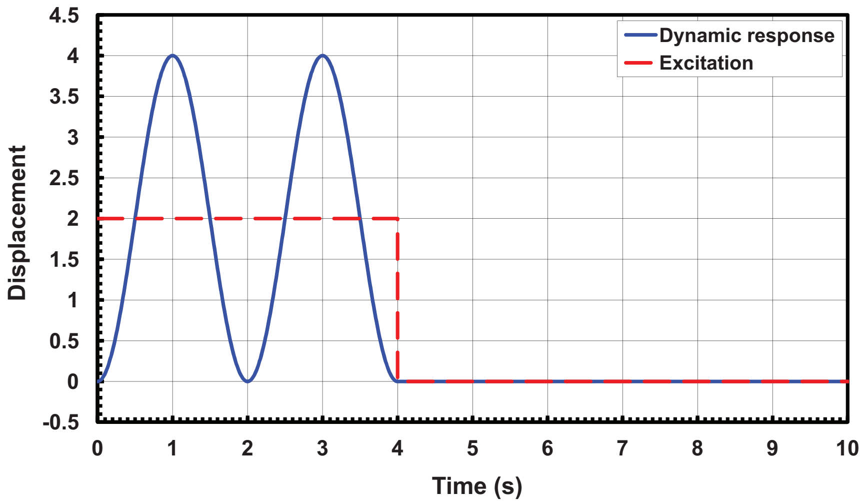
- Rectangular pulse: T_n=2s, t_1=3s (t_1/T_n=1.50), F_o/k=2, ζ=0%



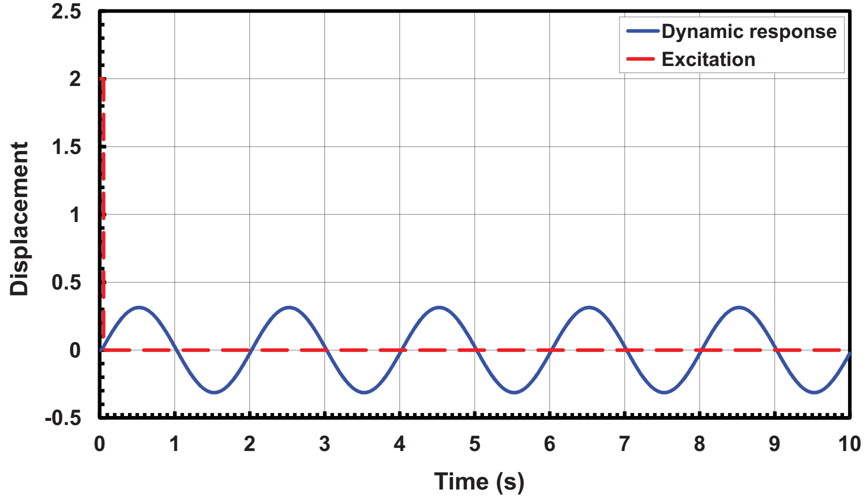
- Rectangular pulse: T_n=2s, t_1=3.5s (t_1/T_n=1.75), F_o/k=2, ζ=0%



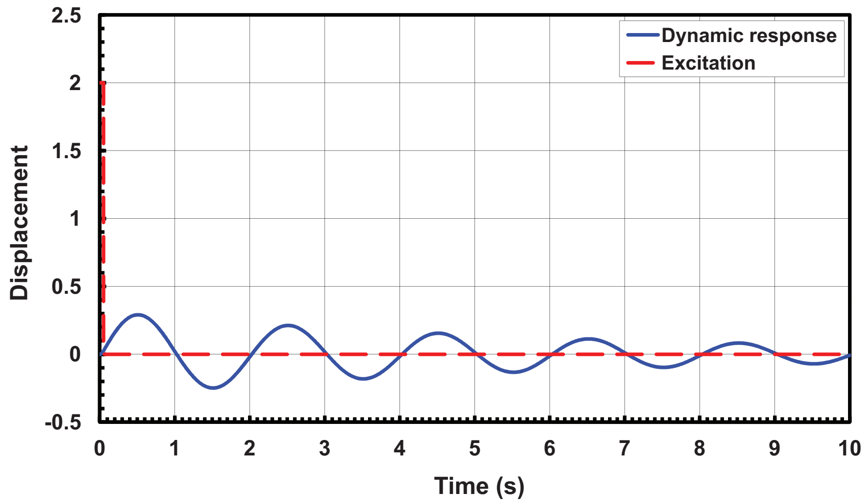
- Rectangular pulse: T_n=2s, t_1=4s (t_1/T_n=2.00), F_o/k=2, ζ=0%



- Short rectangular pulse: T_n=2s, t_1=0.05s, F_o/k=2, ζ=0%



- Short rectangular pulse: T_n=2s, t_1=0.05s, F_o/k=2, ζ=5%



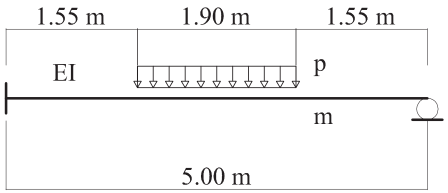
6.2.3 Example "blast action"

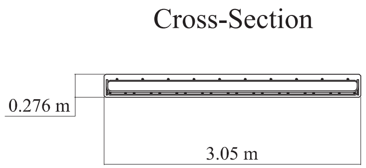
- Test

- Modelling option 1

Within a simplified modelling approach, it is assumed that the slab remains elastic during loading. Sought is the maximum deflection of the slab due to the explosion.

- Simplified system





Mass: $m = 3.05 \cdot 0.276 \cdot 2.45 = 2.06\mathrm{t/m}$

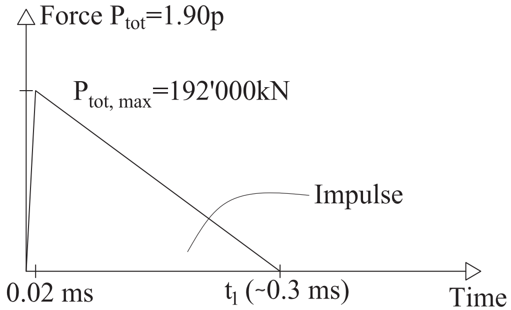
Concrete $f_c' = 41.4\mathrm{MPa}$, $E_c = 5000 \cdot \sqrt{f_c'} = 32172\mathrm{MPa}$

Stiffness $I_o = (3050 \cdot 276^3)/12 = 5344\times10^6\mathrm{mm}^4$

$E_cI_o = 171.9\mathrm{kNm}^2$

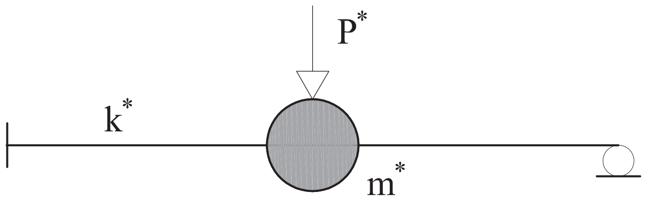
$E_cI = 0.30E_cI_o = 52184\mathrm{kNm}^2$ (due to cracking!)

- Action



$t_1 \approx 0.3\mathrm{ms}$ is by sure much shorter than the period $T_n = 64\mathrm{ms}$ of the slab (see Equation (6.51)). Therefore, the excitation can be considered as short.

- Equivalent modal SDoF system (see Section "Modelling")



Ansatz for the deformed shape:

$$\psi = C1 \cdot \sin(\beta x) + C2 \cdot \cos(\beta x) + C3 \cdot \sinh(\beta x) + C4 \cdot \cosh(\beta x) \tag{6.40}$$

Boundary conditions:

$$\psi(0) = 0 \ , \ \psi(L) = 0 \ , \ \psi'(0) = 0 \ , \ \psi''(L) = 0 \quad (6.41)$$

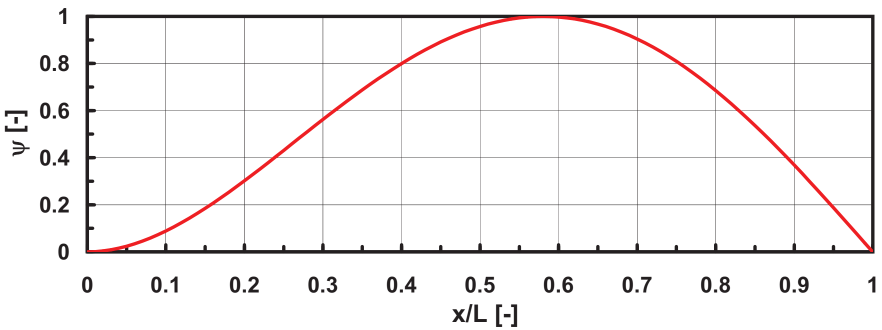
By means of the mathematics program "Maple" Equation (6.40) can be solved for the boundary conditions (6.41) and we get:

$$1.508 \cdot \psi = \sin(\beta x) - \sinh(\beta x) + \frac{[\sin(\beta L) + \sinh(\beta L)] \cdot [\cos(\beta x) - \cosh(\beta x)]}{-\cos(\beta L) - \cosh(\beta L)} \quad (6.42)$$

with

$$\beta L = 3.927 \quad (6.43)$$

The shape of the function ψ is:



And with the equations given in Section "Modelling", the modal properties of the equivalent SDoF system are determined:

$$m^* = \int_0^L m \cdot \psi^2 \cdot dx = 0.439 mL \quad (6.44)$$

$$k^* = \int_0^L (EI \cdot (\psi'')^2 \cdot dx) = 104.37 \cdot \frac{EI}{L^3} \quad (6.45)$$

$$P^* = \int_{L_1=1.55m}^{L_2=3.45m} (p \cdot \psi \cdot dx) = 0.888 \cdot P_{tot} \quad (6.46)$$

For this example, the modal properties characterizing the equivalent modal SDoF system are:

$$m^* = 0.439 \cdot 2.06 \cdot 5 = 4.52t \quad (6.47)$$

$$k^* = 104.37 \cdot \frac{52184}{5^3} = 43571 kN/m \quad (6.48)$$

$$P^* = 0.888 \cdot 192000 = 170496 kN \quad (6.49)$$

$$\omega = \sqrt{k^*/m^*} = \sqrt{43571/4.52} = 98.18 rad/s \quad (6.50)$$

$$T_n = 2\pi/\omega = 0.064s \quad (6.51)$$

The maximum elastic deformation of the SDoF system can be calculated using the modal pulse as follows:

$$I^* = 0.5 \cdot P^* \cdot t_0 = 0.5 \cdot 170496 \cdot 0.3 \times 10^{-3} = 25.6 kNs \quad (6.52)$$

The initial velocity of the free vibration is:

$$v_0 = \frac{I^*}{m^*} = \frac{25.6}{4.52} = 5.66 m/s \quad (6.53)$$

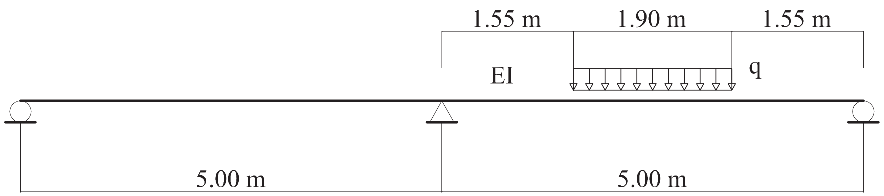
The maximum elastic deflection is:

$$\Delta_{m,e} = v_0/\omega = 5.66/98.18 = 0.058m \quad (6.54)$$

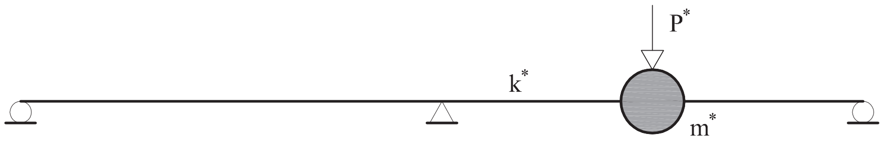
- Modelling option 2

Within a simplified modelling approach, it is assumed that the slab remains elastic during loading. Sought is the maximum deflection of the slab due to the explosion.

- Simplified system



- Equivalent modal SDoF system (see Section “Modelling”)



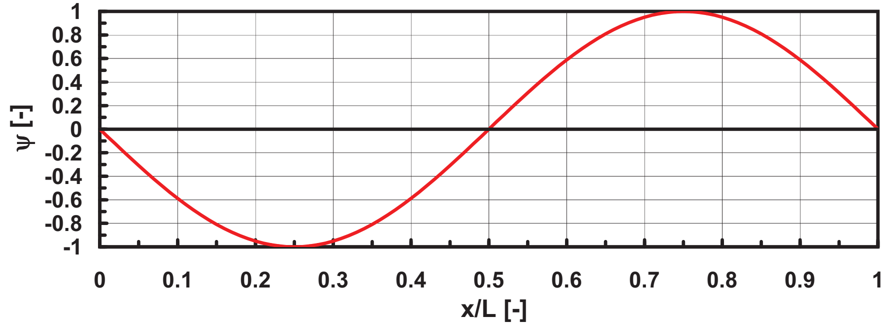
Ansatz for the deformed shape:

$$\psi = -\sin\left(\frac{2\pi x}{L}\right) \tag{6.55}$$

Boundary conditions:

$$\psi(0) = 0\ ,\ \psi(L) = 0\ ,\ \psi''(0) = 0\ ,\ \psi''(L) = 0 \tag{6.56}$$

The shape of the function ψ is:



And with the equations given in Section “Modelling”, the modal properties of the equivalent SDoF system are determined:

$$m^* = \int_0^L m \cdot \psi^2 \cdot dx = 0.5mL \tag{6.57}$$

$$k^* = \int_0^L (EI \cdot (\psi'')^2 \cdot dx) = 8\pi^4 \cdot \frac{EI}{L^3} = 779.27 \cdot \frac{EI}{L^3} \tag{6.58}$$

$$P^* = \int_{L_1=6.55m}^{L_2=8.45m} (p \cdot \psi \cdot dx) = 0.941 \cdot P_{tot} \tag{6.59}$$

For this example, the modal properties characterizing the equivalent modal SDoF system are:

$$m^* = 0.5 \cdot 2.06 \cdot 10 = 10.3t \tag{6.60}$$

$$k^* = 779.27 \cdot \frac{52184}{10^3} = 40666kN/m \tag{6.61}$$

$$P^* = 0.941 \cdot 192000 = 180672kN \tag{6.62}$$

$$\omega = \sqrt{k^*/m^*} = \sqrt{40666/10.3} = 62.83rad/s \tag{6.63}$$

$$T_n = 2\pi/\omega = 0.10\mathrm{s} \tag{6.64}$$

The maximum elastic deformation of the SDoF system can be calculated using the modal pulse as follows:

$$I^* = 0.5 \cdot P^* \cdot t_0 = 0.5 \cdot 180672 \cdot 0.3\times10^{-3} = 27.1\mathrm{kNs} \tag{6.65}$$

The initial velocity of the free vibration is:

$$v_0 = \frac{I^*}{m^*} = \frac{27.1}{10.3} = 2.63\mathrm{m/s} \tag{6.66}$$

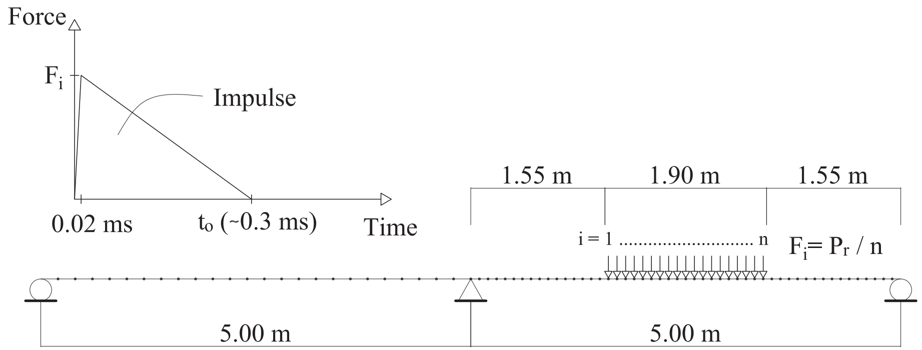
The maximum elastic deflection is:

$$\Delta_{m,e} = v_0/\omega = 2.63/62.83 = 0.042\mathrm{m} \tag{6.67}$$

- Modelling option 3

As a third option, the slab is modelled using the commercial finite element software SAP 2000.

- Numerical Model



The distributed load q is replaced by $n = 19$ concentrated forces F_i:

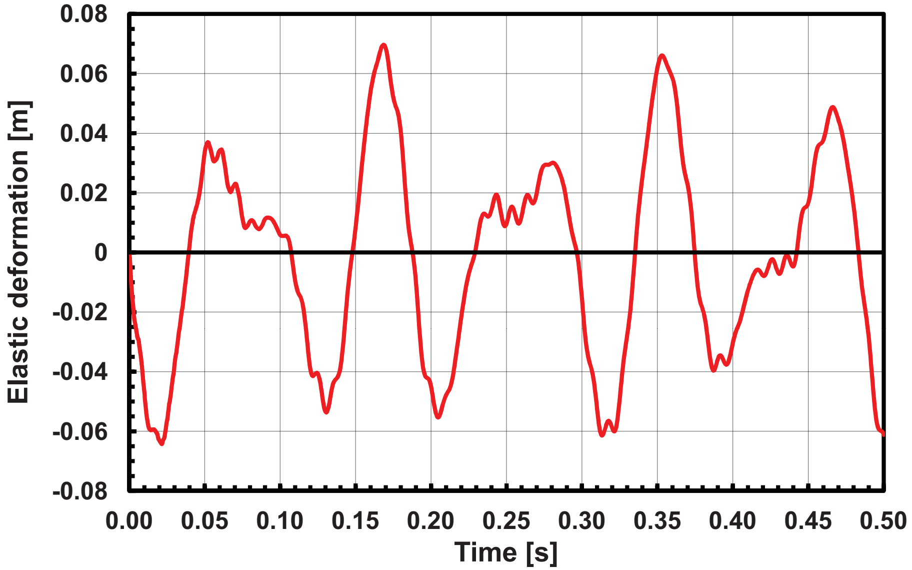
$$F_i = \frac{192000}{19} = 10105\mathrm{kN} \tag{6.68}$$

The first period of the system is:

$$T_1 = 0.100\mathrm{s} \tag{6.69}$$

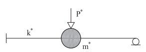
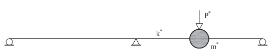
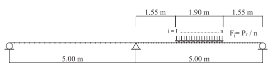
which corresponds to Equation (6.64).

And the time-history of the elastic deflection is:



The effect of the higher modes can be clearly seen!

- Comparison

System	m* [t]	k* [kN/m]	P* [P]	T [s]	$\Delta_{m,e}$ [m]
	4.52	43571	0.888	0.064	0.058
	10.30	40666	0.941	0.100	0.042
	-	-	-	0.100	0.064

Blank Page

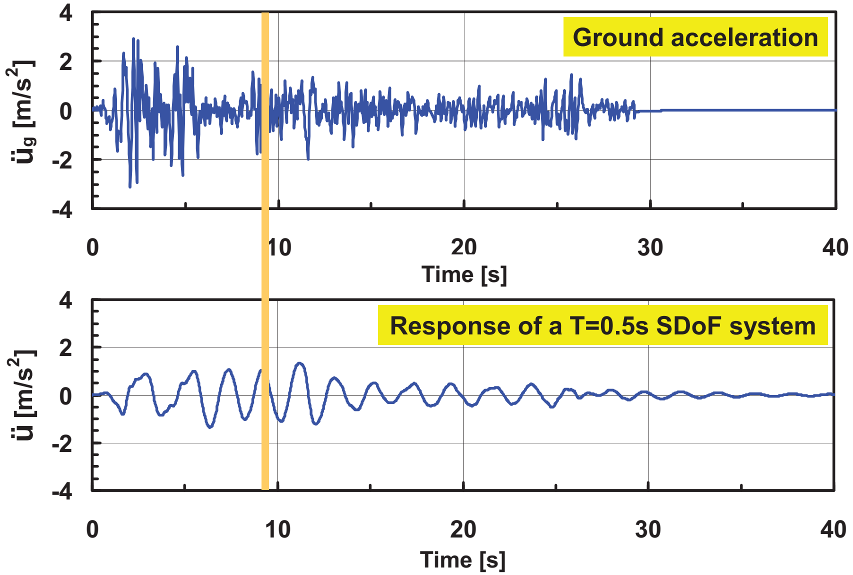
7 Seismic Excitation

7.1 Introduction

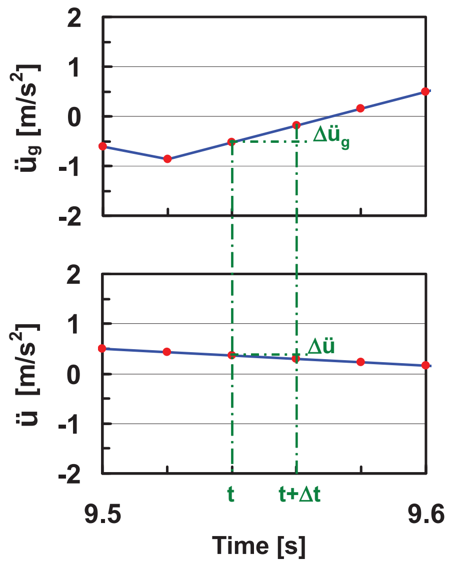
The equation of motion for a base point excitation through an acceleration time-history $\ddot{u}_g(t)$ can be derived from the equilibrium of forces (see Section 2.1.1) as:

$$m\ddot{u} + c\dot{u} + f_s(u,t) = -m\ddot{u}_g \qquad (7.1)$$

where $\ddot{u}$, $\dot{u}$ and u are motion quantities relative to the base point of the SDoF system, while $f_s(u,t)$ is the spring force of the system that can be linear or nonlinear in function of time and space. The time-history of the motion quantities $\ddot{u}$, $\dot{u}$ and u for a given SDoF system are calculated by solving Equation (7.1).

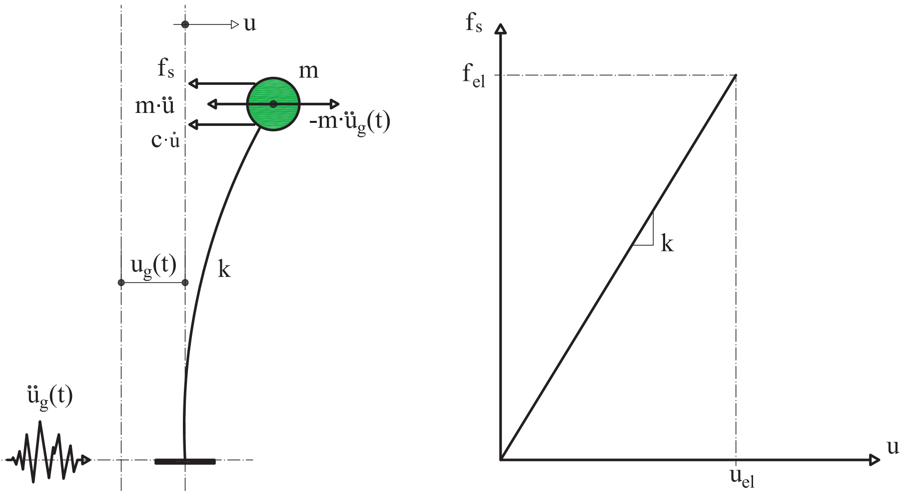


From the previous figure it can be clearly seen that the time-history of an earthquake ground acceleration can not be described by a simple mathematical formula. Time-histories are therefore usually expressed as sequence of discrete sample values and hence Equation (7.1) must therefore be solved numerically.



The sample values of the ground acceleration $\ddot{u}_g(t)$ are known from beginning to end of the earthquake at each increment of time Δt ("time step"). The solution strategy assumes that the motion quantities of the SDoF system at the time t are known, and that those at the time $t + \Delta t$ can be computed. Calculations start at the time $t = 0$ (at which the SDoF system is subjected to known initial conditions) and are carried out time step after time step until the entire time-history of the motion quantities is computed, like e.g. the acceleration shown in the figure on page 7-1.

7.2 Time-history analysis of linear SDoF systems



In the case of a linear SDoF system Equation (7.1) becomes:

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g \tag{7.2}$$

and by introducing the definitions of natural circular frequency $\omega_n = \sqrt{k/m}$ and of damping ratio $\zeta = c/(2m\omega_n)$, Equation (7.1) can be rearranged as:

$$\ddot{u} + 2\zeta\omega_n u + \omega_n^2 u = -\ddot{u}_g \tag{7.3}$$

The response to an arbitrarily time-varying force can be computed using:

- Convolution integral ([Cho11] Chapter 4.2)
- Numerical integration of the differential equation of motion ([Cho11] Chapter 5)

Course “Fundamentals of Structural Dynamics” An-Najah 2013

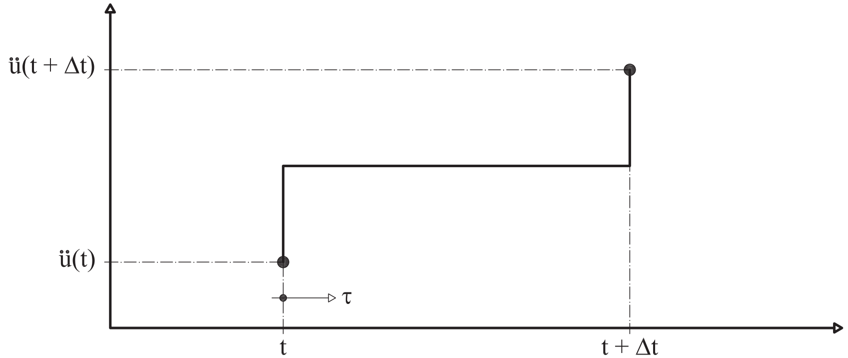
7.2.1 Newmark’s method (see [New59])

- Incremental formulation of the equation of motion

$$m\Delta\ddot{u} + c\Delta\dot{u} + k\Delta u = -m\Delta\ddot{u}_g \tag{7.4}$$

$${}^{t+\Delta t}u = {}^{t}u + \Delta u \;,\; {}^{t+\Delta t}\dot{u} = {}^{t}\dot{u} + \Delta\dot{u} \;,\; {}^{t+\Delta t}\ddot{u} = {}^{t}\ddot{u} + \Delta\ddot{u} \tag{7.5}$$

- Assumption of the acceleration variation over the time step:



$$\ddot{u}(\tau) = \frac{1}{2}({}^{t}\ddot{u} + {}^{t+\Delta t}\ddot{u}) = {}^{t}\ddot{u} + \frac{\Delta\ddot{u}}{2} \tag{7.6}$$

$$\dot{u}(\tau) = {}^{t}\dot{u} + \int_t^{\tau} \ddot{u}(\tau)d\tau = {}^{t}\dot{u} + \left({}^{t}\ddot{u} + \frac{\Delta\ddot{u}}{2}\right)(\tau - t) \tag{7.7}$$

$$u(\tau) = {}^{t}u + \int_t^{\tau} \dot{u}(\tau)d\tau = {}^{t}u + \int_t^{\tau}\left[{}^{t}\dot{u} + \left({}^{t}\ddot{u} + \frac{\Delta\ddot{u}}{2}\right)(\tau - t)\right]d\tau \tag{7.8}$$

$$u(\tau) = {}^t u + {}^t \dot{u}(\tau - t) + \left({}^t \ddot{u} + \frac{\Delta\ddot{u}}{2}\right)\frac{(\tau - t)^2}{2} \tag{7.9}$$

The increments of acceleration, velocity and displacement during the time step are:

$$\Delta\ddot{u} = {}^{t+\Delta t}\ddot{u} - {}^t\ddot{u} = \Delta\ddot{u} \tag{7.10}$$

$$\Delta\dot{u} = {}^{t+\Delta t}\dot{u} - {}^t\dot{u} = \left({}^t\ddot{u} + \frac{\Delta\ddot{u}}{2}\right)\Delta t \tag{7.11}$$

$$\Delta u = {}^t\dot{u}\Delta t + \left({}^t\ddot{u} + \frac{\Delta\ddot{u}}{2}\right)\frac{\Delta t^2}{2} \tag{7.12}$$

Introducing the parameters γ and β into Equations (7.11) and (7.12) for $\Delta\dot{u}$ and Δu, respectively, can be generalized as follows:

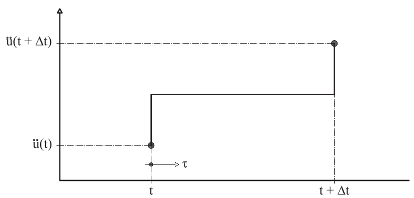
$$\Delta\dot{u} = ({}^t\ddot{u} + \gamma\Delta\ddot{u})\Delta t \tag{7.13}$$

$$\Delta u = {}^t\dot{u}\Delta t + ({}^t\ddot{u} + 2\beta\Delta\ddot{u})\frac{\Delta t^2}{2} \tag{7.14}$$

where different values of the parameters γ and β correspond to different assumptions regarding the variation of the acceleration within the time step:

Average Acceleration:

$$\beta = \frac{1}{4}, \gamma = \frac{1}{2}$$

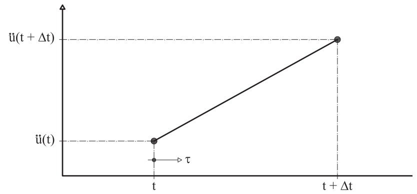


$$\frac{\Delta t}{T} \le \infty$$

Course "Fundamentals of Structural Dynamics" *An-Najah 2013*

Linear Acceleration:

$$\beta = \frac{1}{6}, \gamma = \frac{1}{2}$$



$$\frac{\Delta t}{T} \le 0.551$$

It is important to note that the "average acceleration"-method is unconditionally stable, while the "linear acceleration"-method is only stable if the condition $\Delta t / T \le 0.551$ is fulfilled.

However, the "linear acceleration"-method is typically more accurate and should be preferred if there are no stability concerns. For a discussion on stability and accuracy of the Newmark's methods see e.g. [Cho11] and [Bat96].

- Solution of the differential equation: **Option 1**

Substituting Equations (7.13) and (7.14) into Equation (7.4) gives Equation (7.15), which can be solved for the only remaining variable $\Delta\ddot{u}$:

$$(m + c\gamma\Delta t + k\beta\Delta t^2)\Delta\ddot{u} = -m\Delta\ddot{u}_g - c\,{}^t\ddot{u}\Delta t - k\left({}^t\dot{u}\Delta t + {}^t\ddot{u}\frac{\Delta t^2}{2}\right) \tag{7.15}$$

or in compact form:

$$\tilde{m}\Delta\ddot{u} = \Delta\tilde{p} \tag{7.16}$$

Substituting $\Delta\ddot{u}$ into Equations (7.11) and (7.12) gives the increments of the velocity $\Delta\dot{u}$ and of the displacement Δu. In conjunction with Equation (7.5), these increments yield the dynamic response of the SDoF system at the end of the time step $t + \Delta t$.

- Solution of the differential equation: **Option 2**

Equation (7.14) can be transformed to:

$$\Delta\ddot{u} = \frac{\Delta u}{\beta\Delta t^2} - \frac{{}^t\dot{u}}{\beta\Delta t} - \frac{{}^t\ddot{u}}{2\beta} \tag{7.17}$$

and substituting Equation (7.17) into (7.13) we obtain:

$$\Delta\dot{u} = \frac{\gamma\Delta u}{\beta\Delta t} - \frac{\gamma\,{}^t\dot{u}}{\beta} + \Delta t\left(1 - \frac{\gamma}{2\beta}\right){}^t\ddot{u} \tag{7.18}$$

Substituting Equations (7.17) and (7.18) into Equation (7.4) gives Equation (7.19), which can be solved for the only remaining variable Δu:

$$\left(k + \frac{m}{\beta\Delta t^2} + \frac{\gamma c}{\beta\Delta t}\right)\Delta u = -m\Delta\ddot{u}_g + \left(\frac{m}{\beta\Delta t} + \frac{\gamma c}{\beta}\right){}^t\dot{u} + \left(\frac{m}{2\beta} - \Delta t\left(1 - \frac{\gamma}{2\beta}\right)c\right){}^t\ddot{u} \tag{7.19}$$

or in compact form:

$$\bar{k}\Delta u = \Delta\bar{p} \tag{7.20}$$

Substituting Δu into Equations (7.18) and (7.17) gives the increments of the velocity $\Delta\dot{u}$ and of the acceleration $\Delta\ddot{x}$. In conjunction with Equation (7.5), these increments yield the dynamic response of the SDoF system at the end of the time step $t + \Delta t$.

For linear systems we have:

- m, c and k are constant throughout the whole time-history.
- $\tilde{m}$ in Equation (7.15), as well as $\bar{k}$ in Equation (7.20), are also constant and have to be computed only once.

7.2.2 Implementation of Newmark's integration scheme within the Excel-Table "SDOF_TH.xls"

Equation (7.15) is implemented in the Excel-Table as follows:

$$\underbrace{(m + c\gamma\Delta t + k\beta\Delta t^2)}_{meq}\underbrace{\Delta\ddot{u}}_{da} = \underbrace{-m\Delta\ddot{u}_g}_{\Delta F(t)} - c\underbrace{{}^t\ddot{u}\Delta t}_{dv} - k\underbrace{\left({}^t\dot{u}\Delta t + {}^t\ddot{u}\frac{\Delta t^2}{2}\right)}_{dd}$$

- In the columns **C** to **E** the so-called "**predictors**" **dd**, **dv** and **da** are computed first:

$$dd = {}^t\dot{u}\Delta t + {}^t\ddot{u}\frac{\Delta t^2}{2} \qquad \text{("delta-displacement")}$$

("**d**elta-**d**isplacement")

$$dv = {}^t\ddot{u}\Delta t$$

("**d**elta-**v**elocity")

$$da = \frac{-m\Delta\ddot{u}_g - c\cdot dv - k\cdot dd}{meq} = \Delta\ddot{u}$$

("**d**elta-**a**cceleration")

- Afterwards, in the columns **F** to **H** the ground motion quantities at the time step $t + \Delta t$ are computed by means of so-called "**correctors**:

$${}^{t+\Delta t}\ddot{u} = {}^t\ddot{u} + da$$

$${}^{t+\Delta t}\dot{u} = {}^t\dot{u} + \underbrace{dv + (\mathbf{da}\cdot\boldsymbol{\gamma}\cdot\mathbf{\Delta t})}_{\Delta\dot{u}}$$

$${}^{t+\Delta t}u = {}^t u + \underbrace{dd + (\mathbf{da}\cdot\boldsymbol{\beta}\cdot\mathbf{\Delta t^2})}_{\Delta u}$$

- Finally, in column **I** the absolute acceleration $\ddot{u}_{abs}$ at the time step $t + \Delta t$ is computed as follows:

$$^{t+\Delta t}\ddot{u}_{abs} = {}^{t+\Delta t}\ddot{u} + {}^{t+\Delta t}\ddot{u}_g$$

Observations about the use of the Excel-Table

- Only the yellow cells should be modified:
  - The columns **A** and **B** contain the time vector and the ground acceleration $\ddot{u}_g(t)$ at intervals Δt; for this ground motion the response of a Single-Degree-of-Freedom (SDoF) system will be computed. To compute the response of the SDoF system for a different ground motion $\ddot{u}_g(t)$, the time and acceleration vector of the new ground motion have to be pasted into columns **A** and **B**.
  - For a given ground motion $\ddot{u}_g(t)$, the response of a linear SDoF system is only dependent on its period $T = 2\pi/\omega_n$ and its damping ζ. For this reason, the period T and the damping ζ can be chosen freely in the Excel-Table.
  - The mass m is only used to define the actual stiffness of the SDoF system $k = m \cdot \omega_n^2$ and to compute from it the correct spring force $f_s = k \cdot u$. However, f_s is not needed in any of the presented plots, hence the defaults value $m = 1$ can be kept for all computations.
  - In the field “Number of periods” (cell **V19**) one can enter the number of periods T_i for which the response of the SDoF is to be computed in order to draw the corresponding response spectra.
- The response spectra are computed if the button **“compute response spectra”** is pressed. The macro pastes the different periods T_i into cell **S3**, computes the response of the SDoF system, reads the maximum response quantities from the cells **F6**, **G6**, **H6** and **I6** and writes these value into the columns **L** to **P**.

7.2.3 Alternative formulation of Newmark’s Method.

The formulation of the Newmark’s Method presented in Section 7.2.1 corresponds to an incremental formulation. It is possible to rearrange the methodology to obtain a total formulation.

The equation of motion at the time $t + \Delta t$ can be written as:

$$m\,{}^{t+\Delta t}\ddot{u} + c\,{}^{t+\Delta t}\dot{u} + k\,{}^{t+\Delta t}u = -m\,{}^{t+\Delta t}\ddot{u}_g \qquad (7.21)$$

where

$$^{t+\Delta t}\ddot{u} = {}^{t}\ddot{u} + \Delta\ddot{u} \qquad (7.22)$$

$$^{t+\Delta t}\dot{u} = {}^{t}\dot{u} + \Delta\dot{u} \qquad (7.23)$$

Using the expressions for $\Delta\ddot{u}$ and $\Delta\dot{u}$ given by Equations (7.17) and (7.18), the acceleration and the velocity at the time $t + \Delta t$ can be written as:

$$^{t+\Delta t}\ddot{u} = \frac{1}{\beta\Delta t^2}({}^{t+\Delta t}u - {}^{t}u) - \frac{1}{\beta\Delta t}{}^{t}\dot{u} - \left(\frac{1}{2\beta} - 1\right){}^{t}\ddot{u} \qquad (7.24)$$

$$^{t+\Delta t}\dot{u} = \frac{\gamma}{\beta\Delta t}({}^{t+\Delta t}u - {}^{t}u) + \left(1 + \frac{\gamma}{\beta}\right){}^{t}\dot{u} + \Delta t\left(1 - \frac{\gamma}{2\beta}\right){}^{t}\ddot{u} \qquad (7.25)$$

Introducing Equations (7.24) and (7.25) into Equation (7.21) and solving for the only unknown $^{t+\Delta t}u$ we obtain:

$$\bar{k} \cdot {}^{t+\Delta t}u = {}^{t+\Delta t}\bar{p} \qquad (7.26)$$

where:

$$\bar{k} = k + a_1 \tag{7.27}$$

$$^{t+\Delta t}\bar{p} = -m\,^{t+\Delta t}\ddot{u}_g + a_1\,^{t}u + a_2\,^{t}\dot{u} + a_3\,^{t}\ddot{u} \tag{7.28}$$

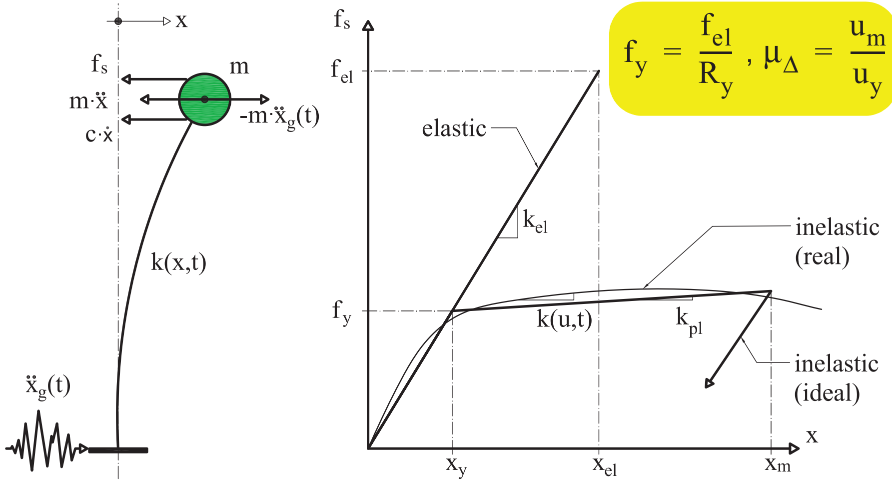
$$a_1 = \frac{m}{\beta \Delta t^2} + \frac{\gamma c}{\beta \Delta t} \tag{7.29}$$

$$a_2 = \frac{m}{\beta \Delta t} + \left(\frac{\gamma}{\beta} - 1\right)c \tag{7.30}$$

$$a_3 = \left(\frac{1}{2\beta} - 1\right)m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c \tag{7.31}$$

This formulation corresponds to the implementation of Newmark's method presented in [Cho11].

7.3 Time-history analysis of nonlinear SDoF systems



$$f_y = \frac{f_{el}}{R_y}, \quad \mu_\Delta = \frac{u_m}{u_y}$$

- Strength f_y of the nonlinear SDoF system

$$f_y = \frac{f_{el}}{R_y} = \frac{k_{el} \cdot u_{el}}{R_y} \tag{7.32}$$

  - R_y = **force reduction factor**
  - f_{el} = maximum spring force f_s that a linear SDoF system of the same period T and damping ζ would experience if submitted to the same ground motion $\ddot{u}_g$

- Maximum deformation u_m of the nonlinear SDoF system

$$u_m = \mu_\Delta \cdot u_y, \qquad \text{hence} \qquad \mu_\Delta = u_m / u_y \tag{7.33}$$

  - u_y = yield displacement
  - μ_Δ = **displacement ductility**

7.3.1 Equation of motion of nonlinear SDoF systems

In the equation of motion for a base point excitation through an acceleration time-history $\ddot{u}_g(t)$

$$m\ddot{u} + c\dot{u} + f_s(u,t) = -m\ddot{u}_g \qquad (7.34)$$

of a nonlinear SDoF system, the spring force $f_s(u,t)$ is no longer constant and varies in function of time and location.

Most structural components are characterised by a continuously curved force-deformation relationship like the one shown by means of a thin line on the right of the figure on page 7-12, which however is often approximated by a bilinear curve (thick line in the same figure).

When the loading of the nonlinear SDoF system is cyclic, then f_s is no longer an unambiguous function of the location u and also for this reason Equation (7.34) shall be solved incrementally.

For this reason $f_s(u,t)$ must be described in such a way, that starting from a known spring force $f_s(u,t)$ at the time t, the still unknown spring force $f_s(x + \Delta u, t + \Delta t)$ at the time $t + \Delta t$ can easily be computed.

This description of the cyclic force-deformation relationship is known as **hysteretic rule**. In the literature many different hysteretic rules for nonlinear SDoF systems are available (See e.g. [Saa91]).

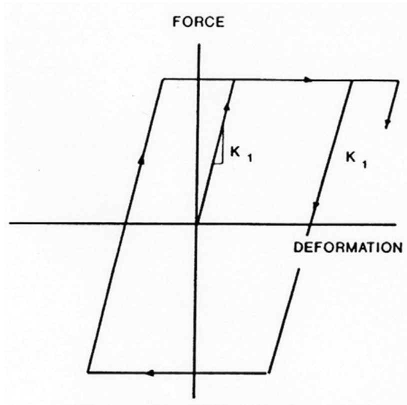
In the following section a few hysteretic rules are presented and discussed.

Course “Fundamentals of Structural Dynamics” *An-Najah 2013*

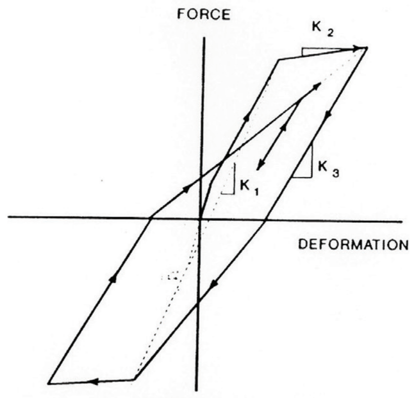
7.3.2 Hysteretic rules

The next figure shows typical hysteretic rules (or models) for nonlinear SDoF systems.

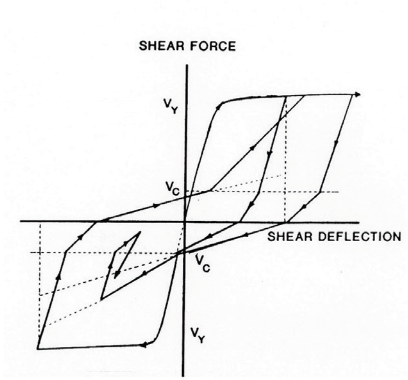
Elasto-plastic model



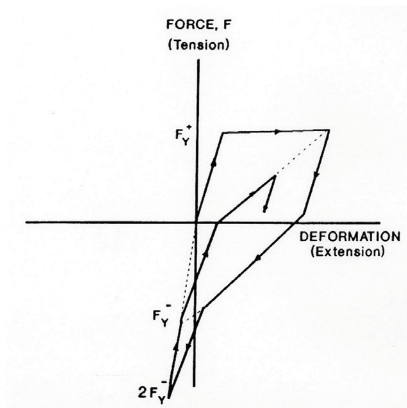
Takeda-Model



Oczabe-Saatcioglu-Model

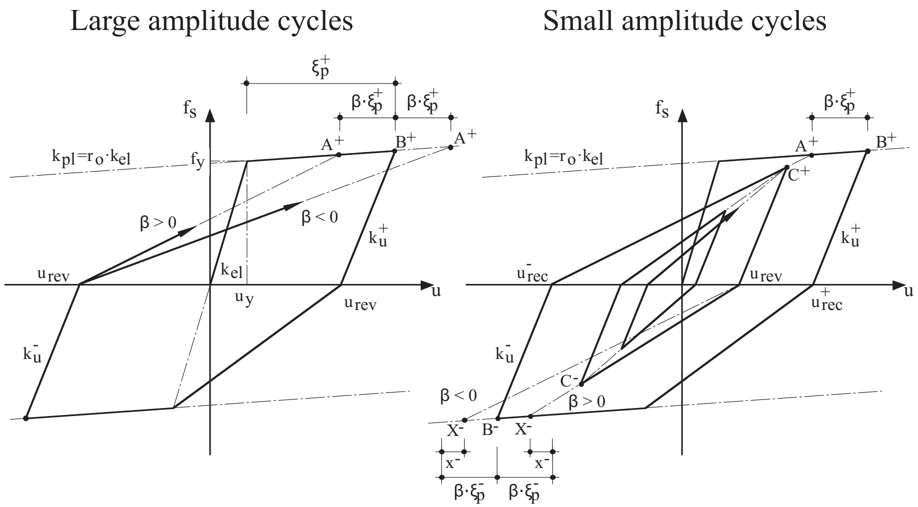


Kabeyasawa-Model



In the following the often used Takeda hysteretic model is discussed in some detail.

The Takeda model was first described in [TSN70] and later modified by various authors. The assumed force-deformation relationship shown in the following figure was derived from the moment-curvature relationship described in [AP88].



The initial loading follows the bilinear force-deformation relationship for monotonic loading mentioned in the previous section. The exact definition of this so-called skeleton curve depends on the structural element at hand. For example, in the case of Reinforced Concrete (RC) structural walls the elastic stiffness k_{el} corresponds to 20 to 30% of the uncracked stiffness, while the plastic stiffness $k_{pl} = r_o \cdot k_{el}$ is approximated assuming an hardening factor $r_o = 0.01 \ldots 0.05$.

Unloading occurs along a straight line with stiffness k_u. This unloading stiffness is computed by means of Equation (7.35) as a function of the elastic stiffness k_{el} and taking into softening effects in proportion of the previously reached maximum displacement ductility μ_Δ. The parameter α controls the unloading stiffness reduction and varies from structural element to structural element.

$$k_u^+ = k_{el} \cdot (\max\{\mu_\Delta^+\})^{-\alpha}, \qquad k_u^- = k_{el} \cdot (\max\{|\mu_\Delta^-|\})^{-\alpha} \quad (7.35)$$

Reloading follows a straight line which is defined by the force reversal point $(u_{rev}, 0)$ and the point **A**. The location of point **A** is determined according to the figure on the previous page as a function of the last reversal point **B**, the plastic strain ξ_p and the damage influence parameter β. The parameter β allows taking into account softening effects occurring during the reloading phase.

Again, in the case of RC walls meaningful parameters α and β lay in the following ranges: $\alpha = 0.2 \ldots 0.6$ and $\beta = 0.0 \ldots -0.3$.

These rules, which are valid for cycles with large amplitude, are typically based on observations of physical phenomena made during experiments.

On the other hand, rules for small amplitude cycles are based on engineering considerations rather than on exact observations. They are designed to provide reasonable hysteresis lops during an earthquake time-history, thus avoiding clearly incorrect behaviours like e.g. negative stiffnesses.

The rules for small amplitude cycles are shown on the right of the figure on the previous page.

If a reloading phase starts from a force reversal point $(u_{rev},0)$ laying between the two extreme force reversal point $(u_{rec}^{+},0)$ and $(u_{rec}^{-},0)$, then reloading does no longer occurs towards point **A**, but towards a newly calculated point **X**, which lies between points **A** and **B**. The position of point **X** is calculated using the auxiliary variables x^{+} and x^{-} defined in Equation (7.36).

$$x^{+} = \left(\frac{u_{rev} - u_{rec}^{+}}{u_{rec}^{-} - u_{rec}^{+}}\right) \cdot \beta\xi_p^{+}, \qquad x^{-} = \left(\frac{u_{rev} - u_{rec}^{-}}{u_{rec}^{+} - u_{rec}^{-}}\right) \cdot \beta\xi_p^{-} \qquad (7.36)$$

When a load reversal occurs before point **X** is reached, a new point **C** is defined as a temporary maximum and minimum. The reloading in the subsequent cycles, which are smaller than the temporary maximum and minimum is then always in the direction of point **C**.

These rules for cycles with a small amplitude are a simplification of those described in [AP88], however they lead to very satisfactory results and can be programmed very easily and efficiently.

7.3.3 Newmark's method for inelastic systems

The Newmark's numerical method discussed in Sections 7.2.1 to 7.2.3 can be easily modified for application to nonlinear systems. The following modifications are required:

- The mass m and the damping c are typically constant throughout the whole time-history.
- The stiffness k changes during the time-history, hence $\tilde{m}$, respectively $\bar{k}$, are no longer constant.
- If the stiffness changes within the time step iterations are needed (e.g. Newton-Raphson).
- For nonlinear systems the second solution strategy presented in Section 7.2.1 (**Option 2**) has the advantage that the factors

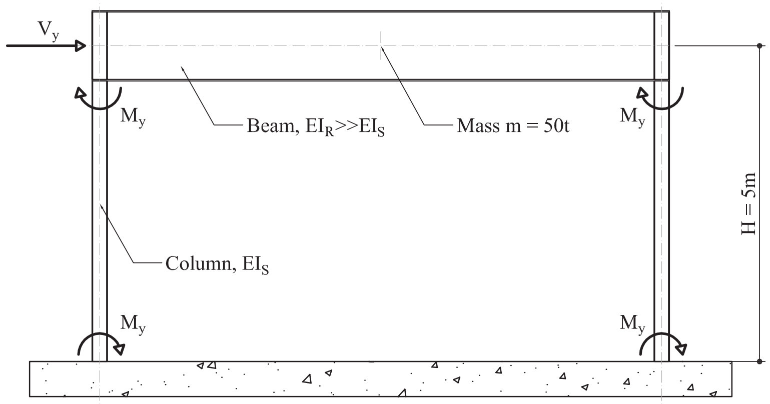
  $$a = \left(\frac{m}{\beta\Delta t} + \frac{\gamma c}{\beta}\right) \text{ and } b = \frac{m}{2\beta} - \Delta t\left(1 - \frac{\gamma}{2\beta}\right)c$$

  on the RHS of Equation (7.19) are constant throughout the whole time-history and do not need to be recomputed at every time step.

Remark

For the actual implementation of the nonlinear version of Newmark's time stepping method, it is suggested to use the formulation presented in Section 7.4.4 in conjunction with the Newton-Raphson algorithm described in Section 7.4.3.

7.3.4 Example 1: One-storey, one-bay frame

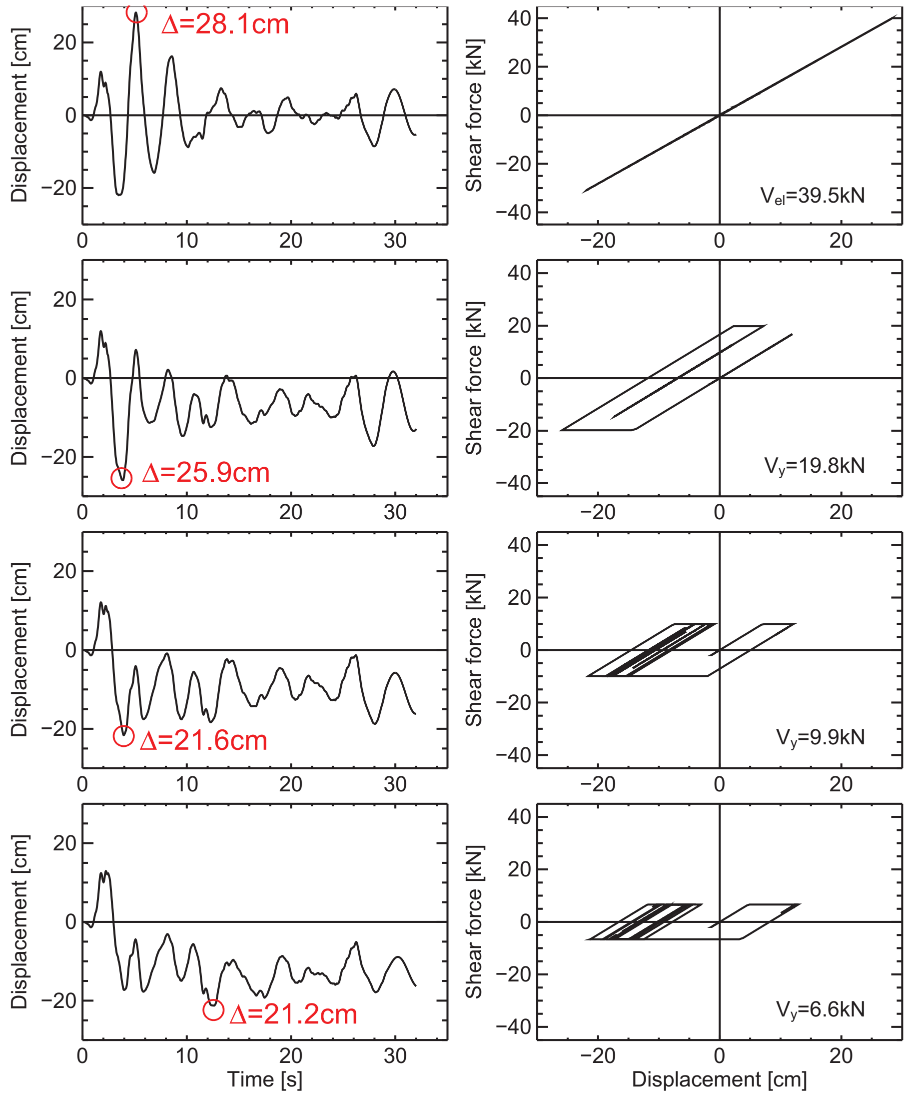


$$k = 2\cdot\frac{12EI_s}{H^3}\ ,\ T = 2\pi\sqrt{\frac{m}{k}}\ ,\ V_y = 2\cdot\frac{M_y}{H/2}\ ,\ \Delta_y = \frac{V_y}{k} \qquad (7.37)$$

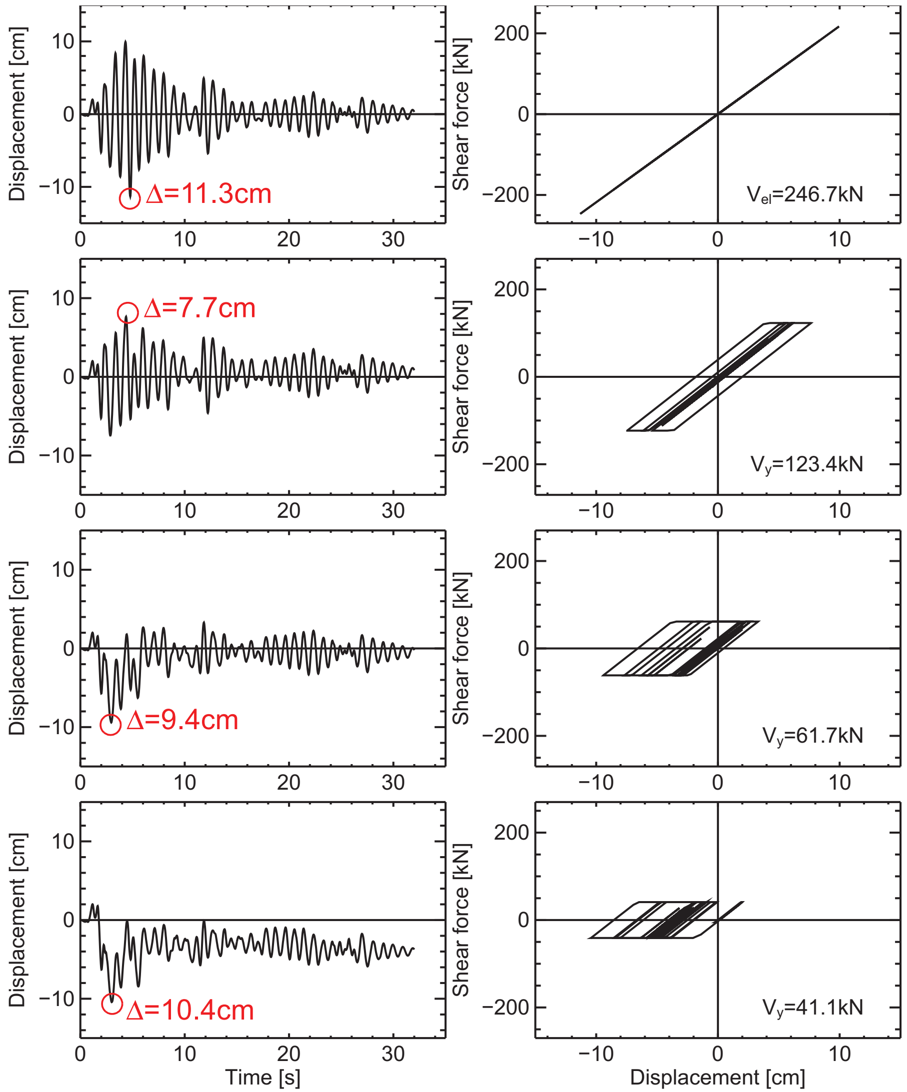
- Parameters

Columns	k [kN/m]	T [s]	f_y [MPa]	V_y [kN]	V_y/V_{el} [-]	Δ_y [cm]
HEA 100	141	3.75	**595**	39.5	1.00	28.1
HEA 100	141	3.75	**298**	19.8	0.50	14.0
HEA 100	141	3.75	**149**	9.9	0.25	7.0
HEA 100	141	3.75	**99**	6.6	0.167	4.7
HEA 220	2181	0.95	**543**	246.7	1.00	11.3
HEA 220	2181	0.95	**272**	123.4	0.50	5.7
HEA 220	2181	0.95	**136**	61.7	0.25	2.8
HEA 220	2181	0.95	**91**	41.1	0.167	1.9
IPE 550	27055	0.27	**185**	411.4	1.00	1.52
IPE 550	27055	0.27	**93**	205.7	0.50	0.76
IPE 550	27055	0.27	**43**	102.9	0.25	0.38
IPE 550	27055	0.27	**31**	68.6	0.167	0.25

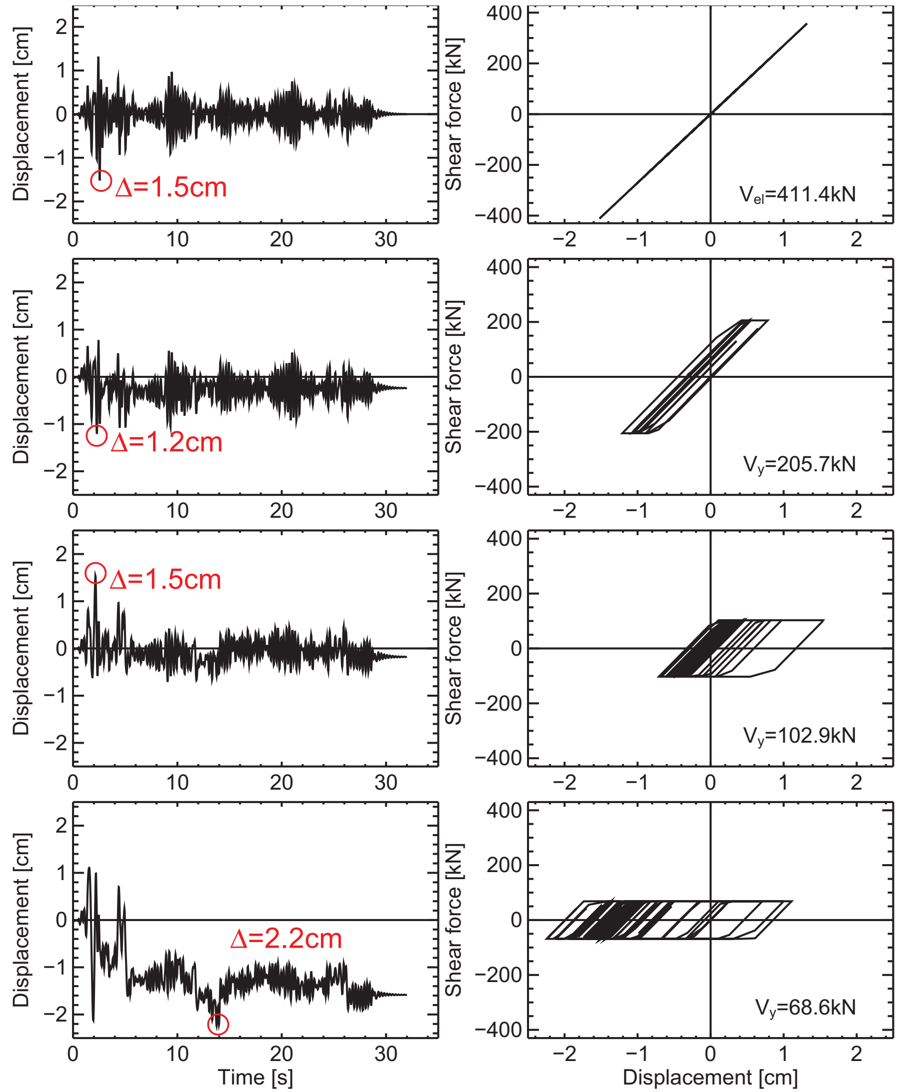
- Frames with $T = 3.75\text{s}$, $\zeta = 5\%$ subjected to “El Centro”



- Frames with $T = 0.95\text{s}$, $\zeta = 5\%$ subjected to "El Centro"



- Frames with $T = 0.27\text{s}$, $\zeta = 5\%$ subjected to "El Centro"



7.3.5 Example 2: A 3-storey RC wall

As a second example, the behaviour of the RC wall WDH4 presented in Section 7.6.3 is simulated. Wall WDH4 is actually a 3-DoF system and its behaviour is simulated by means of an equivalent SDoF system. For this reason the relative displacement of the SDoF system shall be multiplied by the participation factor $\tilde{\Gamma} = 1.291$ to obtain an estimation of top displacement of WDH4.

To simulate the behaviour of WDH4 a nonlinear SDoF system with Takeda hysteretic model is used. The parameter used to characterise the SDoF system are:

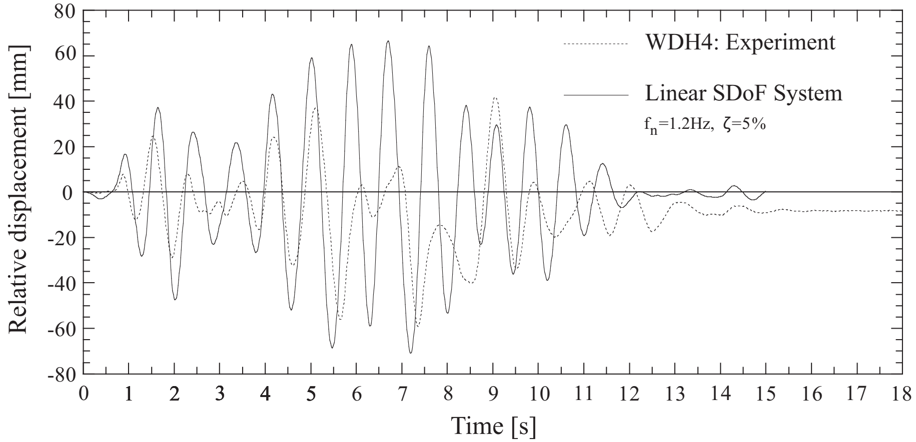
$$\zeta = 5\%, \; f_n = 1.2\text{Hz}, \; R_y = 2.5, \; r_o = 0.03, \\ \alpha = 0.2, \; \beta = -0.15 \tag{7.38}$$

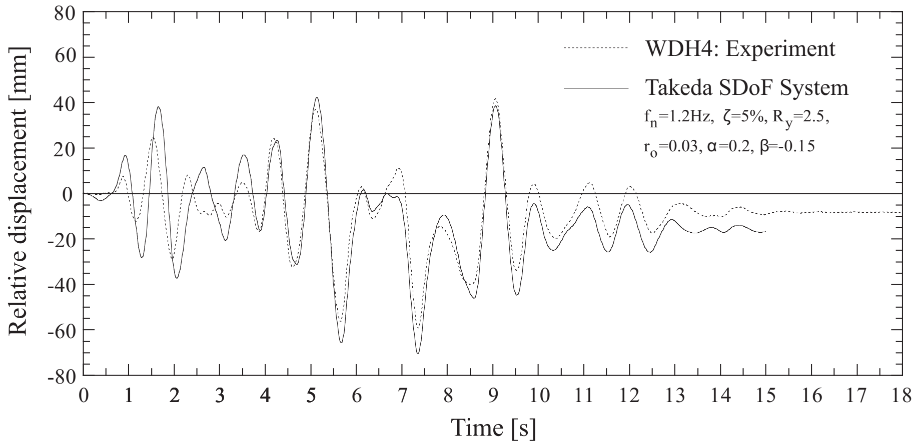
where ζ is the damping rate, f_n is the natural frequency of the SDoF system for elastic deformations (i.e. with $k = k_{el}$), and R_y is the force reduction factor. The parameters given in (7.38) were subsequently adjusted to obtain the best possible match between the simulation and the experiment. In a first phase, a calculation using a linear SDoF system is performed. The latter has the same damping rate and natural frequency as the Takeda SDoF system and allows an estimation of the maximum elastic spring force f_{el} and of the maximum elastic deformation u_{el}. The yield force f_y and the yield displacement u_y of the Takeda SDoF system are then estimated using the force reduction factor R_y as follows:

$$R_y = \frac{f_{el}}{f_y} = \frac{u_{el}}{u_y} \tag{7.39}$$

Time-history of the top displacement

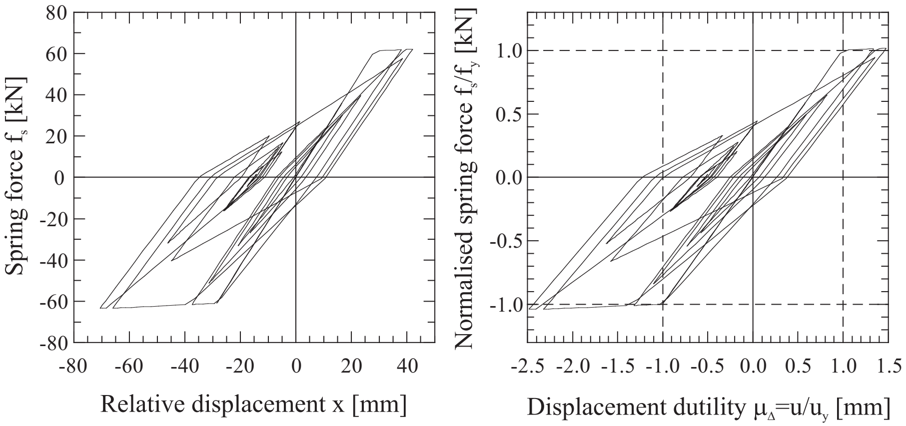
The time-history of the top displacement (see below) shows that: (i) plastic phenomena affect the behaviour of the wall WDH4 significantly, and that (ii) the Takeda-SDoF System is able can describe the global behaviour of the wall WDH4 quite accurately.





Hysteretic behaviour of the nonlinear SDoF System

Force-deformation relationship of the Takeda-SDoF System subjected to the same ground motion as Wall WDH4. In both diagrams the same curve is plotted: On the left in absolute units and on the right in normalised units.



7.4 Solution algorithms for nonlinear analysis problems

In this section the more general case of system with multiple degrees of freedom is discussed. SDoF system can be seen as a special case thereof.

7.4.1 General equilibrium condition

The general equilibrium condition for elastic and inelastic static and dynamic systems is:

$$\mathbf{F}(t) = \mathbf{R}(t) \quad (7.40)$$

In this equation $\mathbf{F}(t)$ is the time-dependent vector of the internal forces of all DoFs of the structure and $\mathbf{R}(t)$ the time-dependent vector of the external forces.

The vector $\mathbf{R}(t)$ depends on the problem analysed and is known.

7.4.2 Nonlinear static analysis

For linear-elastic systems the internal forces can be computed by means of Equation (7.41):

$$\mathbf{F} = \mathbf{KU} \quad (7.41)$$

where $\mathbf{U}$ is the vector of the displacements of the DoF and $\mathbf{K}$ is the stiffness matrix of the structure. Equation (7.40) can therefore be rewritten as

$$\mathbf{KU} = \mathbf{R} \quad (7.42)$$

In Equation (7.42) $\mathbf{R}$ is known and $\mathbf{K}$ is also known, therefore the unknown vector $\mathbf{U}$ can be computed by means of Equation (7.43):

$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{R} \tag{7.43}$$

The equilibrium condition of Equation (7.40) can only be solved for linear-elastic systems by means of Equation (7.43).

For inelastic systems, due to successive yielding of the structure, the stiffness matrix $\mathbf{K}$ is not constant over the course of the loading.

For this reason Equation (7.40) must be solved in **increments** (=small load steps) and **iteratively**. The approach is as follows:

- The nodal displacements ${}^{t}\mathbf{U}$ at the time t are known from the previous load step;
- The nodal displacements ${}^{t+\Delta t}\mathbf{U}$ at the end of the load step Δt are determined by means of n iterations of Equations (7.44) and (7.45).

$$ {}^{t+\Delta t}\mathbf{K}_T^{i-1}\,\Delta\mathbf{U}^i = {}^{t+\Delta t}\Delta\mathbf{R}^{i-1} \tag{7.44}$$

$$ {}^{t+\Delta t}\mathbf{U}^i = {}^{t+\Delta t}\mathbf{U}^{i-1} + \Delta\mathbf{U}^i \tag{7.45}$$

where:

$$ {}^{t+\Delta t}\Delta\mathbf{R}^{i-1} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{i-1} \tag{7.46}$$

and $\mathbf{K}_T$ is the tangent stiffness matrix of the structure.

The actual solution of the equilibrium conditions of Equation (7.40) is often obtained by the numerical method for the iterative solution of nonlinear equations that was originally developed by Newton.

In the next section the so-called Newton-Raphson Algorithm for the solution of Equation (7.40) will be discussed.

7.4.3 The Newton-Raphson Algorithm

The Newton-Raphson Algorithm allows the solution of the loading of nonlinear springs with the following equilibrium condition:

$$\mathbf{F}(\mathbf{U}(t)) = \mathbf{R}(t) \tag{7.47}$$

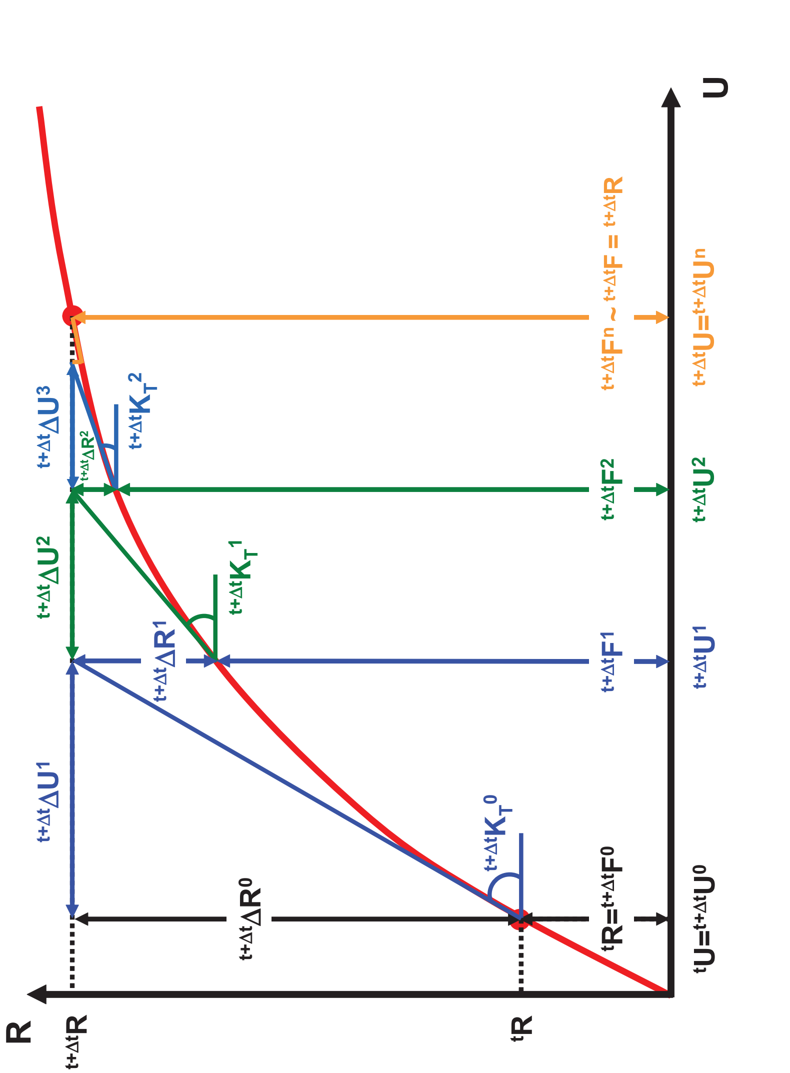
$\mathbf{F}(\mathbf{U}(t)) = \mathbf{R}(t)$ represents the internal spring force, which is a given nonlinear function of $\mathbf{U}(t)$. The external force $\mathbf{R}(t)$ is a function of the time t.

For a system with 1 DOF the solution method of the Newton-Raphson Algorithm can be illustrated by the figure on page 7-29.

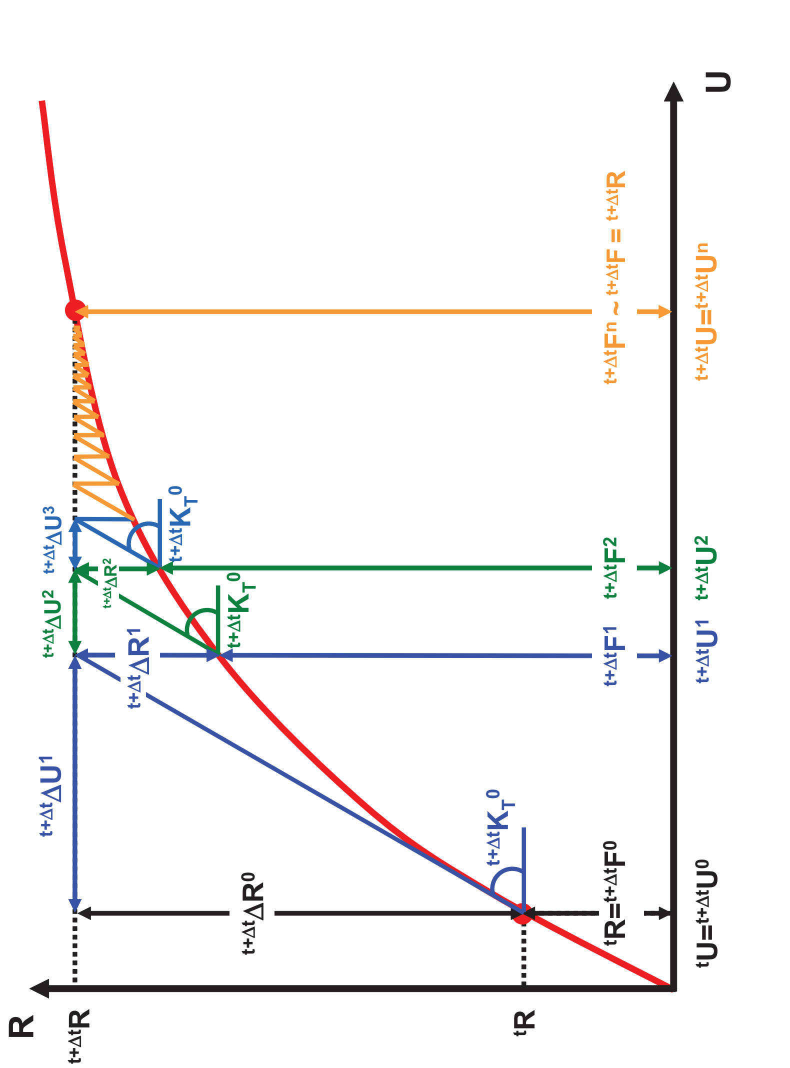
The algorithm consists of the following steps:

0) Up to time step t a solution was obtained and at time step t the system is in equilibrium with $({}^{t}U, {}^{t}R)$;

1) The initial conditions at the beginning of the iteration are determined. The iteration commences with the nodal displacement, the tangent stiffness and the internal force that have resulted at the end of the previous time step t. The external initial loading increment ${}^{t+\Delta t}\Delta R^0$ within the time step is determined by Equation (7.51).



Newton-Raphson algorithm to solve systems of nonlinear equations (according to [AEM86])



Modified Newton-Raphson algorithm to solve systems of nonlinear equations (according to [AEM86])

Displacement: $^{t+\Delta t}U^0 = {}^tU$ (7.48)

Tangent stiffness: $^{t+\Delta t}K_T^0 = {}^tK_T$ (7.49)

Internal force: $^{t+\Delta t}F^0 = {}^tF$ (7.50)

External force: $^{t+\Delta t}\Delta R^0 = {}^{t+\Delta t}R - {}^{t+\Delta t}F^0$ (7.51)

2) Computation of the i^{th}-displacement increment ΔU^i by means of Equation (7.52) (i starts from 1).
For systems with more DoFs, $^{t+\Delta t}K_T^{i-1}$ is a matrix (tangent stiffness matrix) and Equation (7.52) is best solved by means of a $\mathbf{LDL}^T$-decomposition of the matrix $^{t+\Delta t}\mathbf{K}_T^{i-1}$.

$$^{t+\Delta t}K_T^{i-1}\Delta U^i = {}^{t+\Delta t}\Delta R^{i-1} \quad (7.52)$$

3) Computation of the displacement $^{t+\Delta t}U^i$ at the end of the i^{th}-iteration

$$^{t+\Delta t}U^i = {}^{t+\Delta t}U^{i-1} + \Delta U^i \quad (7.53)$$

4) Computation of the internal force $^{t+\Delta t}F^i$ and the new external force (residual force) $^{t+\Delta t}R^i$

$$^{t+\Delta t}\Delta R^i = {}^{t+\Delta t}R - {}^{t+\Delta t}F^i \quad (7.54)$$

5) If ΔU^i and/or $^{t+\Delta t}\Delta R^i$ are so small that they can be neglected: Continue with Step 7;

6) Determine the new tangent stiffness $^{t+\Delta t}K_T^i$ and repeat from Step 2;

7) If the analysis time has not yet ended, start a new time step and start again the procedure at Step 1.

Comments on the Newton-Raphson Algorithm

- The Newton-Raphson Algorithm for systems with several or many DoFs follows exactly the same procedure as the algorithm for SDoF systems. Only difference: Scalar values are replaced by the corresponding vectorial quantities.
- Apart from the Newton-Raphson Approach ("Full Newton-Raphson iteration") the "Modified Newton-Raphson iteration" is often applied. This algorithm is illustrated on page 7-30.
  - Unlike in the Newton-Raphson Algorithm, in the Modified Newton-Raphson Algorithm the tangent stiffness matrix $\mathbf{K}_T$ is updated only at the beginning of the time step and is kept constant over all the iterations within this time step.
  - To reach the target displacement $^{t+\Delta t}\mathbf{U}$ more iterations are required for the Modified Newton-Raphson Algorithm than for the Full Newton-Raphson Algorithm. However, these can be computed more quickly since assembling the tangent stiffness matrix $\mathbf{K}_T$ (Step 6) and in particular its $\mathbf{LDL}^T$-decomposition (Step 2) are only required at the beginning of a time step and not at each iteration within the time step. This is particularly advantageous for systems with many DoFs.
- In most FE-analysis programs both Newton-Raphson Algorithms as well as other algorithms are typically combined in a general solver in order to obtain a successful convergence of the iteration process for many structural analysis problems.
- Other algorithms for the solution of the equilibrium conditions can be found in Chapters 8 (static analysis) and 9 (dynamic analysis) of [Bat96].

Convergence criteria

- A in-depth discussion of the convergence criteria can be found in [Bat96] and [AEM86]. This section provides only a short overview.
- In Step 5) criteria are required in order to decide whether convergence of the iteration was obtained. Possible convergence criteria can be based on **displacements**, **force** or **energy** considerations.
- Since within the time step the unknown target displacement ${}^{t+\Delta t}\mathbf{U}$ needs to be determined, it makes sense to prescribe that the target displacement is reached within a certain tolerance interval. For this reason a possible **displacement criterion** for the convergence is:

$$\frac{\left\|\Delta \mathbf{U}^i\right\|}{\left\|{}^{t+\Delta t}\mathbf{U}\right\|} \le \varepsilon_D \tag{7.55}$$

  where ε_D is the displacement convergence tolerance.

  - The vector ${}^{t+\Delta t}\mathbf{U}$ is actually unknown and must therefore be approximated. Typically ${}^{t+\Delta t}\mathbf{U}^i$ is used in conjunction with a sufficiently small value of ε_D.
  - It is important to note that in some cases – although the criteria described by Equation (7.55) is satisfied – the wanted target displacement ${}^{t+\Delta t}\mathbf{U}$ has not been reached.
  - This is the case when the computed displacements vary only slightly during one iteration but these small increments are repeated over many iterations.
  - Such a situation can result when the modified Newton-Raphson Algorithm is used (see page 7-30).

  - For this reason the displacement criterion is typically used in conjunction with other convergence criteria.
- A **force criterion**, which checks the residual forces, is given in Equation (7.56).

$$\frac{\left\|{}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^i\right\|}{\left\|{}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F}\right\|} \le \varepsilon_F \tag{7.56}$$

  ε_F is the force convergence tolerance, which checks the magnitude of the residual force after the i^{th}-iteration against the first load increment of the time step.

  - As for the displacement criterion, this force criterion should not be applied on its own because in some cases the target displacement ${}^{t+\Delta t}\mathbf{U}$ may not have been reached. This may happen for systems with small post-yield stiffness.
- The **energy criterion** in Equation (7.57) has the advantage that it checks the convergence of the displacements and the forces simultaneously.

$$\frac{(\Delta \mathbf{U}^i)^T({}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{i-1})}{(\Delta \mathbf{U}^1)^T({}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F})} \le \varepsilon_E \tag{7.57}$$

  ε_E is the energy convergence tolerance, which checks the work of the residual forces of the i^{th}-iteration against the work of the residual forces of the first load increment of the time step.

  Choosing the tolerances ε_D, ε_F or ε_E too large, can yield wrong results, which can lead to the divergence of the solution in the following load steps.

Choosing the tolerances ε_D, ε_F or ε_E too small, results in an unnecessary increase of the required iteration steps. The increased accuracy is typically not useful.

For numerical reasons it can also happen that too small convergence tolerances do not allow to reach convergence at all.

For strongly inelastic systems it is recommended to check the sensitivity of the results to the chosen convergence criteria and the chosen tolerances.

7.4.4 Nonlinear dynamic analyses

Similar to Equation (7.44) the equilibrium condition for nonlinear dynamic analyses is:

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}}^{i} + \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}}^{i} + \left({}^{t+\Delta t}\mathbf{F}^{i-1} + {}^{t+\Delta t}\mathbf{K}_T^{i-1}\Delta\mathbf{U}^{i}\right) = {}^{t+\Delta t}\mathbf{R} \qquad (7.58)$$

$${}^{t+\Delta t}\mathbf{U}^{i} = {}^{t+\Delta t}\mathbf{U}^{i-1} + \Delta\mathbf{U}^{i} \qquad (7.59)$$

For base excitation by means of ground accelerations the vector of the external forces is computed according to Equation (7.60).

$${}^{t+\Delta t}\mathbf{R} = -\mathbf{M}\mathbf{1}\,{}^{t+\Delta t}a_g \qquad (7.60)$$

where $\mathbf{M}$ is the mass matrix of the structure, $\mathbf{1}$ the norm vector with entries of unity for all DoFs in the direction of the excitation and ${}^{t+\Delta t}a_g$ the ground acceleration at the time $t+\Delta t$.

For this type of excitation the differential equation of motion (7.58) has to be integrated **numerically** and – due to the inelastic behaviour of the system – the equation must be solved **iteratively** and **incrementally**.

The numerical time integration of Equation (7.58) is often performed by means of the Newmark's Algorithms ([New 59]). According to these algorithms the displacement vector at the time $t+\Delta t$ is estimated as follows:

$${}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\Delta t + [(1-2\beta)\,{}^{t}\ddot{\mathbf{U}} + 2\beta\,{}^{t+\Delta t}\ddot{\mathbf{U}}]\frac{\Delta t^2}{2} \qquad (7.61)$$

$${}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + [(1-\gamma)\,{}^{t}\ddot{\mathbf{U}} + \gamma\,{}^{t+\Delta t}\ddot{\mathbf{U}}]\Delta t \qquad (7.62)$$

From Equation (7.61):

$${}^{t+\Delta t}\ddot{\mathbf{U}} = \frac{1}{\beta\Delta t^2}({}^{t+\Delta t}\mathbf{U} - {}^{t}\mathbf{U}) - \frac{1}{\beta\Delta t}\,{}^{t}\dot{\mathbf{U}} - \left(\frac{1-2\beta}{2\beta}\right){}^{t}\ddot{\mathbf{U}} \qquad (7.63)$$

Substituting Equation (7.63) into (7.62):

$${}^{t+\Delta t}\dot{\mathbf{U}} = \frac{\gamma}{\beta\Delta t}({}^{t+\Delta t}\mathbf{U} - {}^{t}\mathbf{U}) + \left(1-\frac{\gamma}{\beta}\right){}^{t}\dot{\mathbf{U}} + \left(1-\frac{\gamma}{2\beta}\right){}^{t}\ddot{\mathbf{U}}\Delta t \qquad (7.64)$$

The expressions for the displacement, the velocity and the acceleration at the time $t+\Delta t$ from Equations (7.59), (7.64) and (7.63) can be substituted into the differential equation of motion (7.58) which can then be solved for the only remaining unknown $\Delta\mathbf{U}^{i}$:

$$\begin{aligned}\left({}^{t+\Delta t}\mathbf{K}_T^{i-1} + \frac{1}{\beta\Delta t^2}\mathbf{M} + \frac{\gamma}{\beta\Delta t}\mathbf{C}\right)\Delta\mathbf{U}^{i} = \; & {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{i-1} - \\ & \mathbf{M}\left[\frac{1}{\beta\Delta t^2}({}^{t+\Delta t}\mathbf{U}^{i-1} - {}^{t}\mathbf{U}) - \frac{1}{\beta\Delta t}\,{}^{t}\dot{\mathbf{U}} - \left(\frac{1}{2\beta}-1\right){}^{t}\ddot{\mathbf{U}}\right] - \\ & \mathbf{C}\left[\frac{\gamma}{\beta\Delta t}({}^{t+\Delta t}\mathbf{U}^{i-1} - {}^{t}\mathbf{U}) + \left(1-\frac{\gamma}{\beta}\right){}^{t}\dot{\mathbf{U}} + \left(1-\frac{\gamma}{2\beta}\right){}^{t}\ddot{\mathbf{U}}\Delta t\right]\end{aligned} \qquad (7.65)$$

or in a more compact format:

$$ {}^{t+\Delta t}\underset{\sim}{\mathbf{K}}_T^{i-1}\,\Delta\mathbf{U}^i = {}^{t+\Delta t}\Delta\underset{\sim}{\mathbf{R}}^{i-1} \tag{7.66} $$

Equation (7.66) corresponds exactly to Equation (7.44) and is also solved iteratively by the Newton-Raphson Algorithm.

When dynamic analyses are carried out, typical convergence criteria also consider the inertia forces and, if present, damping forces. Possible, often used convergence criteria are:

$$ \frac{\left\| {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{i-1} - \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}}^{i-1} - \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}}^{i-1} \right\|}{\mathrm{RNORM}} \le \varepsilon_F \tag{7.67} $$

$$ \frac{(\Delta\mathbf{U}^i)^T\left({}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{i-1} - \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}}^{i-1} - \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}}^{i-1}\right)}{(\Delta\mathbf{U}^1)^T\left({}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} - \mathbf{M}\,{}^{t}\ddot{\mathbf{U}} - \mathbf{C}\,{}^{t}\dot{\mathbf{U}}\right)} \le \varepsilon_E \tag{7.68} $$

with

$$ \mathrm{RNORM} = \sum m_{ij} \cdot g \tag{7.69} $$

As alternative, depending at which point during the iteration process convergence is checked, both criteria can be rewritten as:

$$ \frac{\left\| {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{i} - \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}}^{i} - \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}}^{i} \right\|}{\mathrm{RNORM}} \le \varepsilon_F \tag{7.70} $$

$$ \frac{(\Delta\mathbf{U}^i)^T\left({}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{i} - \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}}^{i} - \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}}^{i}\right)}{(\Delta\mathbf{U}^1)^T\left({}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} - \mathbf{M}\,{}^{t}\ddot{\mathbf{U}} - \mathbf{C}\,{}^{t}\dot{\mathbf{U}}\right)} \le \varepsilon_E \tag{7.71} $$

As for static analyses, different convergence criteria exist also for dynamic analyses and a discussion of these can also be found in [Bat96].

7.4.5 Comments on the solution algorithms for nonlinear analysis problems

- Contrary to the analysis of elastic systems, the analysis of inelastic systems is often interrupted before the targeted load or deformation state is reached.
  - This situation arises if in one of the time steps convergence cannot be reached.
- Typical causes for the failing convergence
  - The convergence tolerances are too small or too large;
  - The chosen values for the parameters of the solution algorithm are not appropriate;
  - The solution algorithm is not suitable.
    Typical example: If special measures are not taken, the algorithms of the Newton family are not able to solve a system with a global negative post-yield stiffness;
- Apart from the algorithms by Newmark, many other algorithms have been developed for solving the differential equation of motion (7.58) (e.g. "Houbolt Method", "Wilson θ Method", "α-Method"). These methods are described in detail in [Bat96]. The "α-Method" allows to introduce numerical damping, which can be useful. The "α-Method" is described in detail in [HHT77].
- Choice of the time step Δt for **static** analyses:
  - For static analyses the time t has no physical meaning. For this reason the size of the time step Δt can be chosen almost arbitrarily.
  - As long as the algorithm converges and the variation of the external loads is captured correctly, the size of the time step Δt has only a minor influence on the results.

- However, when geometric nonlinearities are considered or when the constitutive laws are a function of the strain history, care should be taken also for static analyses when choosing the size of the time step.
- The time step size influences the convergence of the algorithm: For small Δt the algorithm converges more quickly; however, more steps are required.
- Time steps of a variable size can be advantageous. If the system is elastic or almost elastic, large times steps can be chosen. If the system is close to its capacity, small time steps should be chosen.
- Certain analysis programs (see for example [HKS03]) determine the time step size within chosen limits as a function of the convergence and the number of required iterations.

- Choice of the time step Δt for **dynamic** analyses
  - The objective of the dynamic analyses is the solution of the differential equation of motion (7.58) between the time t and the time $t + \Delta t$. For this reason the choice of the time step Δt plays always an important role regarding the accuracy of the solution.
  - The accuracy of the integration of the differential equation of motion (7.58) depends on the chosen time-integration algorithm and on the ratio given by Equation (7.72), where T_n are the natural periods of the system. To capture the motion components due to higher modes the time step Δt has therefore to be reduced.

$$\frac{\Delta t}{T_n} \tag{7.72}$$

  - Certain time-integration algorithms can become instable if the time step size Δt is too large.

  - An in-depth discussion on the accuracy of time-integration methods and on so-called "conditionally stable" integration methods can be found in [Bat96].
  - Example:
    The time-integration method by Newmark with $\gamma = 1/2$ and $\beta = 1/6$ (linear variation of the acceleration over the length of the time step, see Section 7.2.1) is only stable if the criterion given in Equation (7.73) is met for all natural periods T_n of the system. For systems with many DoFs, higher modes can be especially problematic and a very short time step is generally required when this time-integration method is used.

$$\frac{\Delta t}{T_n} \le 0.551 \tag{7.73}$$

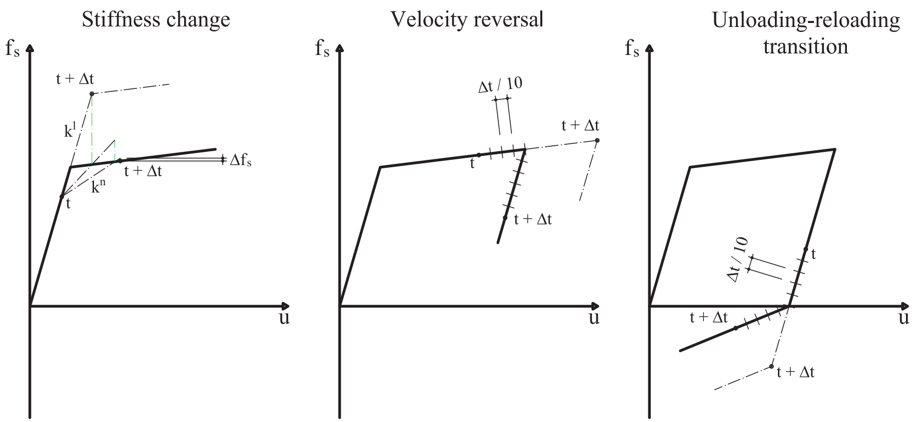
  - For this reason the "unconditionally stable" time-integration algorithm by Newmark with $\gamma = 1/2$ and $\beta = 1/4$ is often used in seismic engineering.

7.4.6 Simplified iteration procedure for SDoF systems with idealised rule-based force-deformation relationships

In the case that the hysteretic behaviour is described by a sequence of straight lines, a so called "idealized rule-based force-deformation relationships" like the Takeda model presented in Section 7.3.2, it is possible to avoid implementing a Newton-Raphson Iteration strategy. In this case adjustments are needed in the case of:

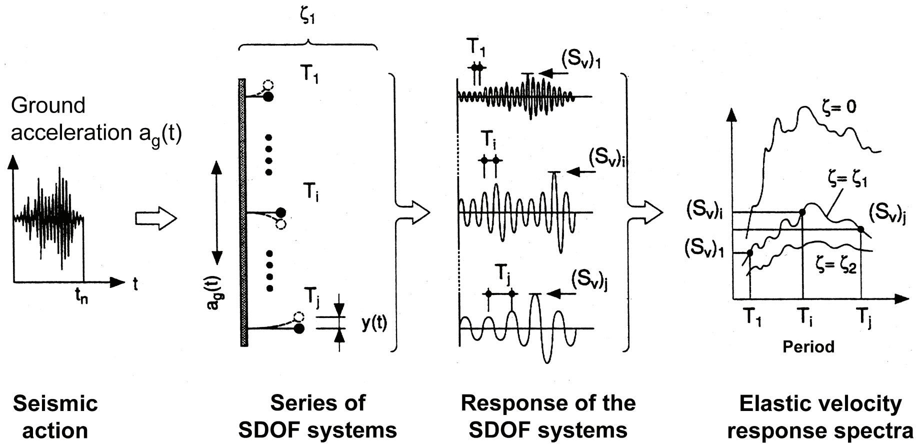
- Stiffness change during loading
- Velocity reversal
- Transition between unloading an reloading.

In first case a secant stiffness can be iteratively computed until the target point lays on the backbone curve, while in the second and third case it is often enough to reduce the size of the time step to limit error. These strategies are shown in the following figure:



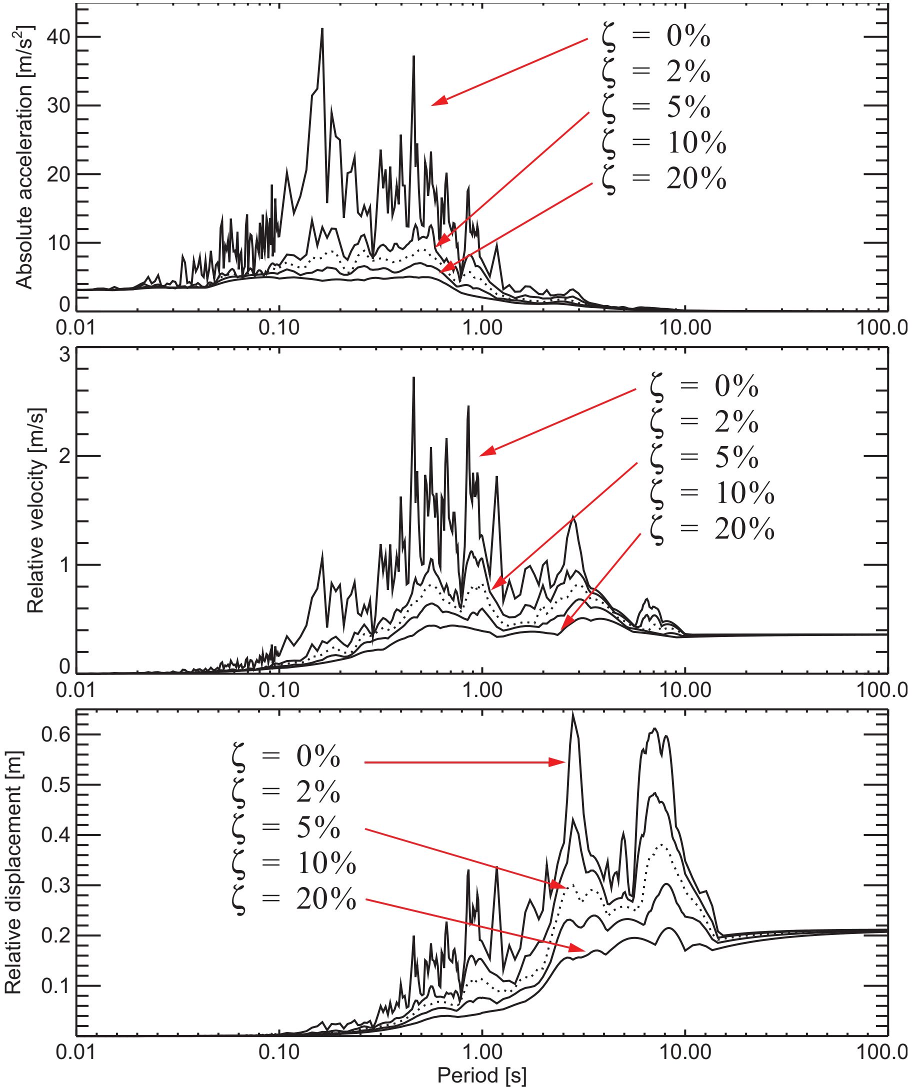
7.5 Elastic response spectra

7.5.1 Computation of response spectra

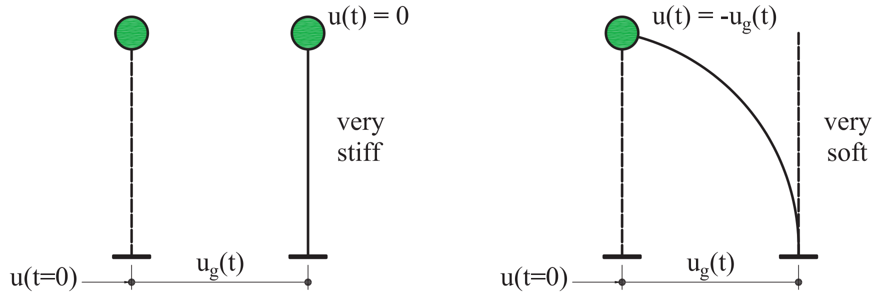


- **Response spectra** are used to represent the seismic demand on structures due to a ground motion record and **design spectra** are used for the seismic design of structures.
- **Response spectra** shall be computed for all periods and damping rates likely to be found in structures.
- Unless specified otherwise, the response spectra presented in the following belongs to the north-south component of the May 18, 1940 "El Centro" Earthquake (see [Cho11]).
- Additional ground motion records can be downloaded for free from:
  1) http://db.cosmos-eq.org/scripts/default.plx
  2) http://peer.berkeley.edu/nga/

- “El Centro”: Linear response spectra

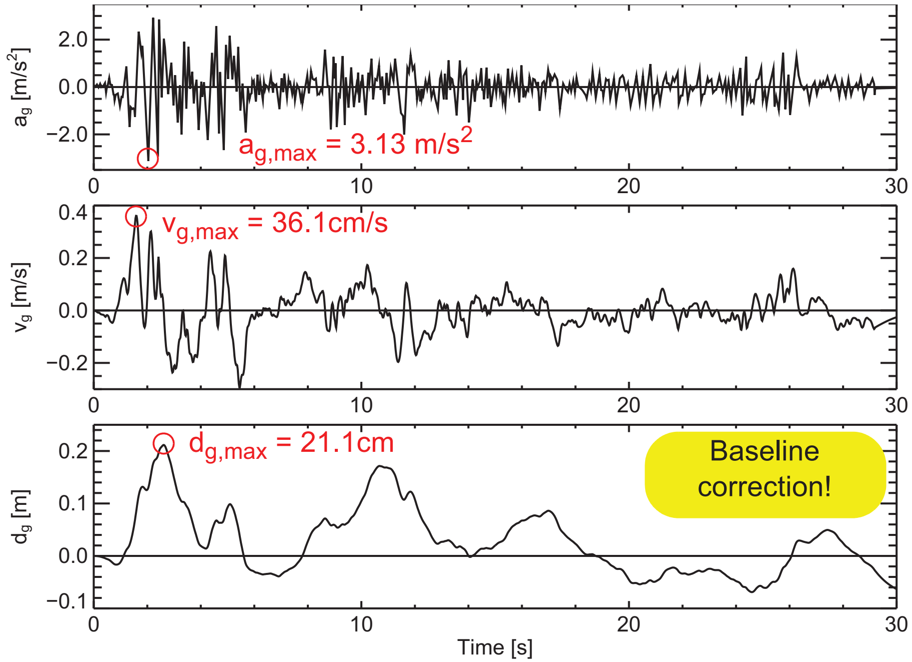


- Limits of response spectra



$$\ddot{u}_a + 2\zeta\omega\dot{u} + \omega^2 u = 0 \qquad \ddot{u}_a + 2\zeta\omega\dot{u} + \omega^2 u = 0$$

$$u = 0 \Rightarrow \ddot{u}_a(t) = \ddot{u}(t) + \ddot{u}_g(t) = \ddot{u}_g(t) \qquad \ddot{u}(t) = -\ddot{u}_g(t) \Rightarrow u(t) = -u_g(t)$$



7.5.2 Pseudo response quantities

- Pseudo-velocity S_{pv}

$$S_{pv} = \omega S_d \tag{7.74}$$

  - S_{pv} has units of a velocity
  - S_{pv} is related to the peak value of the strain energy E_s

$$E_s = \frac{kS_d^2}{2} = \frac{k(S_{pv}/\omega)^2}{2} = \frac{mS_{pv}^2}{2} \tag{7.75}$$

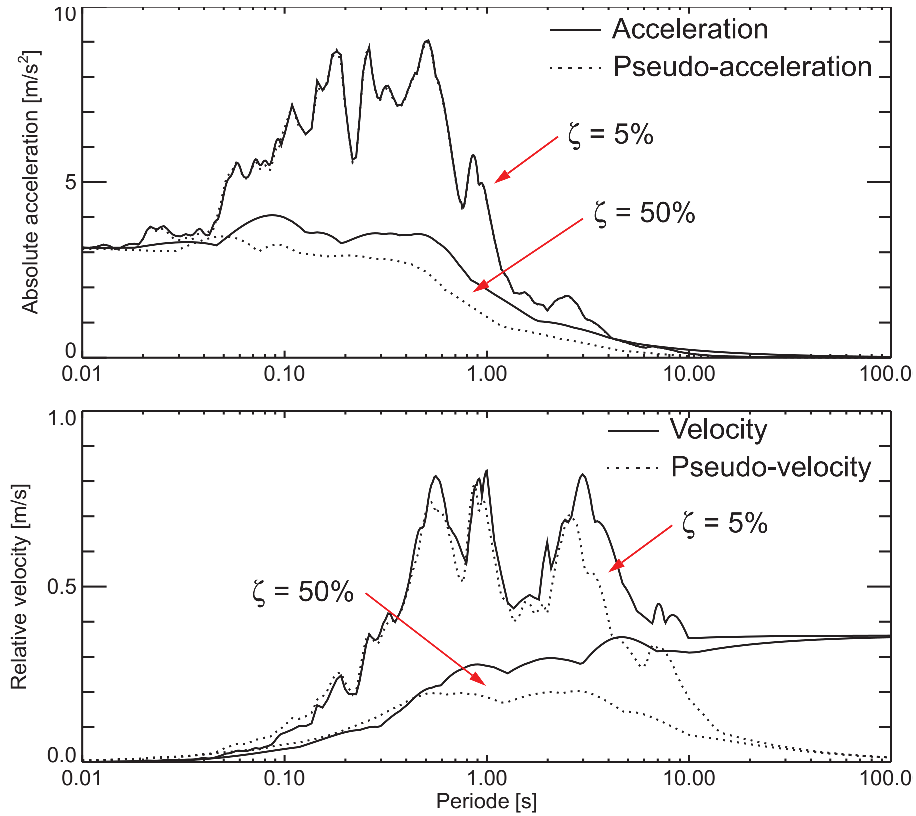
- Pseudo-acceleration

$$S_{pa} = \omega^2 S_d \tag{7.76}$$

  - S_{pa} has units of an acceleration
  - S_{pa} is related to the peak value of the base shear V

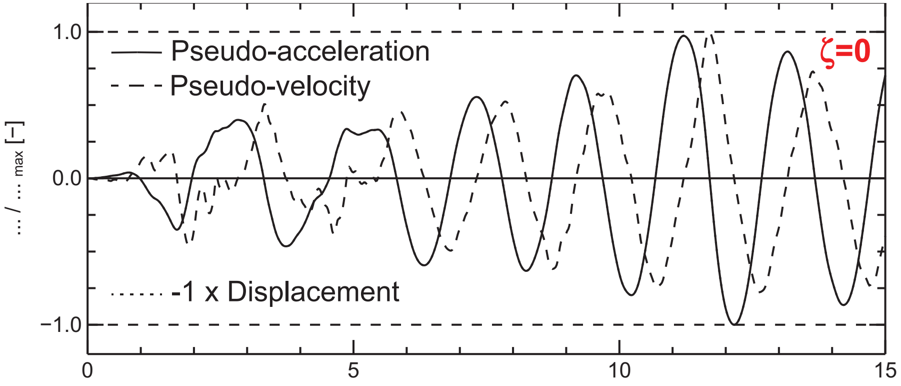
$$V = kS_d = k(S_{pa}/\omega^2) = mS_{pa} \tag{7.77}$$

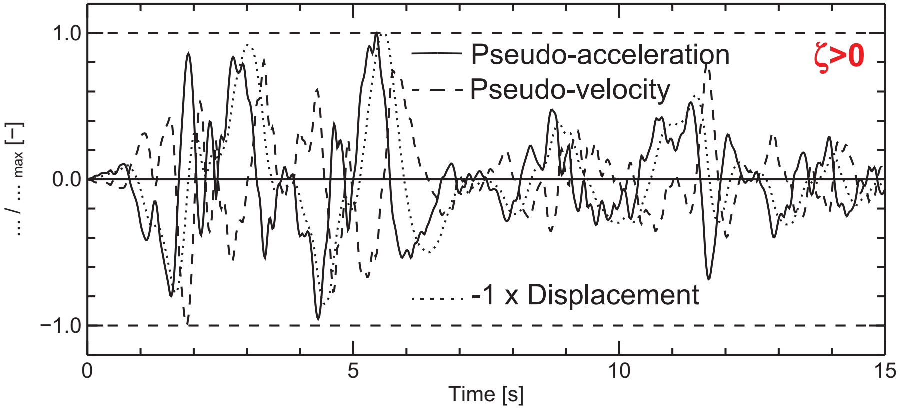
- True vs. Pseudo response quantities



- For $\zeta = 0$ are acceleration and pseudo-acceleration identical.
- For $T \Rightarrow \infty$ the pseudo-velocity tends to zero.
- Pseudo-velocity and pseudo-acceleration match well the true motion of a SDOF system with $\zeta < 20\%$ and $T < 1s$

- Remarks on the Pseudo-acceleration





$$\ddot{u}_a(t) = -\omega^2 u(t) - 2\zeta\omega\dot{u}(t) \qquad (7.78)$$

Time-history of the pseudo-acceleration A(t)

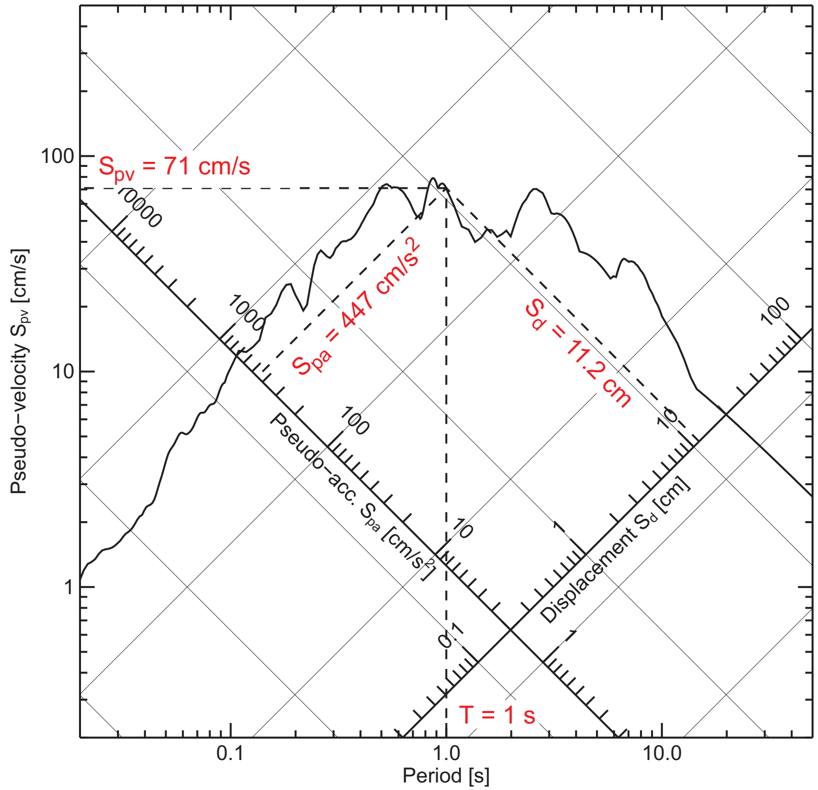
- For $\zeta = 0$: $u(t) = A(t)$
- For $\zeta > 0$: At u_{max}: $u_a = A$ however $A < A_{max}$

  Shift of the location of the maxima through damping

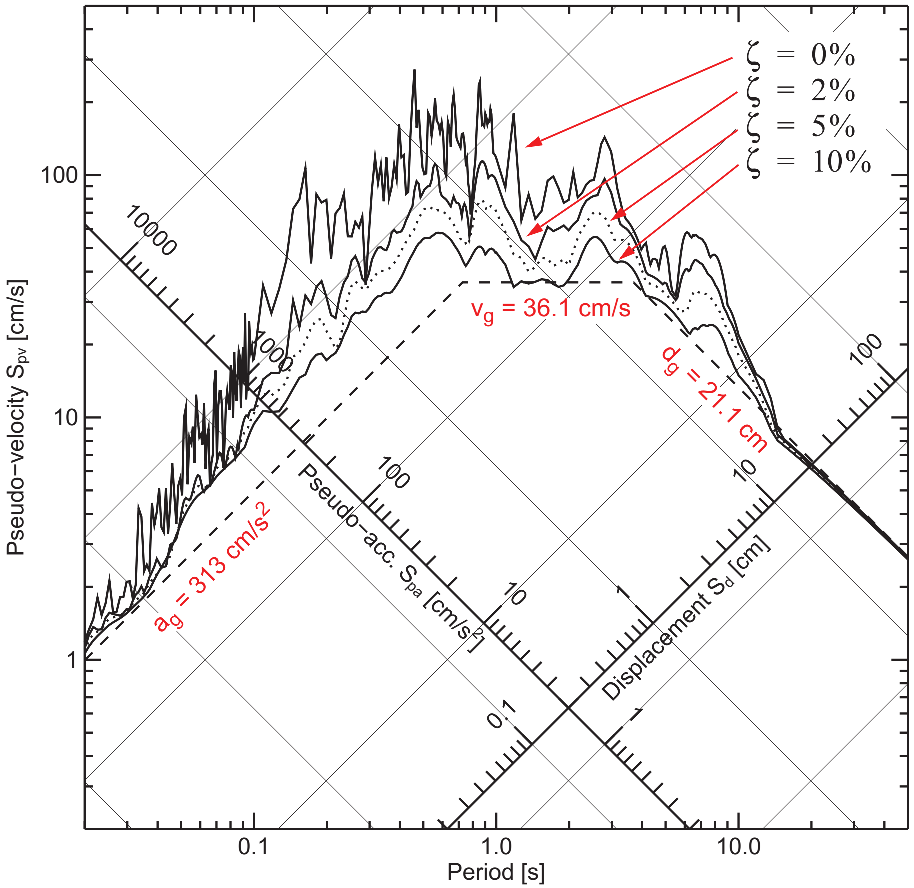
- Combined D-V-A spectra

$$S_{pv} = \omega S_d \qquad \begin{aligned} \log(S_{pv}) &= \log(\omega) + \log(S_d) \\ \log(S_{pv}) &= \log(f) + \log(2\pi) + \log(S_d) \\ \log(S_{pv}) &= -\log(T) + \log(2\pi) + \log(S_d) \end{aligned}$$

$$S_{pv} = \frac{S_{pa}}{\omega} \qquad \begin{aligned} \log(S_{pv}) &= -\log(\omega) + \log(S_{pa}) \\ \log(S_{pv}) &= -\log(f) - \log(2\pi) + \log(S_{pa}) \\ \log(S_{pv}) &= \log(T) - \log(2\pi) + \log(S_{pa}) \end{aligned}$$

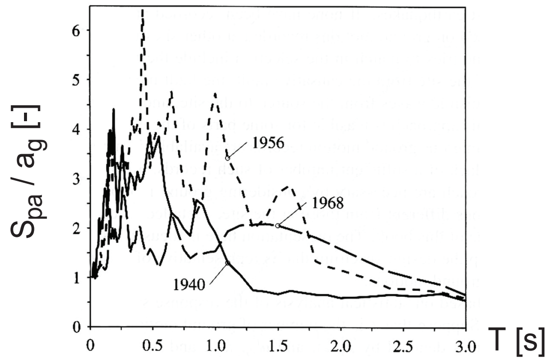


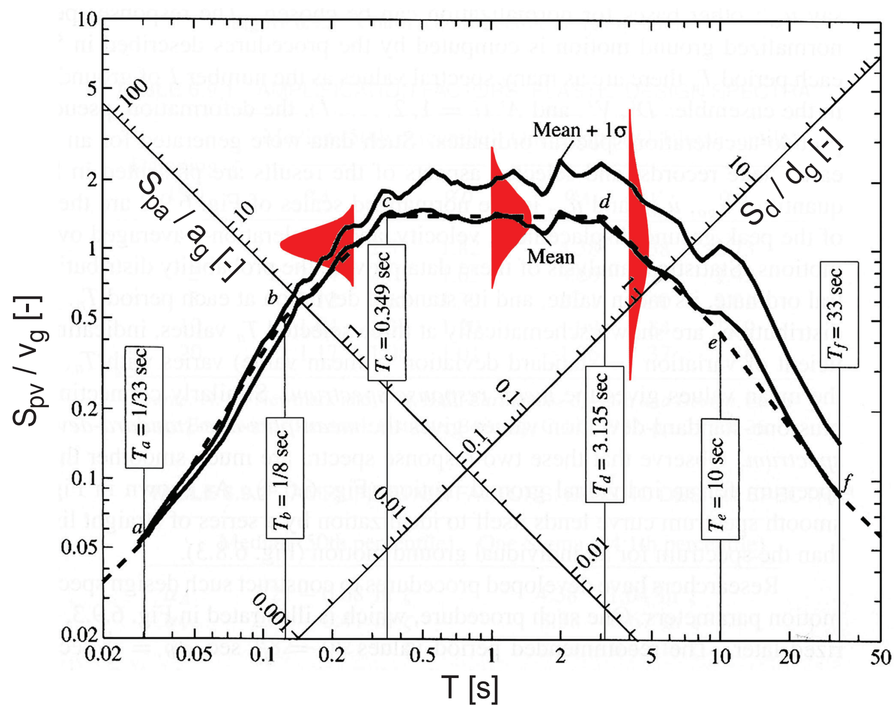
7.5.3 Properties of linear response spectra



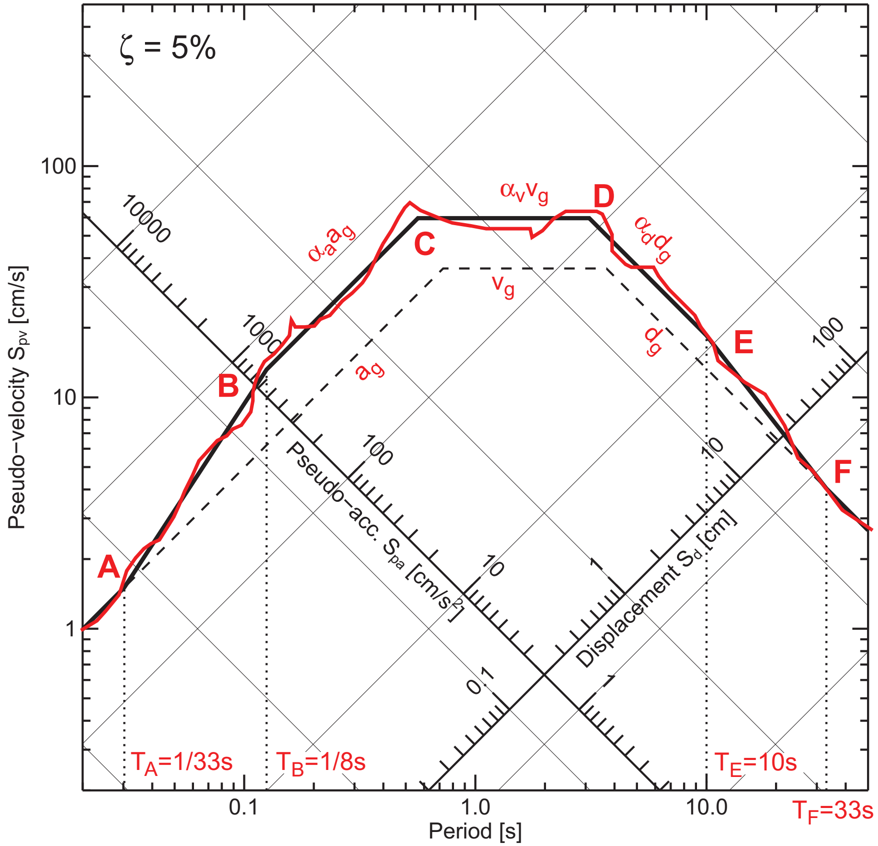
- Response spectra typically show spectral regions where the response is sensitive to different motion quantities, i.e. they show an **acceleration sensitive region** (small periods), a **displacement sensitive region** (large periods) and a **velocity sensitive region** laying in between.

7.5.4 Newmark's elastic design spectra ([Cho11])



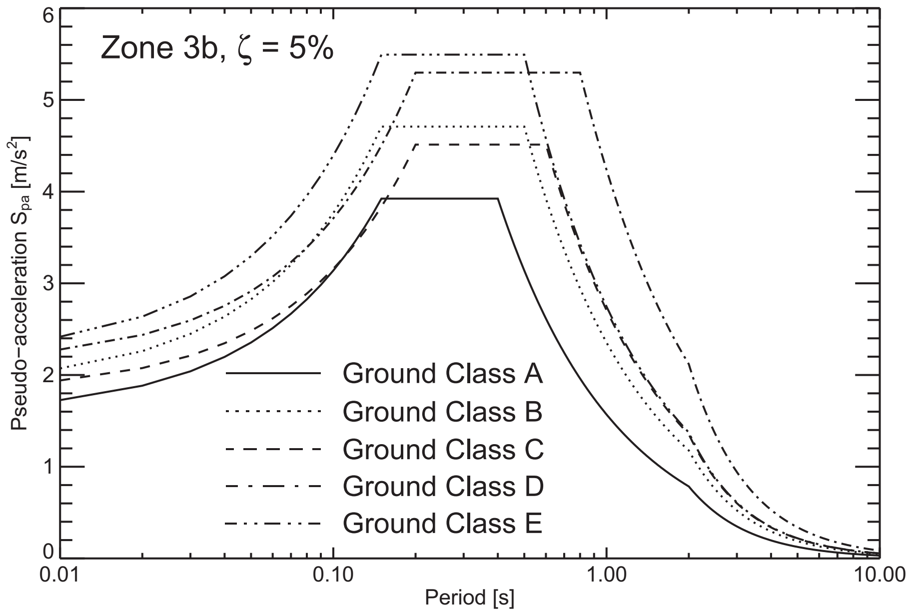


- Newmark's elastic design spectra



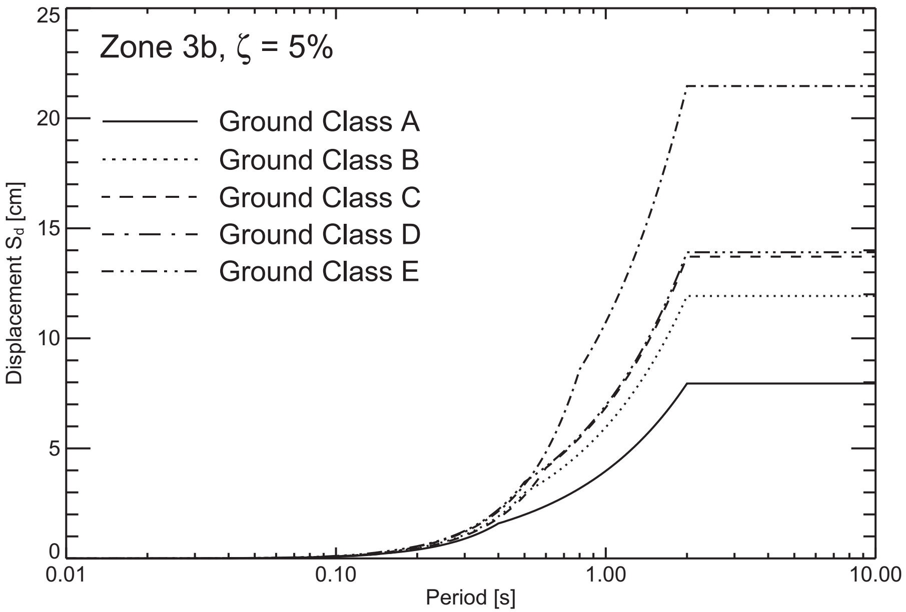
Damping ζ	Median(50%)			One sigma (84%)		
	α_a	α_v	α_d	α_a	α_v	α_d
2%	2.74	2.03	1.63	3.66	2.92	2.42
5%	2.12	1.65	1.39	2.71	2.30	2.01
10%	1.64	1.37	1.20	1.99	1.84	1.69
20%	1.17	1.08	1.01	1.26	1.37	1.38

- Elastic design spectra according to SIA 261 (Art. 16.2.3)



- Ground class **A**: firm or soft rock with a maximum soil cover of 5m
- Ground class **B**: deposit of extensive cemented gravel and sand with a thickness >30m.
- Ground class **C**: deposits of normally consolidated and uncemented gravel and sand with a thickness >30m.
- Ground class **D**: deposits of unconsolidated fine sand, silt and clay with a thickness >30m.
- Ground class **E**: alluvial surface layer of GC C or D, with a thickness of 5 to 30m above a layer of GC A or B.
- Ground class **F**: deposits of structurally-sensitive and organic deposits with a thickness >10m.

- **Displacement** elastic design spectra according to SIA 261

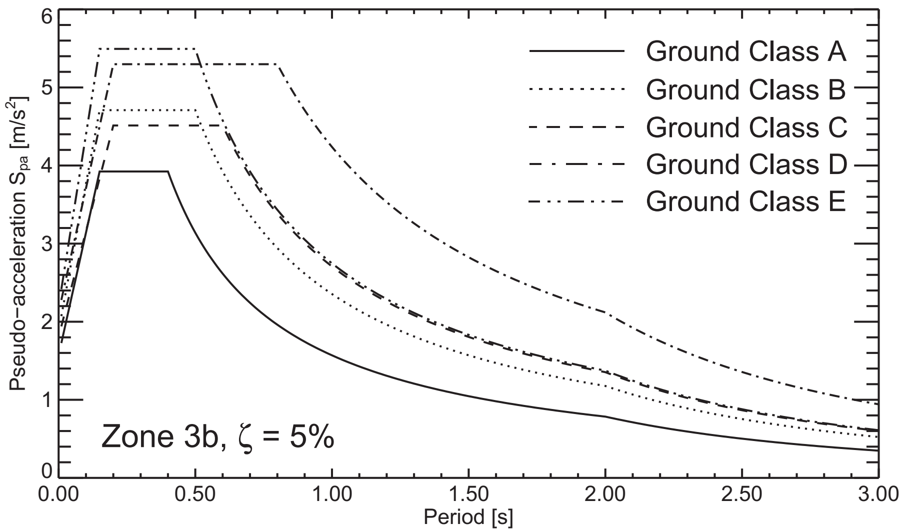


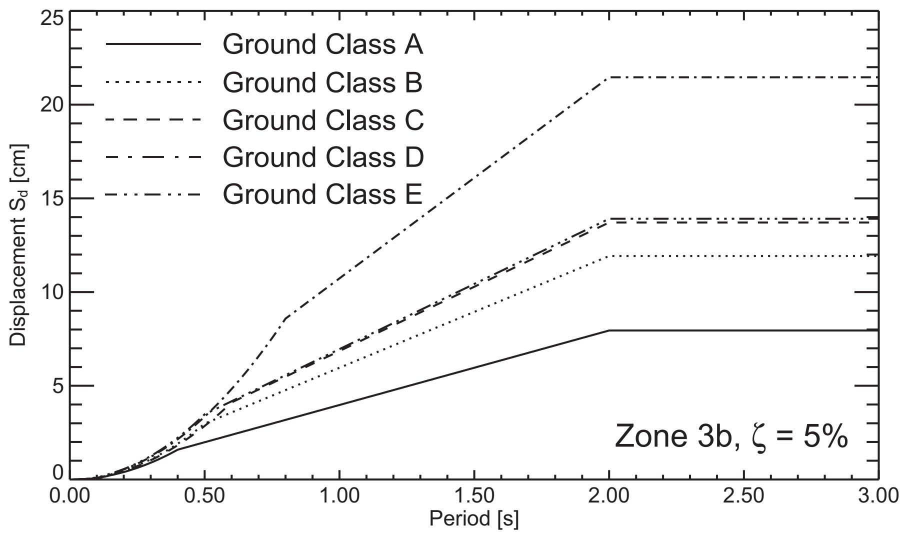
- The displacement spectra are computed from the acceleration spectra using equation (7.79)

$$S_d = \frac{S_{pa}}{\omega^2} \qquad (7.79)$$

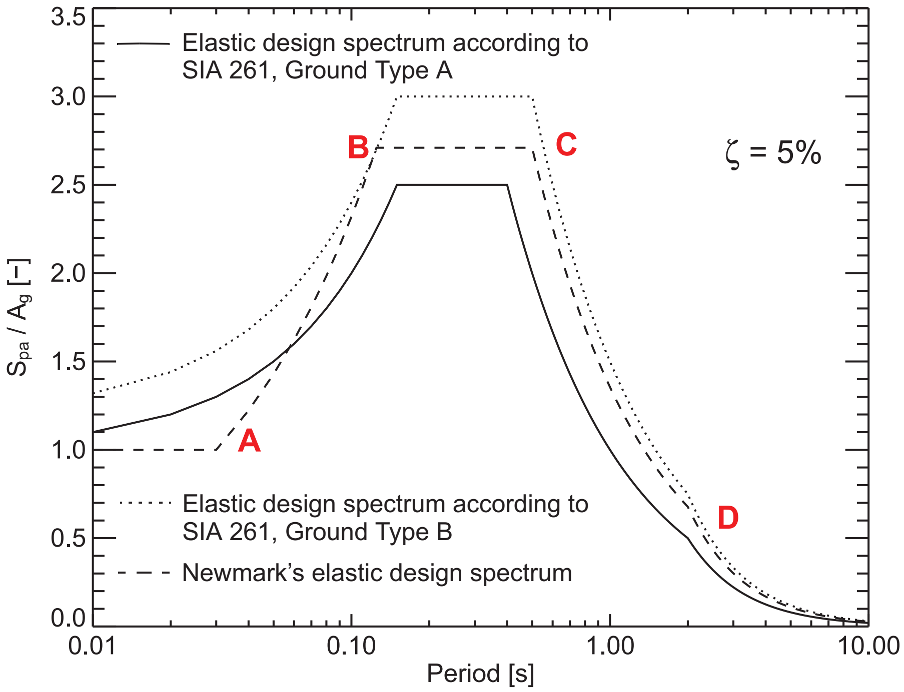
- Displacement spectra are an important design tool (even within force-based design procedures) because they allow a quick estimate of the expected deformations, hence of the expected damage.

- Elastic design spectra according to SIA 261 (linear)





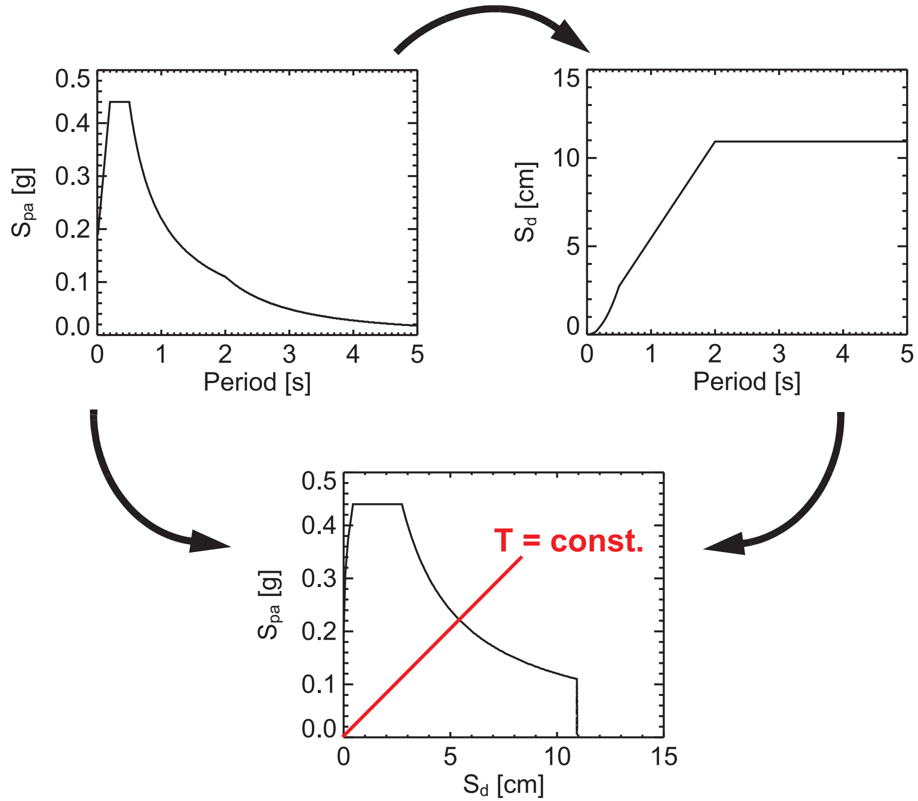
- Elastic design spectra: Newmark vs. SIA 261



- The SIA 261 spectra, like the spectra of the majority of the standards worldwide, were defined using the same principles as Newmark's spectra.
- However, different ground motion were used:
  - SIA 261 takes into account different ground classes;
  - Different seismic sources were considered;
  - A larger number of ground motions was considered.
- Note: in SIA 261 the corner period T_A is not defined.

7.5.5 Elastic design spectra in ADRS-format (e.g. [Faj99]) (Acceleration-Displacement-Response Spectra)

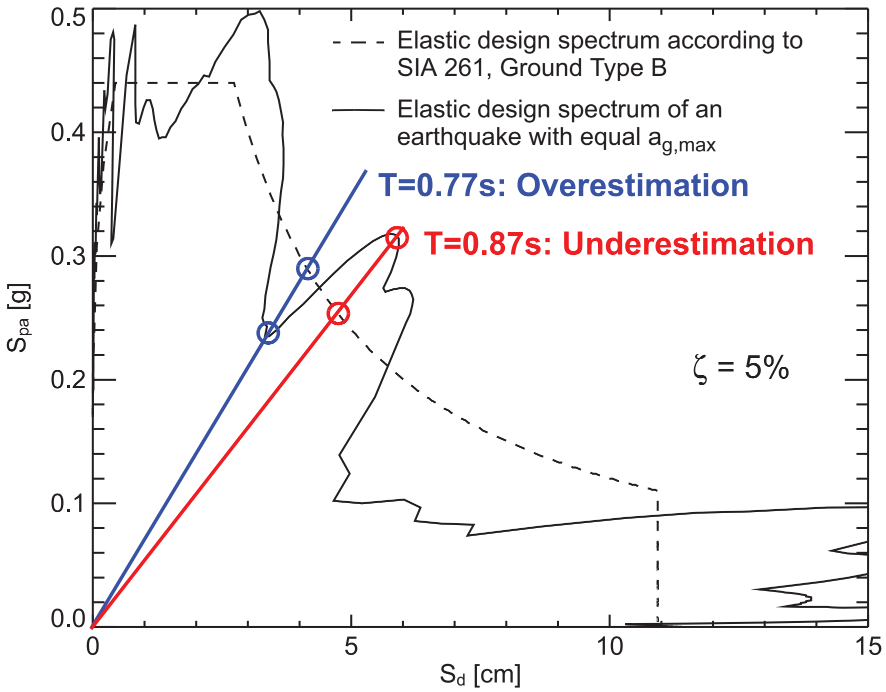
$$S_{pa} = \omega^2 S_d$$



Periods T correspond to lines running through the origin of the axes, because:

$$S_{pa} = \omega^2 S_d \quad \text{and after reorganizing:} \quad T = 2\pi\sqrt{S_d / S_{pa}}$$

- Elastic design spectra in ADRS-format

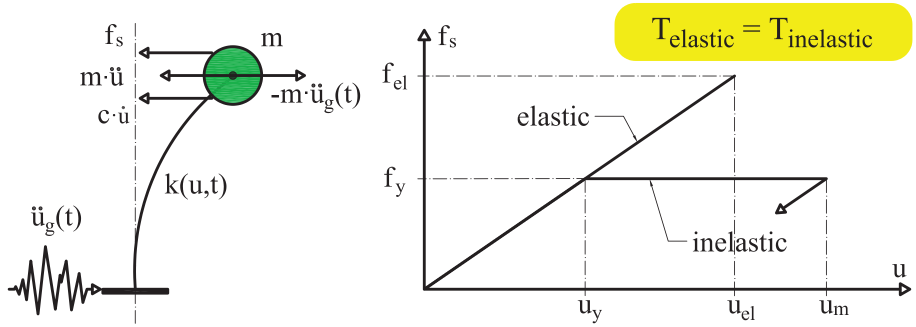


- Design spectra are defined based on averaged response spectra. For this reason, the spectral values of single response spectra may differ significantly from the design spectra.
- **This is a crucial property of design spectra and should be kept in mind during design!**

7.6 Strength and Ductility

7.6.1 Illustrative example

Comparison of the time history analyses of an elastic and an inelastic single-degree-of-freedom system (SDOF system):



Where:

$$R_y = \frac{f_{el}}{f_y}: \text{ Force reduction factor} \tag{7.80}$$

f_{el}: Maximum restoring force that the elastic SDOF system reaches over the course of the seismic excitation $\ddot{u}_g(t)$ (7.81)

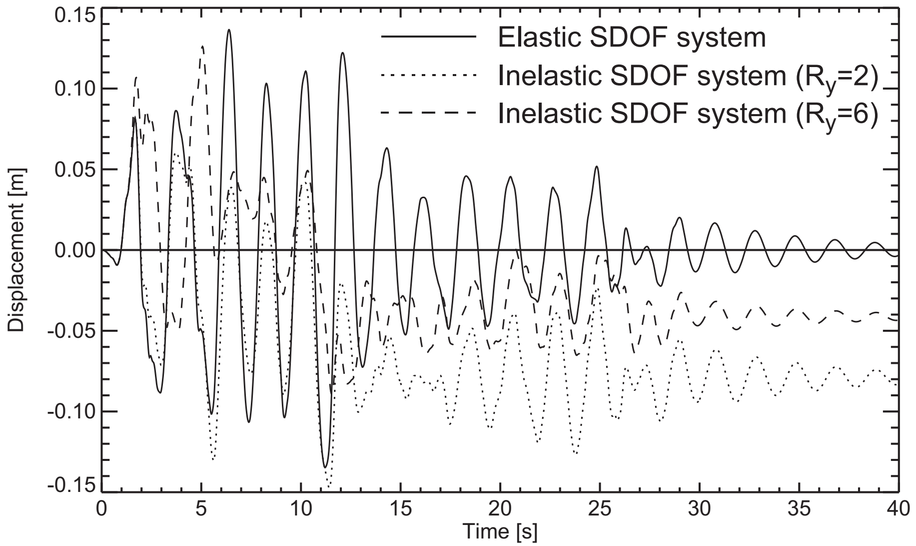
f_y: Yield force of the inelastic SDOF system (7.82)

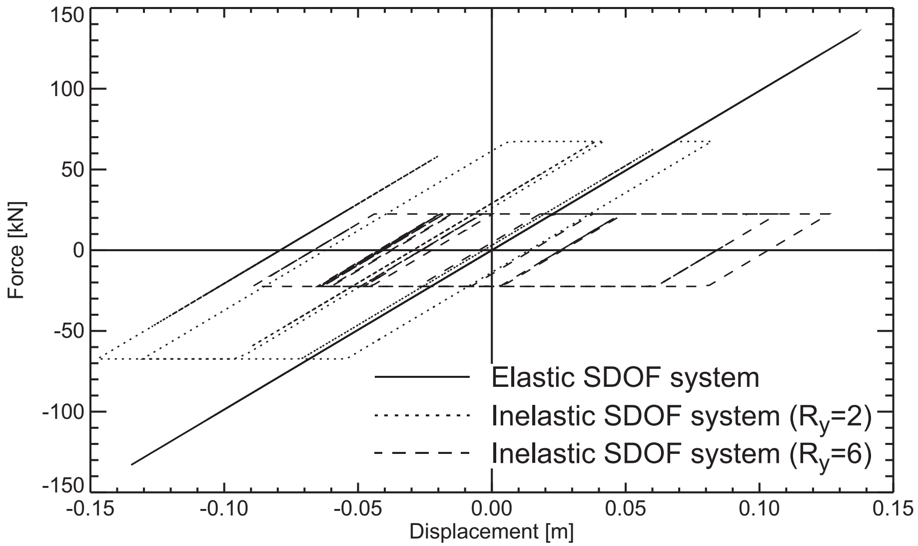
$$\mu_\Delta = \frac{u_m}{u_y}: \text{ Displacement ductility} \tag{7.83}$$

u_m: Maximum displacement that the inelastic SDOF system reaches over the course of the seismic excitation $\ddot{u}_g(t)$ (7.84)

u_y: Yield displacement of the inelastic SDOF system (7.85)

• Results





Quantity	Elastic SDOF	Inela. SDOF R_y=2	Inela. SDOF R_y=6
T [s]	2.0	2.0	2.0
F_{max} [kN]	134.70	67.35	22.45
R_y [-]	–	2.0	6.0
u_y [m]	–	0.068	0.023
u_m [m]	0.136	0.147	0.126
μ_Δ [-]	–	2.16	5.54

• Comments

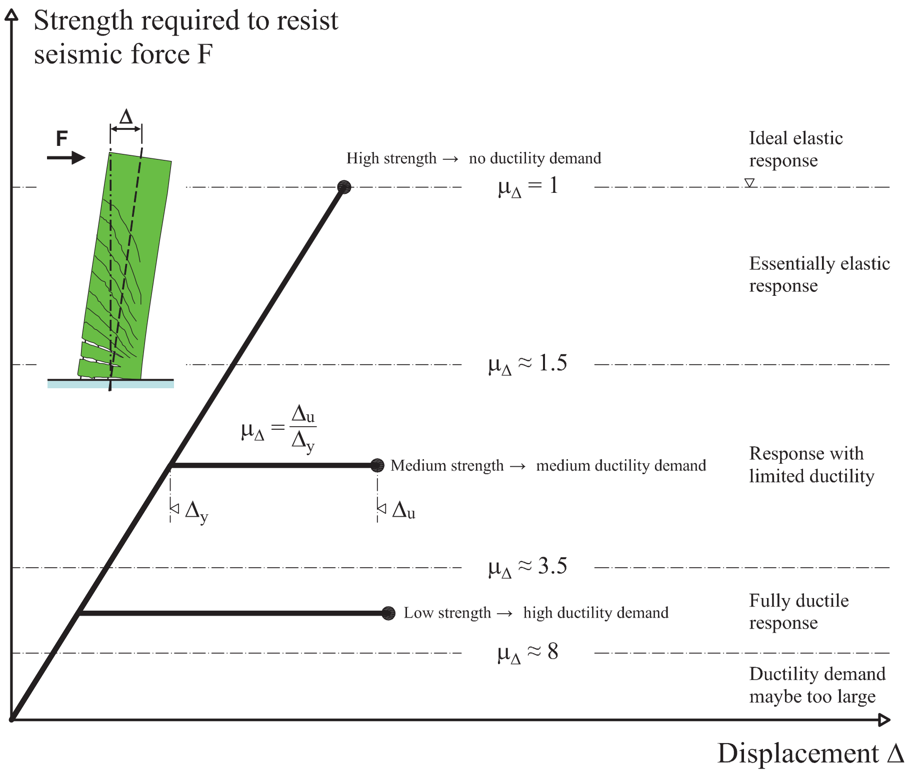
  • Both inelastic SDOF systems show a stable seismic response.

7.6.2 "Seismic behaviour equation"

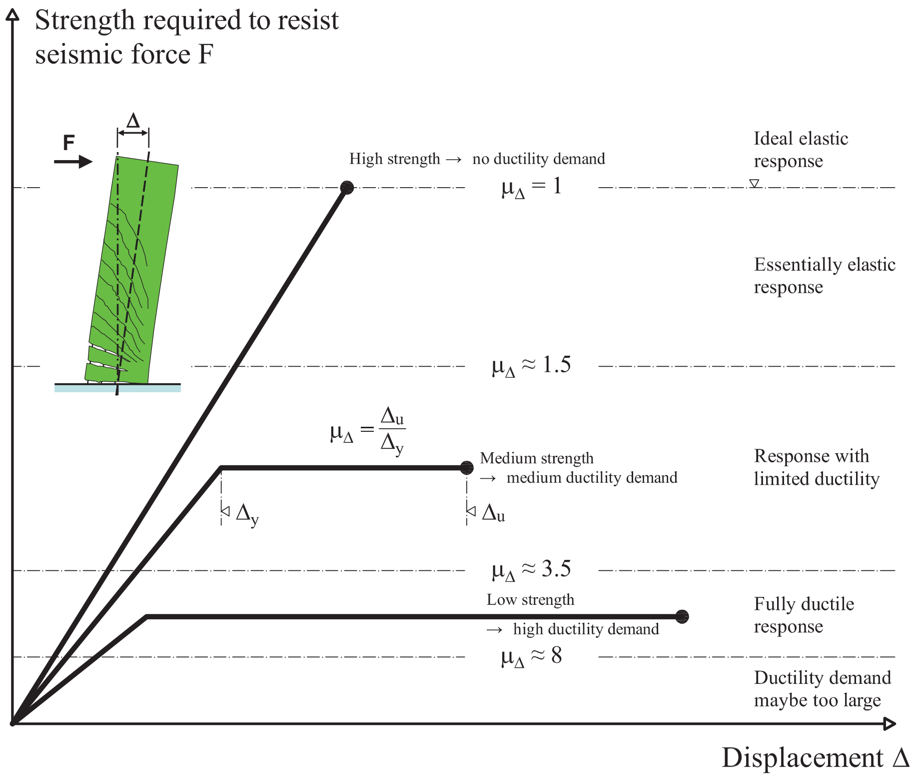
For seismic collapse prevention, the following approximate relationship applies

$$\text{"quality" of seismic behaviour} \approx \text{strength} \times \text{ductility} \qquad (7.86)$$

To survive an earthquake different combination of strength and ductility are possible:

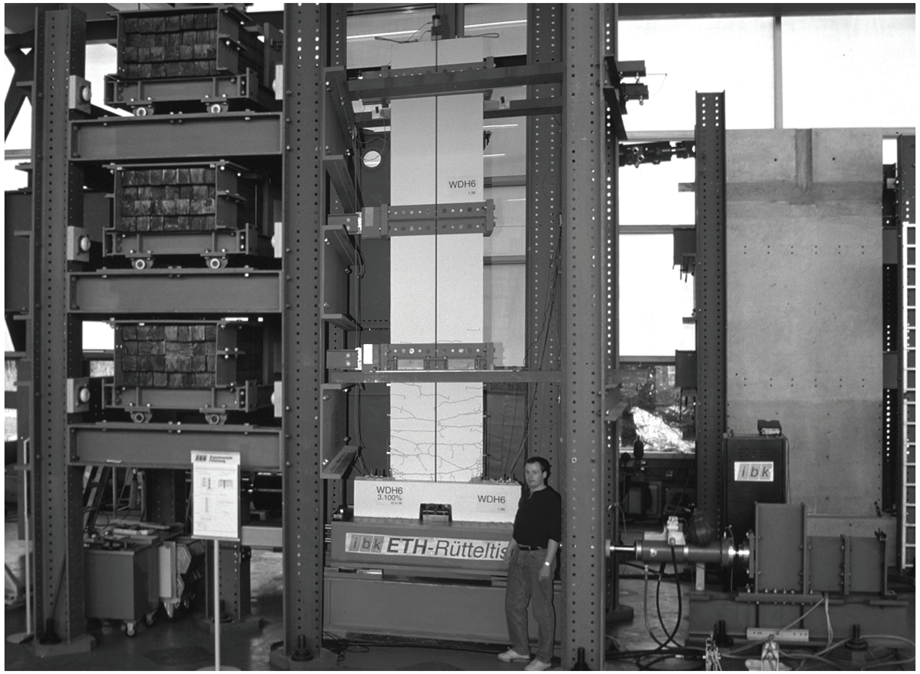


- More realistic representation of the decision possibilities

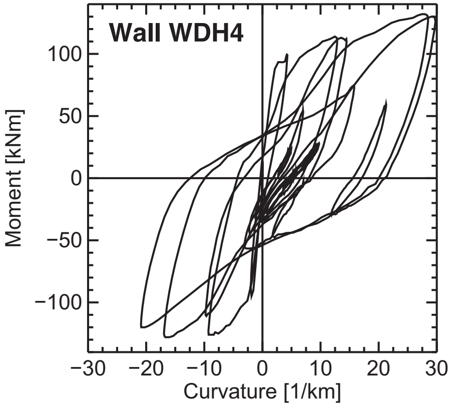


- If the strength of the structure reduces, the stiffness typically reduces too.
- If the masses do not change significantly (which is typically the case), the fundamental period T of the softer structure is longer.
- Structures with a longer fundamental period T are typically subjected to larger deformations, i.e., the deformation demand is larger.

7.6.3 Inelastic behaviour of a RC wall during an earthquake



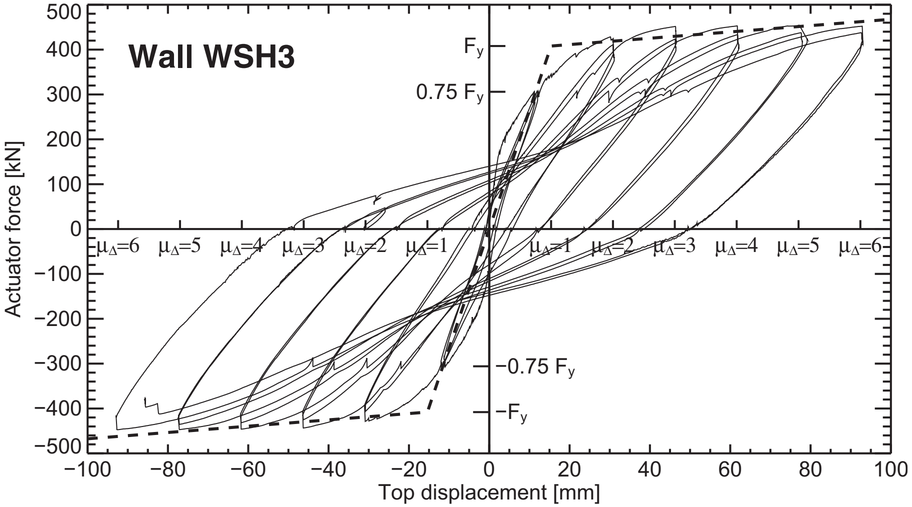
- Moment-curvature-relationship at the base of the plastic hinge zone.
- Despite reaching and exceeding its elastic limit the wall did not collapse.
- The plastic deformation capacity of structures can really be taken into account for seismic design purposes.

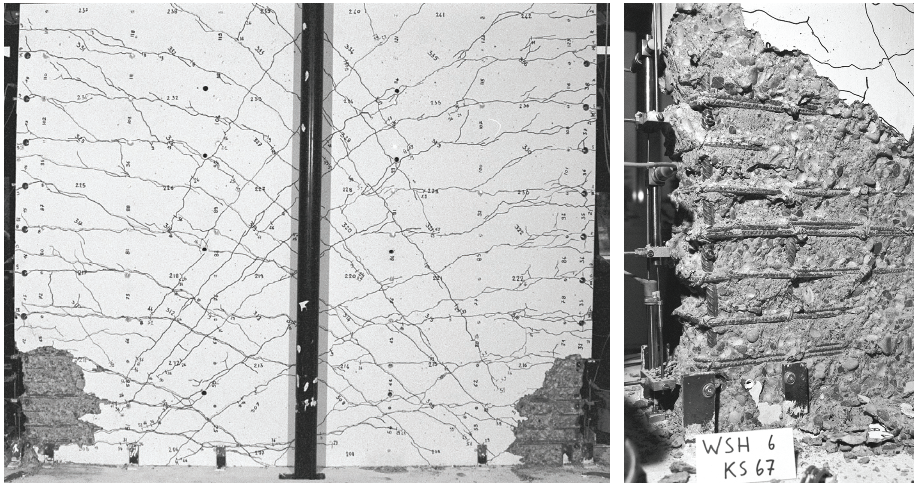


7.6.4 Static-cyclic behaviour of a RC wall

Wall WSH6 [DWB99]

- Hysteretic behaviour of the RC wall under static-cyclic loading



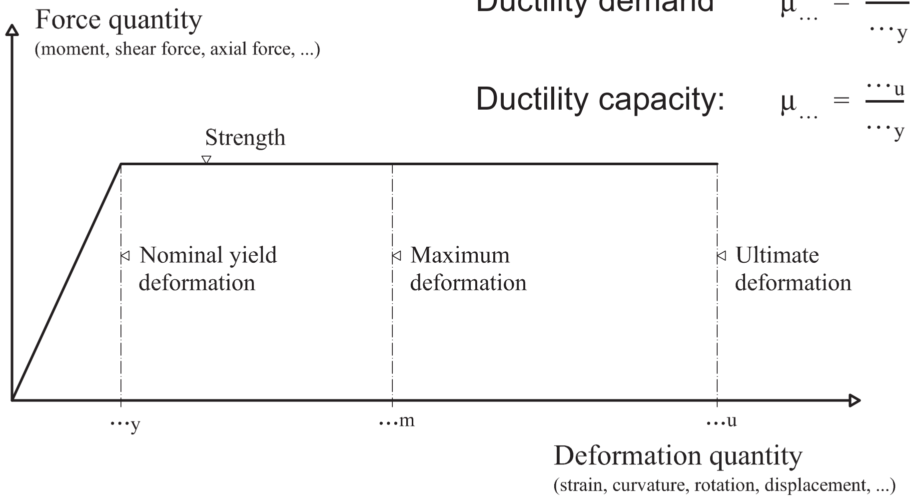


Plastic region of test unit WSH6 (left) and close-up of the left boundary region (right). Both photos were taken at displacement ductility 6.

7.6.5 General definition of ductility

Ductility demand $\mu_{\dots} = \frac{\dots_m}{\dots_y}$

Ductility capacity: $\mu_{\dots} = \frac{\dots_u}{\dots_y}$



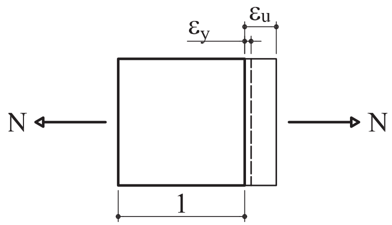
- Comments
  - The **ductility capacity** is a property of the structural member.
  - The **ductility demand** is a result of the seismic excitation and also a function of the dynamic properties of the structure.
  - A structural member survives the earthquake if:

$$\text{Ductility capacity} \geq \text{Ductility demand}$$

  - The structural member fractures when locally the deformation capacity of the structural materials (i.e., their strain capacities) are reached. The ductility capacity is therefore exhausted.

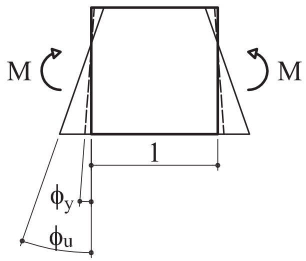
7.6.6 Types of ductilities

strain ductility



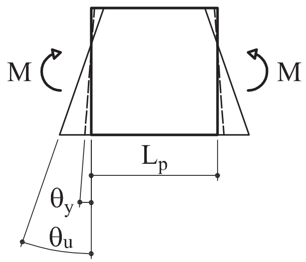
$$\mu_\varepsilon = \frac{\varepsilon_u}{\varepsilon_y}$$

curvature ductility



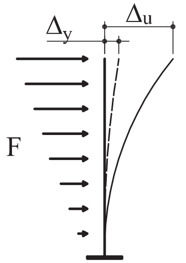
$$\mu_\phi = \frac{\phi_u}{\phi_y}$$

rotation ductility



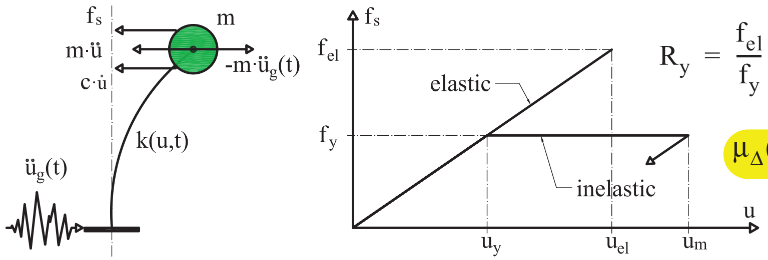
$$\mu_\theta = \frac{\theta_u}{\theta_y}$$

displacement ductility



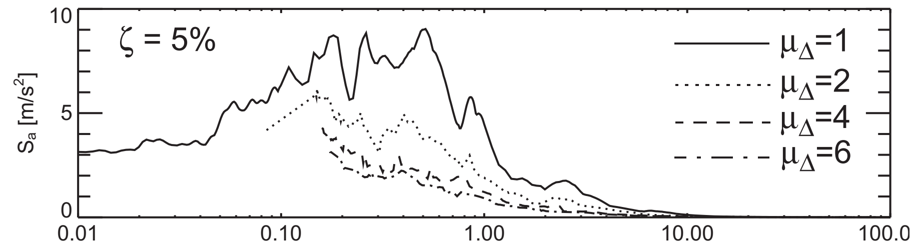
$$\mu_\Delta = \frac{\Delta_u}{\Delta_y}$$

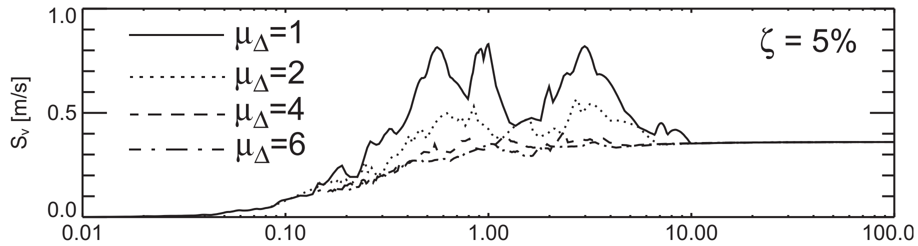
7.7 Inelastic response spectra

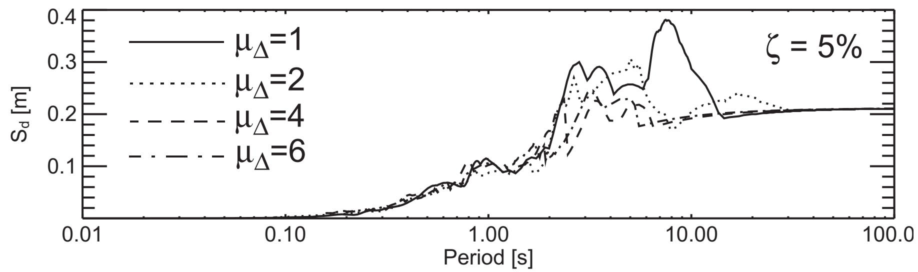


$$R_y = \frac{f_{el}}{f_y}\ ,\ \mu_\Delta = \frac{u_m}{u_y}$$

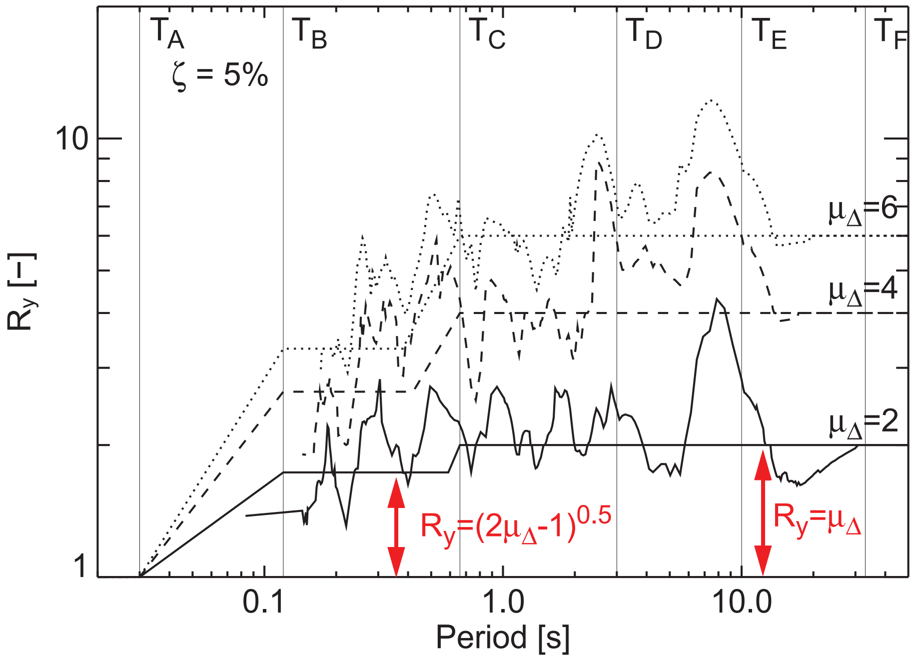
$$\mu_\Delta(R_y) = ?$$



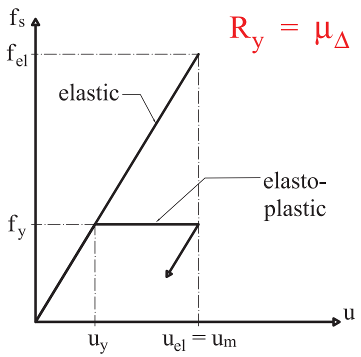




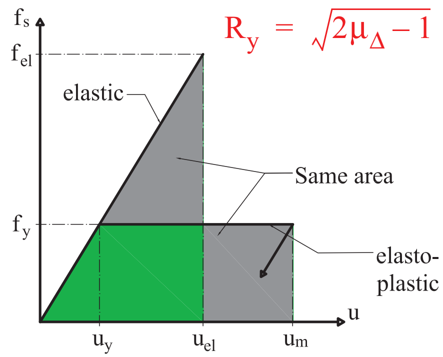
Force reduction factor R_y



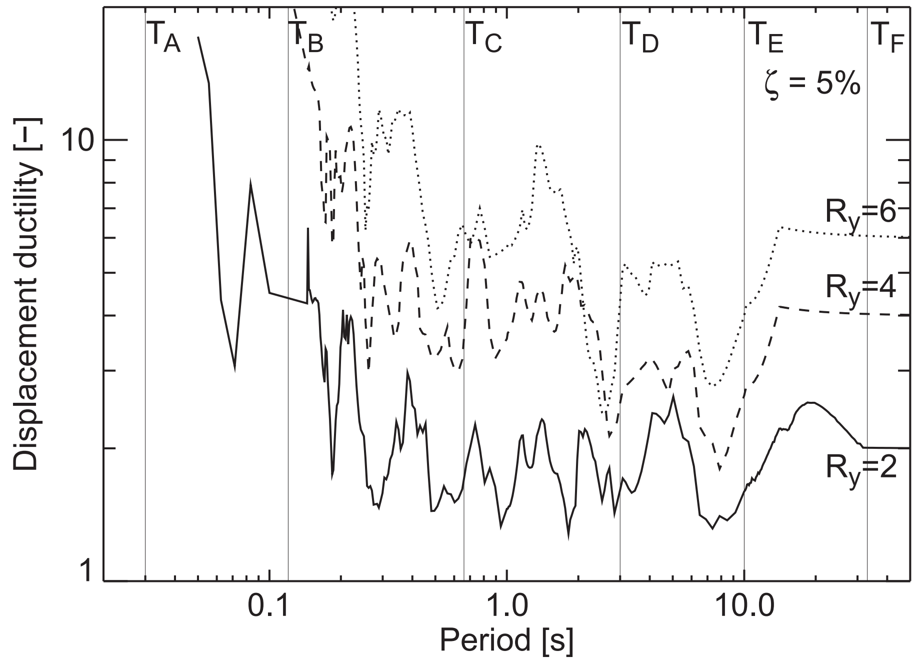
Equal displacement principle



Equal energy principle



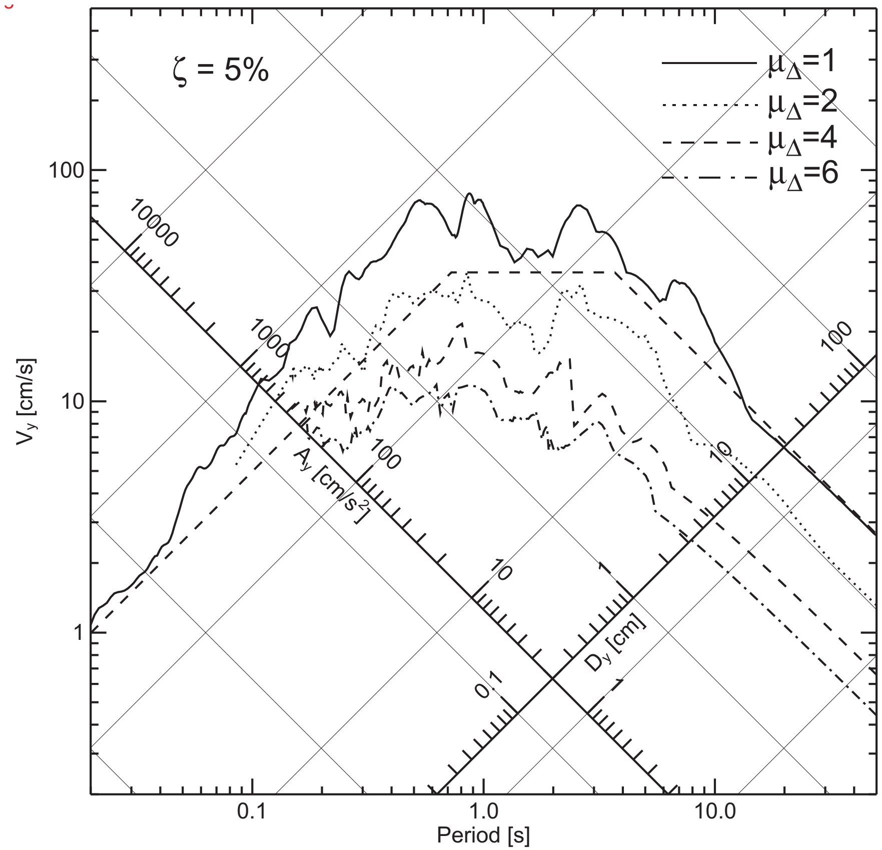
Displacement ductility μ_Δ



- In the small period range, already small reductions of the elastic strength of the SDOF system yield very large ductility demands.
- If the ductility demand is very large, it can be difficult to provide the structure with a sufficiently large ductility capacity. This problem will be further discussed during the design classes.
- Also in the large period range – where the “equal displacement principle” applies – large discrepancies between real and estimated ductility demand can occur.
- The “equal displacement principle” and the “equal energy principle” are “historical” R_y-μ_Δ-T_n relationships. In recent years a lot of research has been done to come up with more accurate formulations (see e.g. works by Krawinkler [KN92], Fajfar [VFF94], Miranda [Mir01], ...)

7.7.1 Inelastic design spectra

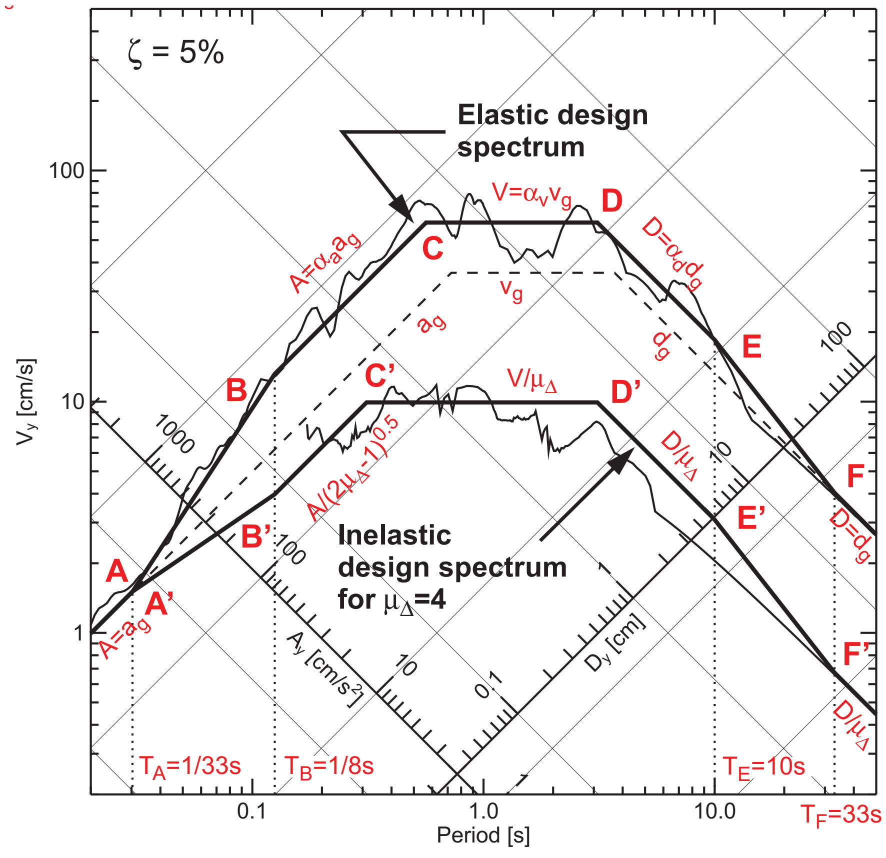
- Inelastic design spectra in combined D-V-A format



Note the new axes: $D_y = u_y$, $V_y = \omega_n u_y$, $A_y = \omega_n^2 u_y$

where: u_y = yield displacement

- Newmark's inelastic design spectra [NH82]



- Maximum displacement of the SDOF system

$$u_m = \mu_\Delta \cdot D_y$$

- Yield strength of the SDOF system:

$$f_y = m \cdot A_y$$

Construction of the spectra using R_y-μ_Δ-T_n relationships

The inelastic design spectra are computed by means of R_y-μ_Δ-T_n relationships:

$$A_y = S_{pa,\ inelastic} = \frac{1}{R_y} \cdot S_{pa,\ elastic} \tag{7.87}$$

$$D = S_{d,\ inelastic} = \frac{\mu_\Delta}{R_y} \cdot S_{d,\ elastic} \tag{7.88}$$

It should be noted that:

$$S_{pa,\ inelastic} \neq \omega^2 \cdot S_{d,\ inelastic} \tag{7.89}$$

- R_y-μ_Δ-T_n relationship according to [NH82]

$$R_y = \begin{cases} 1 & T_n < T_a \\ (2\mu_\Delta - 1)^{\beta/2} & T_a < T_n < T_b \\ \sqrt{2\mu_\Delta - 1} & T_b < T_n < T_{c'} \quad \text{(EE principle)} \\ \dfrac{T_n}{T_c}\mu_\Delta & T_{c'} < T_n < T_c \\ \mu_\Delta & T_n > T_c \quad \text{(ED principle)} \end{cases} \tag{7.90}$$

Where:

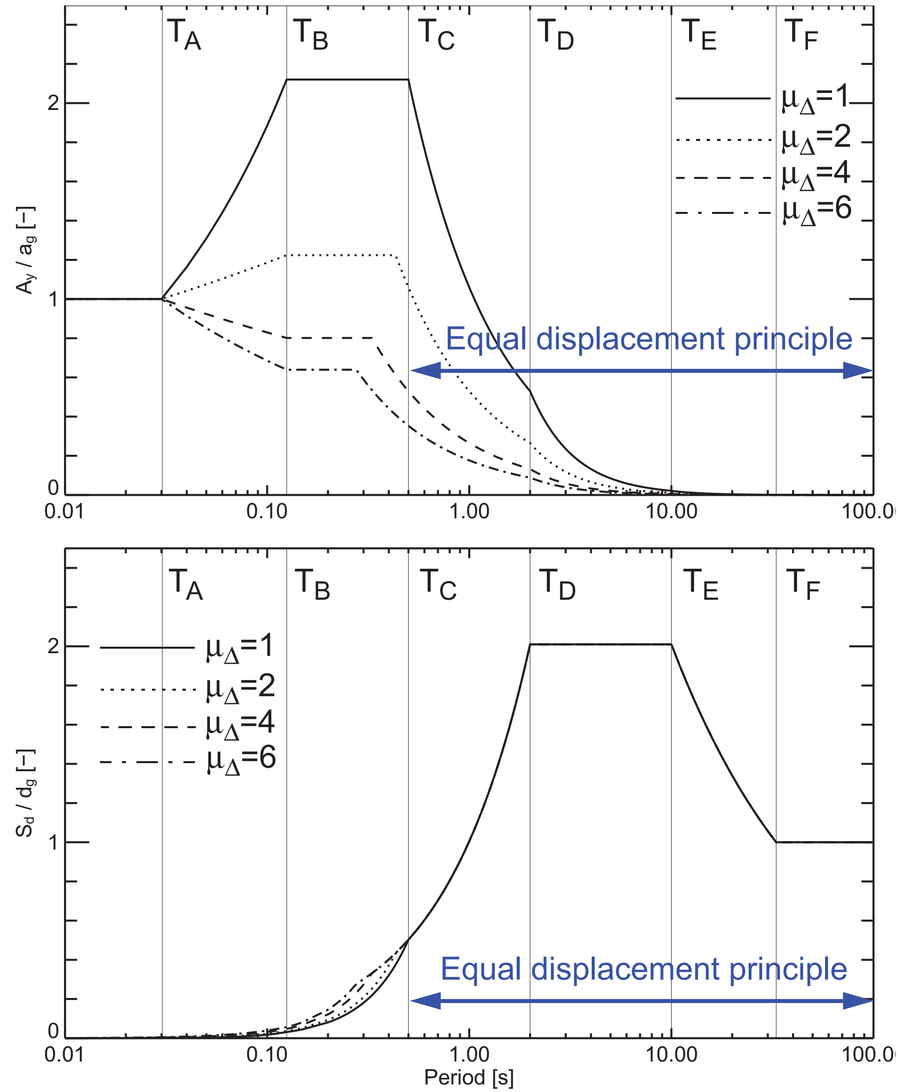
$$\beta = \log(T_n/T_a)/\log(T_b/T_a) \tag{7.91}$$

$$T_a = 1/33\,s,\ T_b = 1/8\,s \tag{7.92}$$

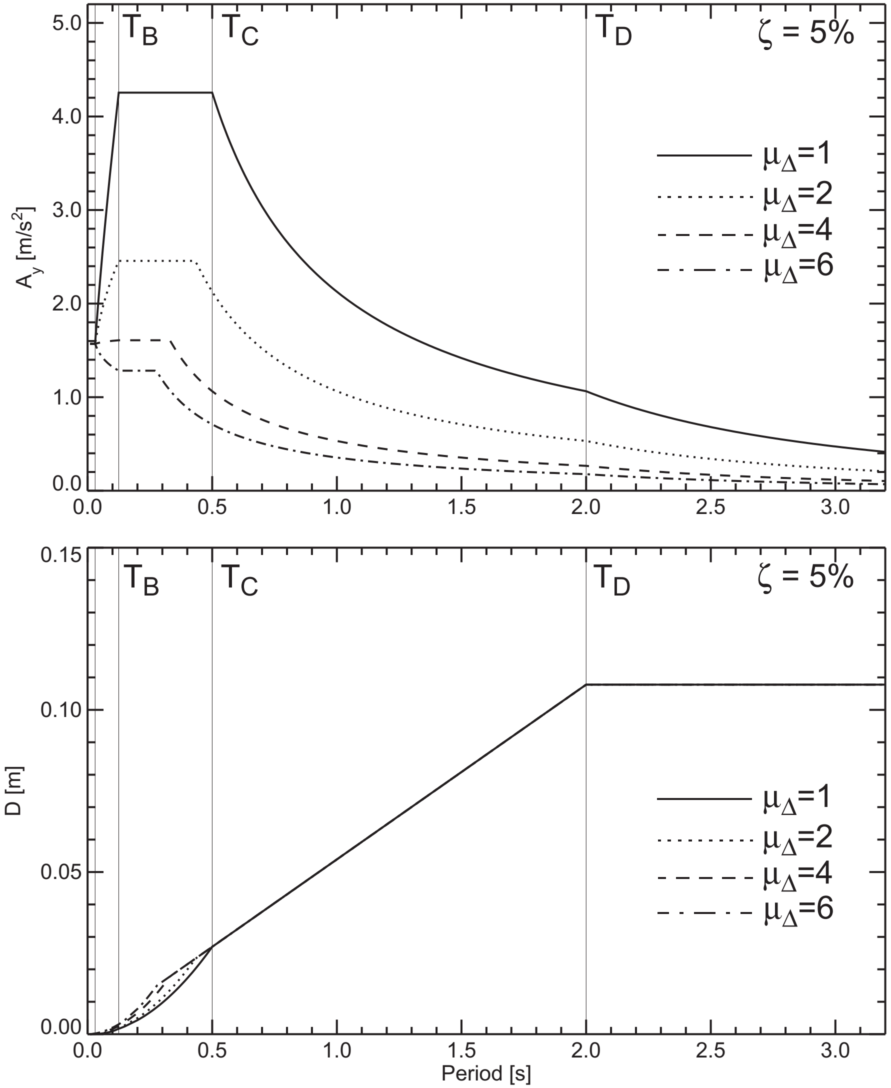
T_c = Corner period between the constant S_{pa} and the constant S_{pv} regions

$T_{c'}$ = Corner period between the constant S_{pa} and the constant S_{pv} regions of the inelastic spectrum

- Inelastic design spectra according to [NH82] (log. x-axis)



- Inelastic design spectra according to [NH82] (linear x-axis)



- R_y-μ_Δ-T_n relationships according to [VFF94]

In [VFF94] R_y-μ_Δ-T_n relationships are defined as follows:

$$R_y = \begin{cases} c_1(\mu_\Delta - 1)^{c_R} \cdot \dfrac{T_n}{T_0} + 1 & T_n \le T_0 \\ c_1(\mu_\Delta - 1)^{c_R} + 1 & T_n > T_0 \end{cases} \tag{7.93}$$

Where: $T_0 = c_2 \cdot \mu_\Delta^{c_T} \cdot T_c \le T_c$ (7.94)

T_c = Corner period between the constant S_{pa} and the constant S_{pv} regions

The parameters c_1, c_2, c_R and c_T are defined as follows for 5% damping:

Model					
Hysteresis	Damping	c_1	c_R	c_2	c_T
Q	Mass	1.0	1.0	0.65	0.30
Q	Tangent stiffness	0.75	1.0	0.65	0.30
Bilinear	Mass	1.35	0.95	0.75	0.20
Bilinear	Tangent stiffness	1.10	0.95	075	0.20

and where the Q-hysteretic rule is a stiffness degrading rule similar to the Takeda-hysteretic rule presented in Section 7.3.2.

The table shows the dependency of the R_y-μ_Δ-T_n relationships both on damping and hysteretic model.

For the Q-hysteretic model and mass-proportional damping, the R_y-μ_Δ-T_n relationships by [VFF94] specialise as:

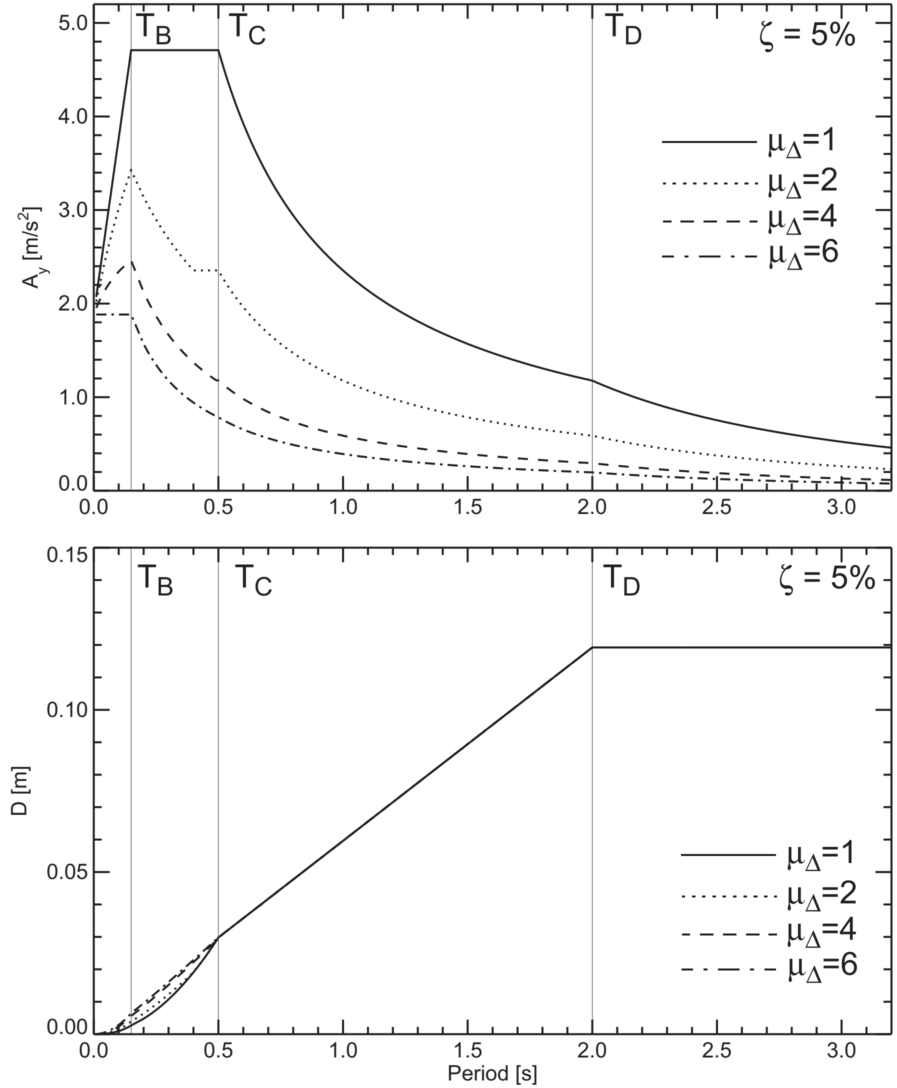
$$R_y = \begin{cases} (\mu_\Delta - 1) \cdot \dfrac{T_n}{T_0} + 1 & T_n \le T_0 \\ \mu_\Delta & T_n > T_0 \quad \text{(ED principle)} \end{cases} \tag{7.95}$$

Where: $T_0 = 0.65 \cdot \mu_\Delta^{0.3} \cdot T_c \le T_c$ (7.96)

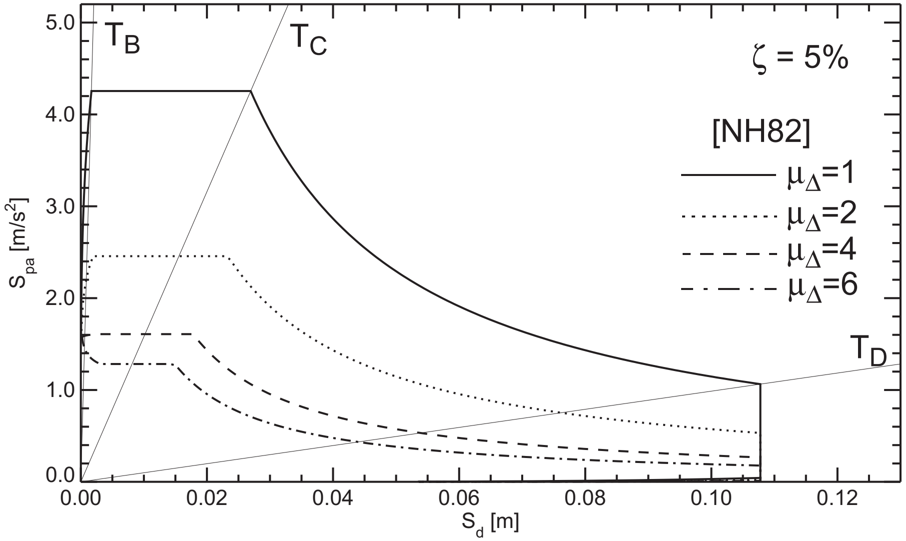
T_c = Corner period between the constant S_{pa} and the constant S_{pv} regions

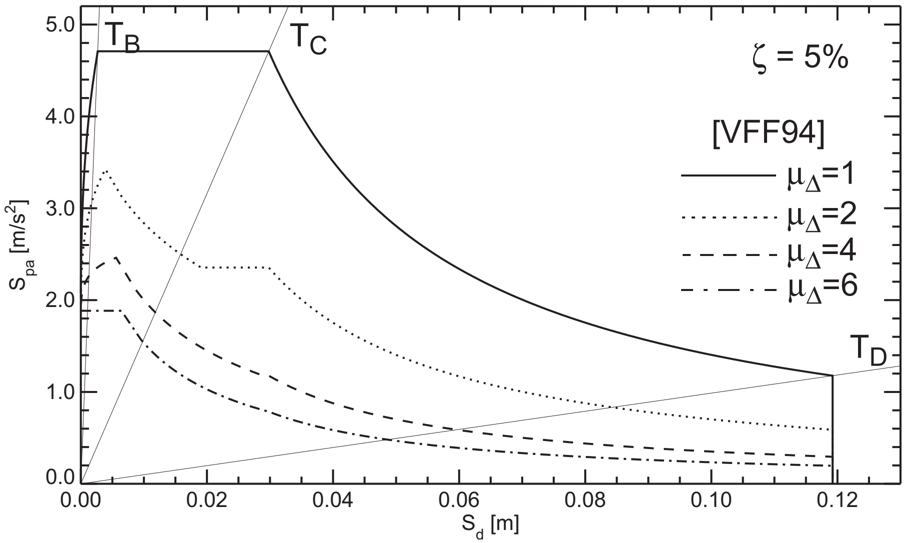
The spectra depicted on the following pages correspond to this case.

- Inelastic design spectra according to [VFF94]



• Inelastic design spectra in ADRS-format



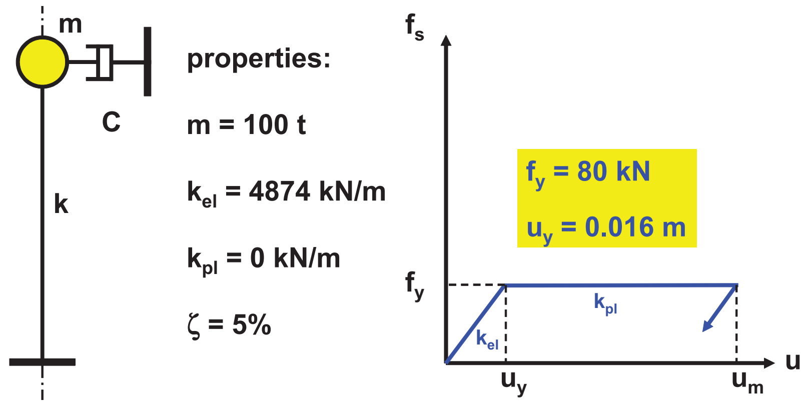


7.7.2 Determining the response of an inelastic SDOF system by means of inelastic design spectra in ADRS-format

In this section the response of two example inelastic SDOF systems is determined by means of inelastic design spectra in ADRS-format.

- SDOF system 1 with T_n = 0.9 s
- SDOF system 2 with T_n = 0.3 s
- The spectra according to [VFF94] will be used (see Section 7.7.1)

• Example 1: SDOF system with T_n= 0.9 s



- Response of the elastic SDOF system 1:

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{100}{4874}} = 0.9s$$

$$S_{pa} = 2.62 m/s^2$$

$$S_d = 0.054 m$$

$$f_{el} = 261.7 kN$$

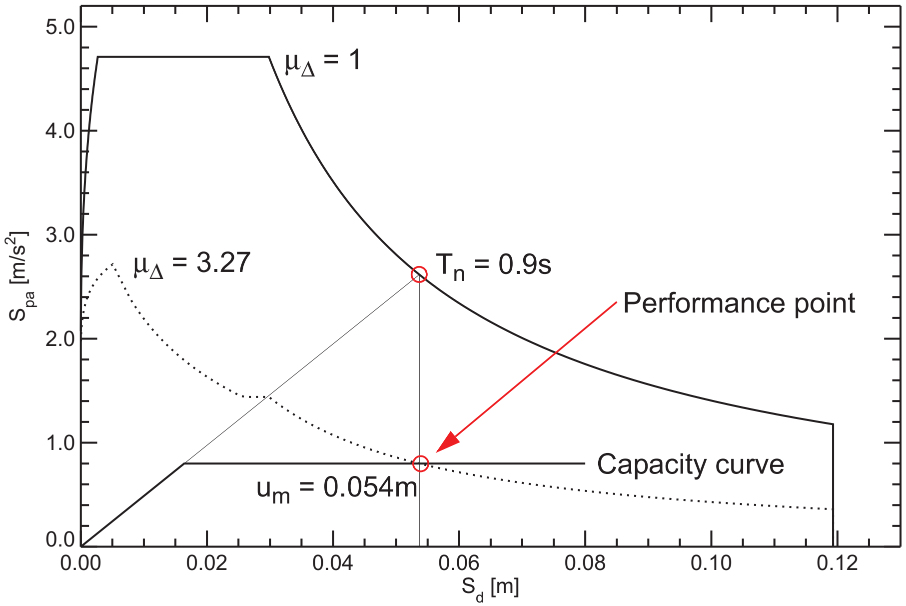
- Response of the inelastic SDOF system 1:

$$R_y = \frac{f_{el}}{f_y} = \frac{261.7}{80} = 3.27$$

$\mu_\Delta = R_y = 3.27$ (From Equation (7.95) since $T_n > T_c = 0.5s$)

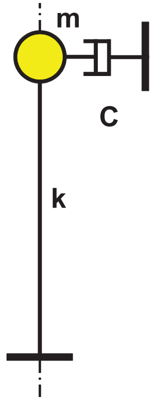
$$u_m = u_y \cdot \mu_\Delta = 0.016 \cdot 3.27 = 0.054m = S_d$$

Representation of the inelastic SDOF system 1 in the inelastic design spectrum in ADRS-format:



- If the force-deformation relationship of the inelastic SDOF system is divided by its mass m, the “capacity curve” is obtained, which can be plotted on top of the spectrum in ADRS-format.
- The capacity curve and the inelastic spectrum intersect in the “performance point”.

- Example 2: SDOF system with T_n= 0.3 s



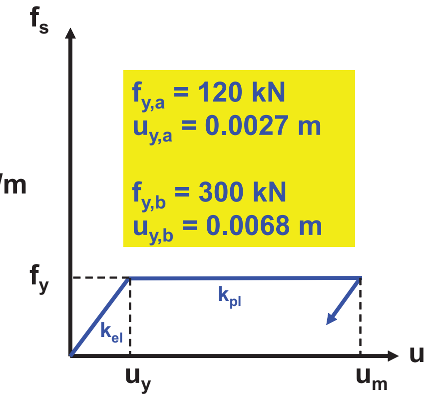
properties:

m = 100 t

k_{el} = 43‘865 kN/m

k_{pl} = 0 kN/m

ζ = 5%



- Response of the elastic SDOF system 2

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{100}{43865}} = 0.3s$$

$$S_{pa} = 4.71 m/s^2$$

$$S_d = 0.011m$$

$$f_{el} = 471kN$$

In this second example two different inelastic SDOF systems will be considered: (a) A SDOF system with a rather low f_y and (b) a SDOF system with a rather high f_y.

- Response of the second inelastic SDOF system 2a

$$R_y = \frac{f_{el}}{f_y} = \frac{471}{120} = 3.93$$

In this case the resulting displacement ductility μ_Δ is so large, that Equation (7.96) $T_0 = T_c = 0.5s$ results. After rearranging Equation (7.95), the displacement ductility μ_Δ can be computed as:

$$\mu_\Delta = (R_y - 1) \cdot \frac{T_c}{T_n} + 1 = (3.93 - 1) \cdot \frac{0.5}{0.3} + 1 = 5.88$$

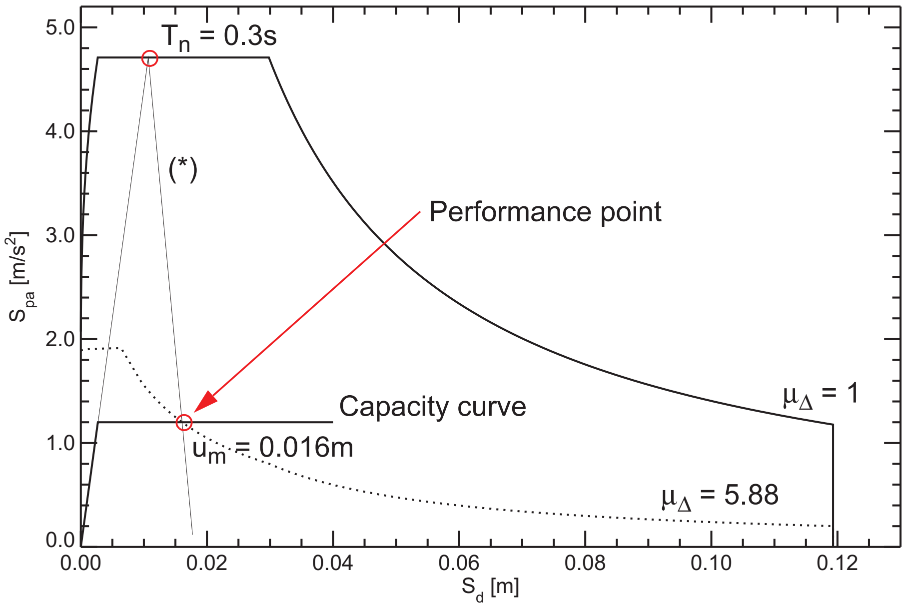
Check that $T_0 > T_c$:

$$T_0 = 0.65 \cdot \mu_\Delta^{0.3} \cdot T_c = 0.65 \cdot 5.88^{0.3} \cdot 0.5 = 0.553\text{s} > T_c$$

The maximum displacement response is therefore:

$$u_m = u_y \cdot \mu_\Delta = 0.0027 \cdot 5.88 = 0.016\text{m} > S_d$$

Representation of the inelastic SDOF system 2a in the inelastic design spectrum in ADRS-format:



- Note that the line (*) is no longer vertical as in Example 1, but inclined according to the equation $\mu_\Delta = (R_y - 1) \cdot (T_c / T_n) + 1$.

Now consider the SDOF system 2b:

- Response of the inelastic SDOF system 2b

$$R_y = \frac{f_{el}}{f_y} = \frac{471}{300} = 1.57$$

In this case the displacement ductility μ_Δ will be such that Equation (7.96) yields $T_0 < T_c = 0.5\text{s}$. To compute μ_Δ insert therefore Equation (7.96) in Equation (7.95). This results in following expression:

$$(\mu_\Delta - 1) \cdot \frac{T_n}{0.65 \cdot \mu_\Delta^{0.3} \cdot T_c} + 1 = R_y \qquad (7.97)$$

Equation (7.97) needs to be solved numerically.

$$\mu_\Delta(T_n = 0.3, T_c = 0.5, R_y = 1.57) = 1.73$$

Check that $T_0 < T_c$:

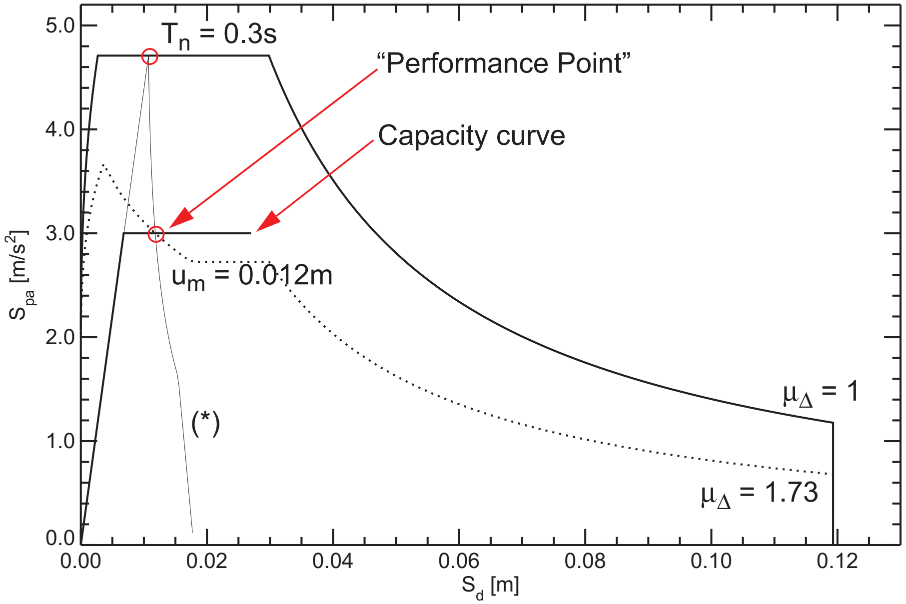
$$T_0 = 0.65 \cdot \mu_\Delta^{0.3} \cdot T_c = 0.65 \cdot 1.73^{0.3} \cdot 0.5 = 0.383\text{s} < T_c$$

$$(\mu_\Delta - 1) \cdot \frac{T_n}{T_0} + 1 = (1.73 - 1) \cdot \frac{0.3}{0.383} + 1 = 1.57 = R_y$$

The maximum displacement response is therefore:

$$u_m = u_y \cdot \mu_\Delta = 0.0068 \cdot 1.73 = 0.012\text{m} > S_d$$

Representation of the inelastic SDOF system 2b in the inelastic design spectrum in ADRS-format:



- Note that the curve (*) is no longer a straight line as in Examples 1 and 2a.
- In Example 2b the curve (*) needs to be computed numerically.
- In Example 2a the curve (*) is only an approximation of the curve (*) in Example 2b. As soon as $T_0 = T_c$ both curves are identical. In Example 2 this is the case if $S_{pa} < 1.6 m/s^2$.
- When $T_0 < T_c$ (i.e. when $S_{pa} > 1.6 m/s^2$) the curve (*)2a predicts larger maximum displacements u_m than curve (*)2b. The difference is, however, small. For this reason, in most cases Equation (7.95) can be approximated as:

$$R_y = \begin{cases} (\mu_\Delta - 1) \cdot \frac{T_n}{T_c} + 1 & T_n \le T_c \\ \mu_\Delta & T_n > T_c \quad \text{(ED principle)} \end{cases} \tag{7.98}$$

This approximation is particularly satisfactory, if the large uncertainties associated with smoothed spectra are considered.

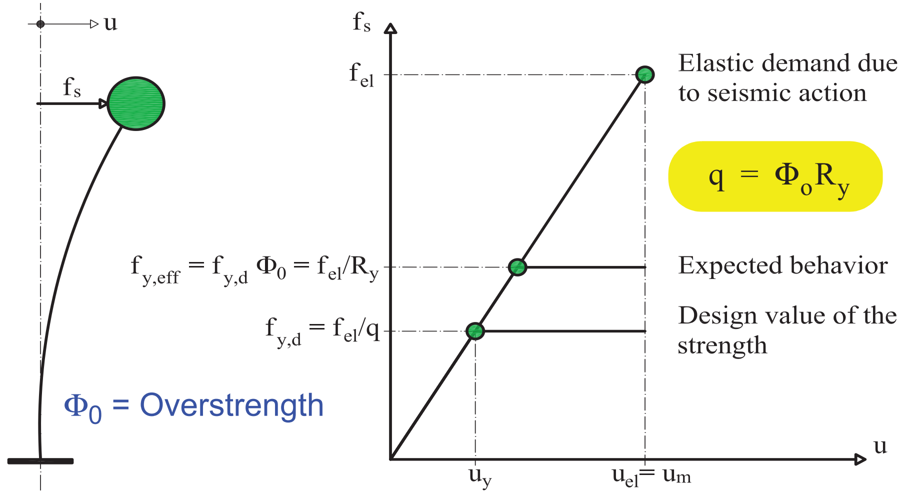
- Comments
  - A discussion of similar examples can be found in [Faj99].
  - For computing the response of inelastic SDOF systems by means of inelastic design spectra, the R_y-μ_Δ-T_n relationships in Section 7.7.1 are sufficient. The spectra in ADRS-format are not absolutely necessary, but they illustrate the maximum response of inelastic SDOF systems very well.
  - R_y-μ_Δ-T_n relationships should only be used in conjunction with smoothed spectra. They should not be used to derive the inelastic response spectra of a single ground motion
  - Remember:
    - Design spectra are very useful tools to design structures for the expected seismic demand. Design spectra represent the average effect of an earthquake with design intensity.
    - If a single earthquake is considered, the spectra may underestimate the seismic demand for a certain period range (... overestimate ...).
    - This characteristic of design spectra should be considered when designing structures: The seismic design should aim at structures that are as robust as possible.

7.7.3 Inelastic design spectra: An important note

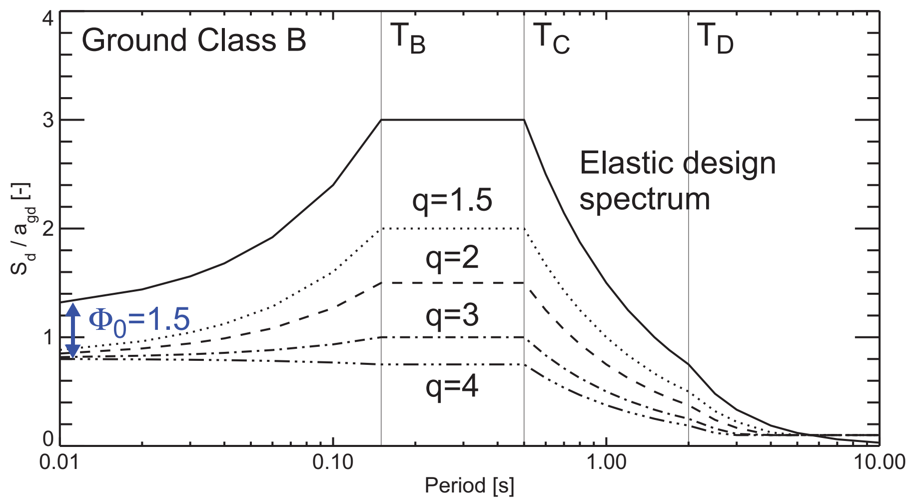
The "equal displacement" and the "equal energy" principles represent a strong simplification of the real inelastic behaviour of SDOF systems.

- Design spectra are a powerful tool to design structures to resist the expected seismic action. On **average**, design spectra are a good representation of the expected peak behaviour of structures.
- However, if **single** ground motions are considered, then it can easily be the case that design spectra significantly underestimate the expected peak behaviour of structures.
- This characteristic of the design spectra shall be taken into account during design by aiming at robust structures.

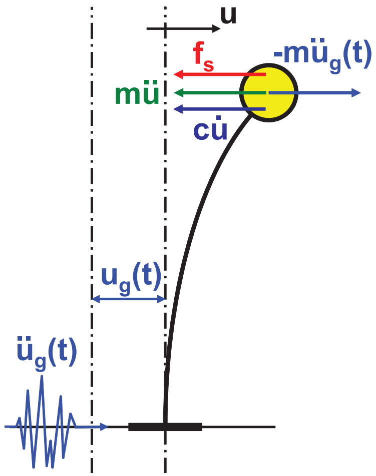
7.7.4 Behaviour factor q according to SIA 261

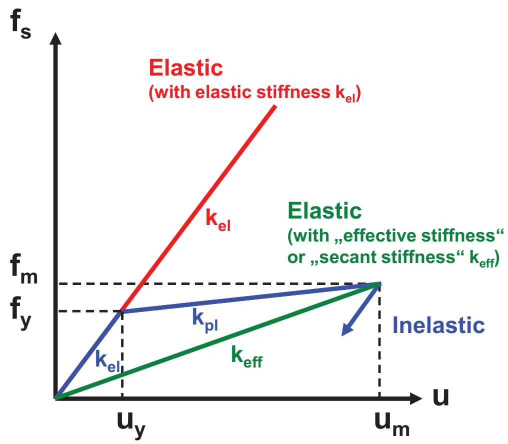


- Design spectra according to SIA 261



7.8 Linear equivalent SDOF system (SDOF$_e$)





It is postulated that the maximum response u_m of an inelastic SDOF system can be estimated by means of a linear equivalent SDOF system (SDOF$_e$). The properties of the SDOF$_e$ are:

Stiffness: $k_{eff} = f_m / u_m$ (7.99)

Damping: ζ_e (7.100)

The differential equation of the SDOF$_e$ is:

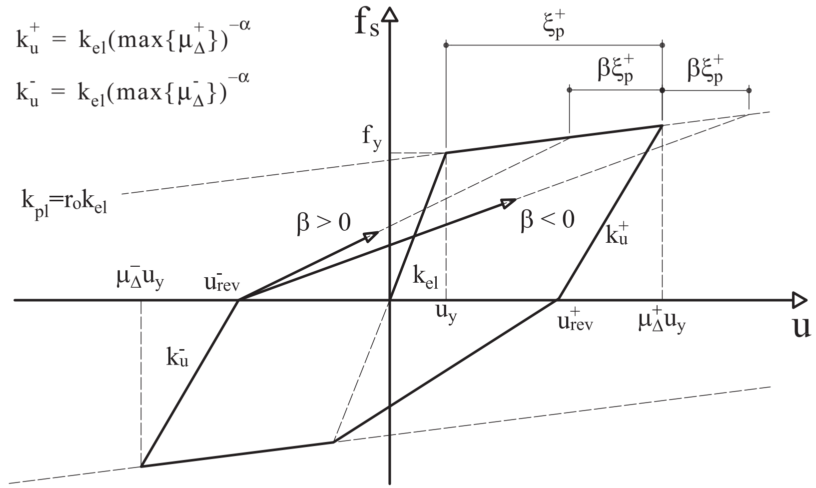
$$\ddot{u}(t) + 2\zeta_e\omega_e\dot{u}(t) + \omega_e^2 u(t) = -\ddot{u}_g(t) \text{ with } \omega_e^2 = k_{eff}/m \quad (7.101)$$

The question is how the viscous damping ζ_e of the SDOF$_e$ can be determined so that $\max(u(t)) = u_m$.

- Example: Inelastic SDOF system with Takeda-hyst. rule [TNS70]

The properties of the inelastic SDOF system are:

- Damping: $\zeta = 5\%$ (constant, proportional to k_{el})
- Mass: $m = 100t$
- Stiffness: $k_{el} = 4874 kN/m$
- Yield force: $f_y = 80kN$
- Hysteresis: Takeda-hysteresis with $r_o = 0.05$, $\alpha = 0.5$, $\beta = 0.0$



The maximum response of the SDOF system when subjected to the NS-component of the 1940 El Centro Earthquake is:

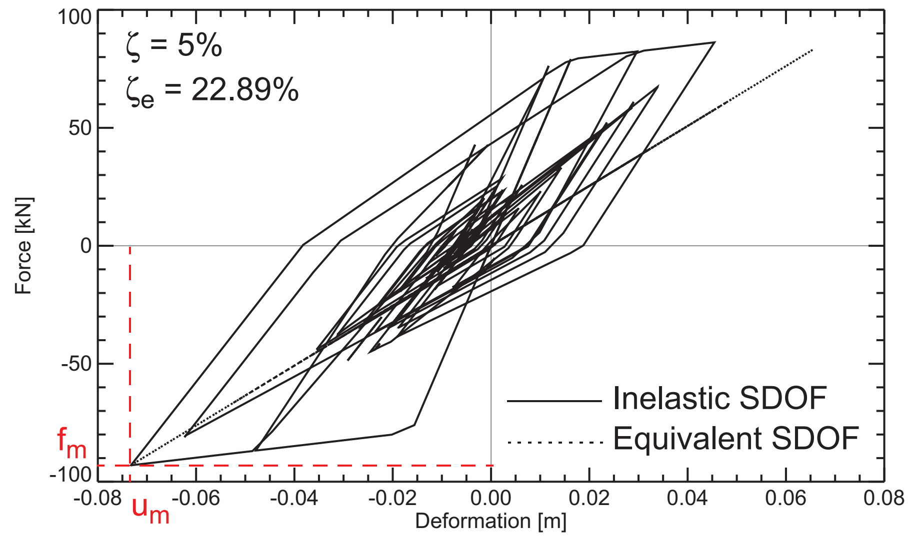
$$x_m = 0.073m \, , \, f_m = 93.0kN$$

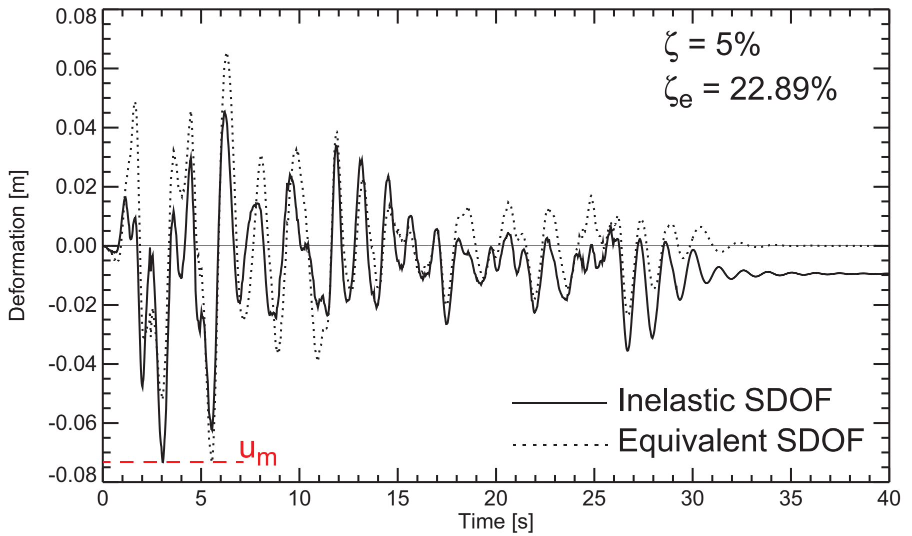
The properties of the corresponding SDOF$_e$ are:

$$k_{eff} = \frac{f_m}{u_m} = \frac{93.0}{0.073} = 1274\frac{kN}{m} \, , \, T_e = 2\pi\sqrt{\frac{m}{k_{eff}}} = 2\pi\sqrt{\frac{100}{1274}} = 1.76s$$

$\zeta_e = 22.89\%$, the viscous damping ζ_e was determined iteratively!

Comparison between inelastic SDOF and $SDOF_e$





Comments regarding the example:

- The damping ζ_e is in general larger than the damping ζ, since ζ_e of the $SDOF_e$ needs to compensate for the hysteretic energy absorption of the inelastic SDOF system.
- However, in rare cases it happens that $\zeta_e < \zeta$. This shows again the difficulties that are associated with the prediction of the seismic response of inelastic SDOF systems.
- In the example, the viscous damping ζ_e was determined iteratively until a value for ζ_e was found for which the response of the $SDOF_e$ system was equal to the maximum response of the inelastic SDOF system. Hence, if a method was available for estimating the viscous damping ζ_e, then the maximum response of the inelastic SDOF system could indeed be estimated by means of the linear equivalent SDOF system.
- The stiffness k_{eff} and the period T_e of the $SDOF_e$ system are only known once the maximum response of the inelastic SDOF system are known. Section 7.8.2 shows how the equivalent viscous damping ζ_e can be estimated without knowing the stiffness k_{eff} and the period T_e of the $SDOF_e$ system a priori.

- Estimating the damping ζ_e

In particular in the sixties significant research has been dedicated to estimating the damping ζ_e (see for example [Jac60], [Jen68] and [IG79]). At that time the interest in linear equivalent systems was big because the numerical computation of the response of inelastic systems was extremely expensive. The basic idea behind estimating the damping ζ_e was:

The inelastic SDOF system dissipates energy due to ζ and due to the inelastic deformations, which are a function of its inelastic force-deformation relationship. The equivalent SDOF system, however, dissipates energy solely due to its viscous damping. For this reason the following relationship applies:

$$\zeta_e = \zeta + \zeta_{eq} \tag{7.102}$$

where ζ_{eq} is the viscous damping equivalent to the hysteretic energy absorption of the inelastic system.

The simplest method for estimating the equivalent viscous damping is to assume that the inelastic system and the linear equivalent system dissipate the same energy within one displacement cycle. According to this assumption [Cho11] defines the equivalent elastic damping as:

$$\zeta_{eq} = \frac{1}{4\pi}\cdot\frac{A_h}{A_e} \tag{7.103}$$

Where:

- A_h: Energy dissipated by the inelastic SDOF system due to the inelastic deformation of the system. The dissipated energy corresponds to the area of the force-displacement hysteresis of the considered displacement cycle;
- A_e: Potential energy of the equivalent SDOF system at maximum displacement:

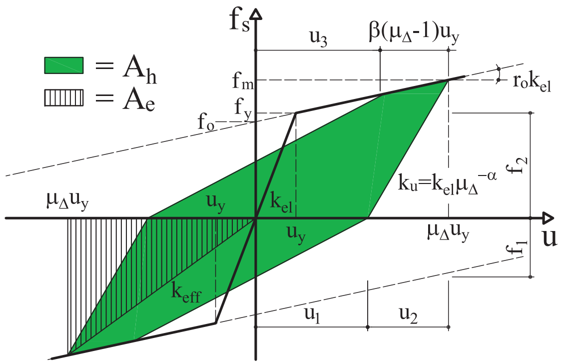
$$A_e = \frac{k_{eff}\cdot u_m^2}{2}$$

The inelastic force-deformation relationship of many structural RC elements can be described by the "Takeda"-hysteresis rule. According to Equation (7.103) the equivalent viscous damping of this hysteresis rule is:

$$\zeta_{eq,Tak} = \frac{1}{4\pi}\cdot\frac{(f_m+f_o)\mu_\Delta u_y + f_1u_1 - f_mu_2 - (f_m+f_o)u_3}{(f_m\mu_\Delta u_y)/2} \tag{7.104}$$

Where:

Course "Fundamentals of Structural Dynamics" **An-Najah 2013**



$$f_m = [1 + r_o(\mu_\Delta - 1)]\cdot f_y$$

$$f_o = (1 - r_o)\cdot f_y$$

$$f_1 = \frac{u_1}{u_1 + u_3}\cdot f_2$$

$$f_2 = [1 + r_o(\mu_\Delta - 1)(1-\beta)]\cdot f_y$$

$$u_1 = \mu_\Delta u_y - u_2$$

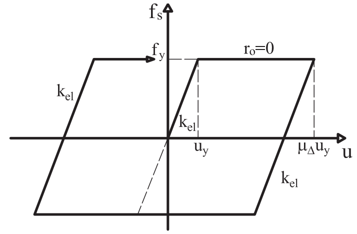
$$u_2 = \frac{1 + r_o(\mu_\Delta - 1)}{\mu_\Delta^{-\alpha}}\cdot u_y$$

$$u_3 = [\mu_\Delta - \beta(\mu_\Delta - 1)]\cdot u_y$$

The equivalent viscous damping of other important hysteresis rules is:

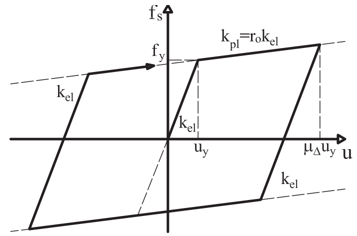
Elasto-plastic (EP) rule:

$$\zeta_{eq,\,EP} = \frac{2}{\pi}\cdot\frac{\mu_\Delta - 1}{\mu_\Delta} \tag{7.105}$$



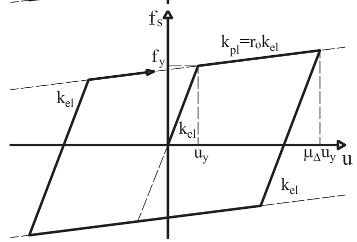
Bilinear (BL) rule:

$$\zeta_{eq,\,BL} = \frac{2}{\pi}\cdot\frac{(\mu_\Delta - 1)(1 - r_o)}{\mu_\Delta(1 + r_o\mu_\Delta - r_o)} \tag{7.106}$$

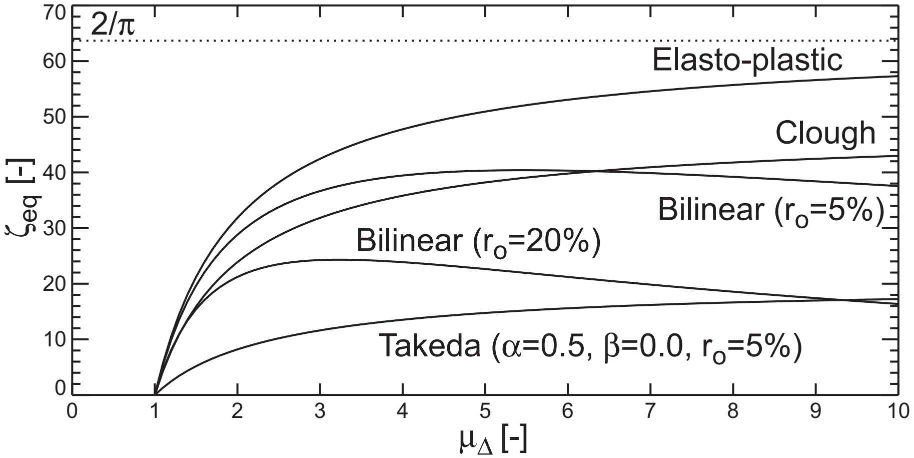


Rule according to Clough (Clo) [CP75]:

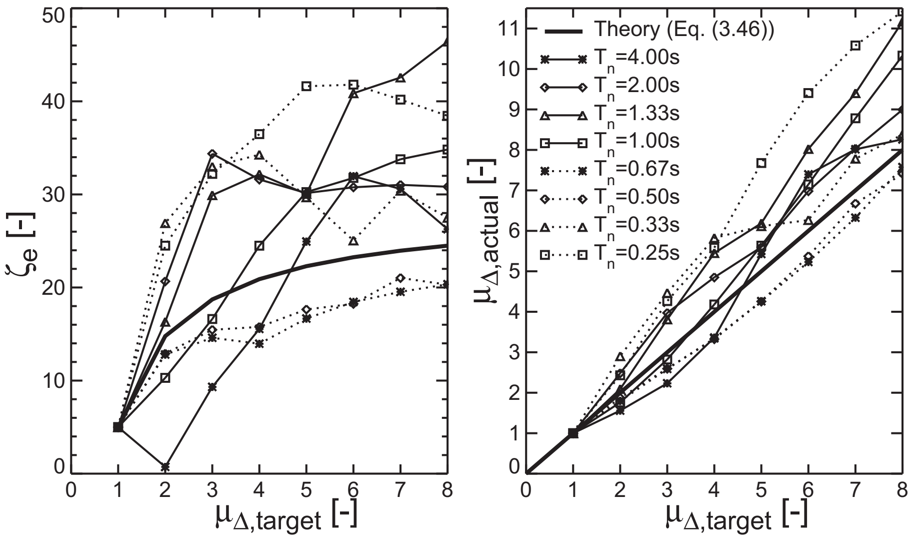
$$\zeta_{eq,\,Clo} = \frac{2}{\pi}\cdot\frac{3}{2\pi}\cdot\frac{\mu_\Delta - 1}{\mu_\Delta} \tag{7.107}$$



ζ_{eq}-μ_{Δ}-relationship for these important hysteresis rules:



The next figure compares the theoretical value for ζ_e for the Takeda-SDOF (Equ. (7.104), r_o=0.05, α=0.5, β=0) with the computed value (for El Centro):



Comments regarding the comparison of the theoretical value with the computed value of ζ_e for the Takeda-SDOF system when excited by the El Centro earthquake:

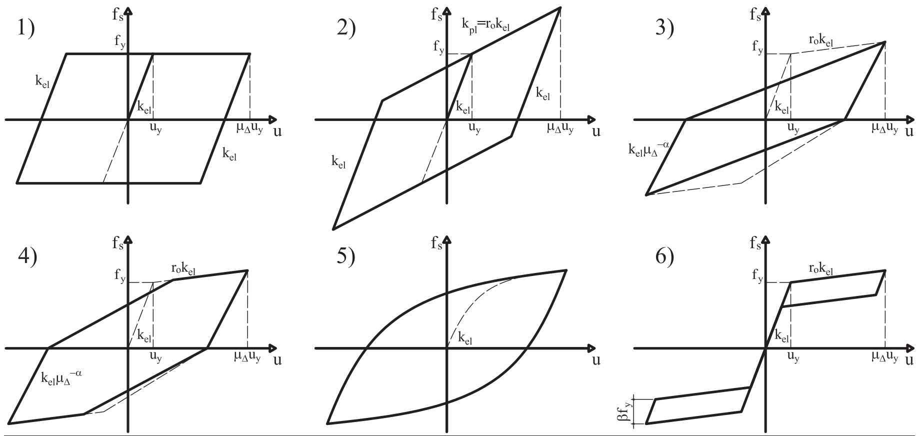
- The computed value of ζ_e was determined iteratively. Eight different inelastic SDOF systems with different periods T_n were considered. The strength of each inelastic SDOF system was varied in such a way that seven different displacement ductilities resulted (μ_{Δ}=2 to 8).
- The results show that ζ_e is not only dependent on μ_{Δ} but also on the period T_n of the SDOF system. This effect is not considered by Equations (7.103) and (7.104), respectively.
- In some cases the difference between the theoretical value and the computed vale for ζ_e is considerable. For this reason there are also considerable differences between $\mu_{\Delta,\,target}$ (target ductility) and $\mu_{\Delta,\,actual}$ (actual ductility obtained from the time-history analysis of the $SDOF_e$ system with the viscous damping ζ_e according to Equation (7.104)).
- Typically these differences increase as the target ductility increases.
- Similar observations were made when the computation of the inelastic spectra was discussed.
- This shows again the difficulties associated with the prediction of the seismic response of inelastic SDOF system.

• Improved estimate for ζ_e

Over the last years some researchers suggested improved formulas for ζ_e by carrying out statistical analyses of time-history responses of inelastic SDOF systems (see [PCK07]). [GBP05] suggest for example the Equation (7.108).

$$\zeta_e = \zeta + \zeta_{eq} \qquad \text{where:} \qquad \zeta_{eq} = a\left(1 - \frac{1}{\mu_{\Delta}^{b}}\right)\left(1 + \frac{1}{(T_e + c)^{d}}\right) \tag{7.108}$$

The constants a to d are:



Hysteresis rule	a	b	c	d
1) Elasto-Plastic (EPP)	0.224	0.336	-0.002	0.250
2) Bilinear, r_o=0.2 (BI)	0.262	0.655	0.813	4.890
3) Takeda Thin (TT)	0.215	0.642	0.824	6.444
4) Takeda Fat (TF)	0.305	0.492	0.790	4.463
5) Ramberg-Osgood (RO)	0.289	0.622	0.856	6.460
6) “Flag-Shaped”, β=0.35 (FS)	0.251	0.148	3.015	0.511

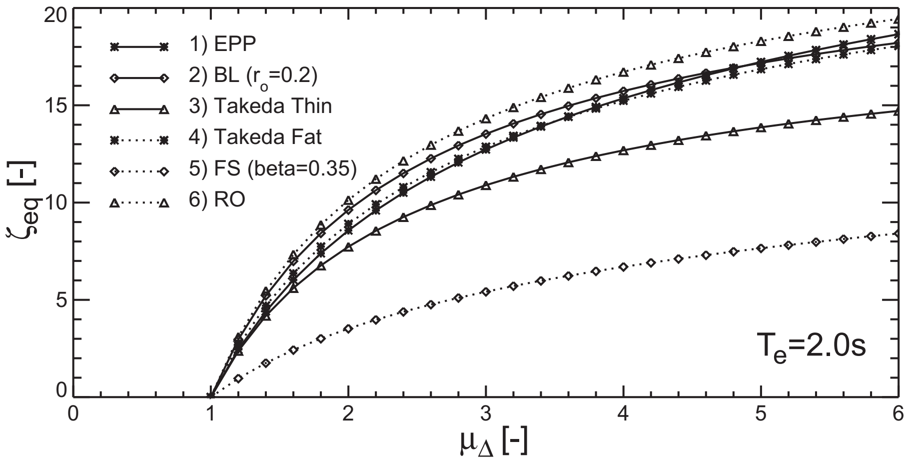
from [GBP05]

The hysteresis rules 1) to 6) were chosen because they can be used to represent the hysteretic behaviour of typical structural types:

- **Elasto-Plastic (EPP):** Hysteretic rule that characterises systems for the seismic isolation of structures (sliding systems that are based on friction).
- **Bilinear, r_o=0.2 (BI):** Hysteretic rule that also characterises systems for the seismic isolation of structures. The value of the post-yield stiffness $r_o k_{pl}$ may vary significantly between different systems.

- **Takeda Thin (TT):** Hysteretic rule that characterises RC structures which lateral stiffness is provided by walls and columns.
- **Takeda Fat (TF):** Hysteretic rule that characterises RC structures which lateral stiffness is provided by frames.
- **Ramberg-Osgood (RO):** Hysteretic rule that characterises ductile steel structures.
- **“Flag-Shaped”, β=0.35 (FS):** Hysteretic rule that characterises prestressed structures with unbonded tendons.

ζ_{eq}-μ_Δ-relationships for the most important hysteresis rules according to [GBP05]:



Important comments:

- With these relationships an in a **statistical sense** improved estimate of the damping ζ_e is obtained.
- For single systems subjected to a specific ground motion differences between the maximum response of the inelastic system and the maximum response of the equivalent SDOF with ζ_e according to these improved ζ_{eq}-μ_Δ-relationships can still be significant!

7.8.1 Elastic design spectra for high damping values

To compute the response of the equivalent SDOF systems, elastic design spectra can be used.

The damping values of equivalent $SDOF_e$ systems are in general larger than the typical 5%. For this reason the design spectra needs to be computed for higher damping values.

The design spectra for higher values of damping are often obtained by multiplying the design spectra for 5% damping with a correction factor η:

$$S_{pa}(T_n,\zeta) = \eta \cdot S_{pa}(T_n,\zeta = 5\%) \tag{7.109}$$

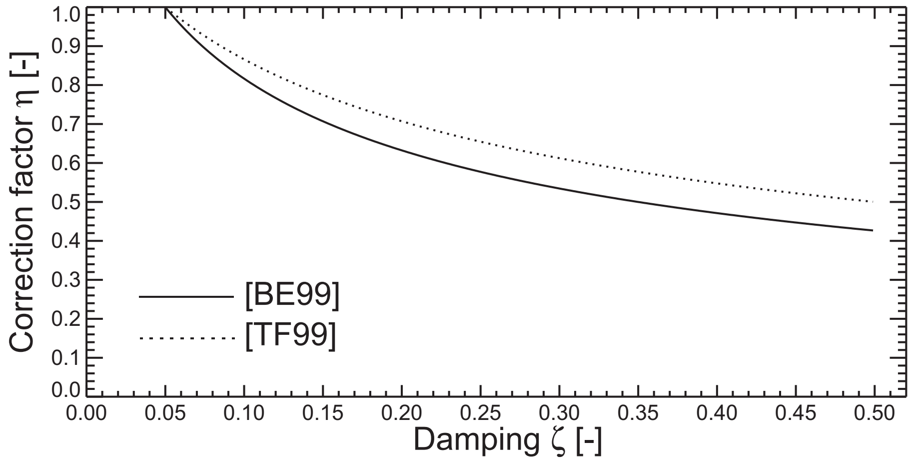
The literature provides different estimates for this correction factor η. Two of these are:

[TF99]: $$\eta = \sqrt{\frac{1.5}{1 + 10\zeta}} \text{ where } 0.05 \le \zeta \le 0.5 \tag{7.110}$$

[BE99]: $$\eta = \sqrt{\frac{1}{0.5 + 10\zeta}} \text{ where } 0.05 \le \zeta \le 0.3 \tag{7.111}$$

Equation (7.111) corresponds to Equation (29) in the Swiss Code SIA 261 [SIA03].

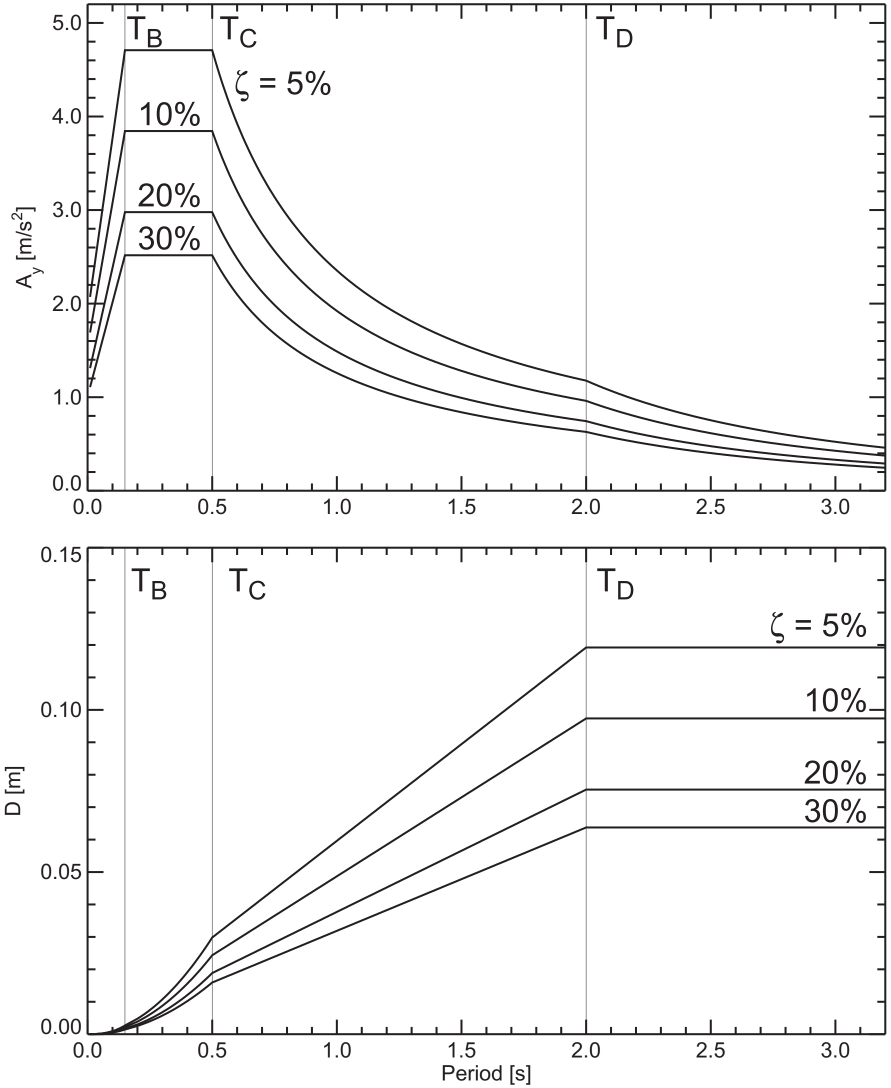
The correction factors η obtained with Equ.s (7.110) and (7.111) are plotted for different damping values ζ in the next figure:



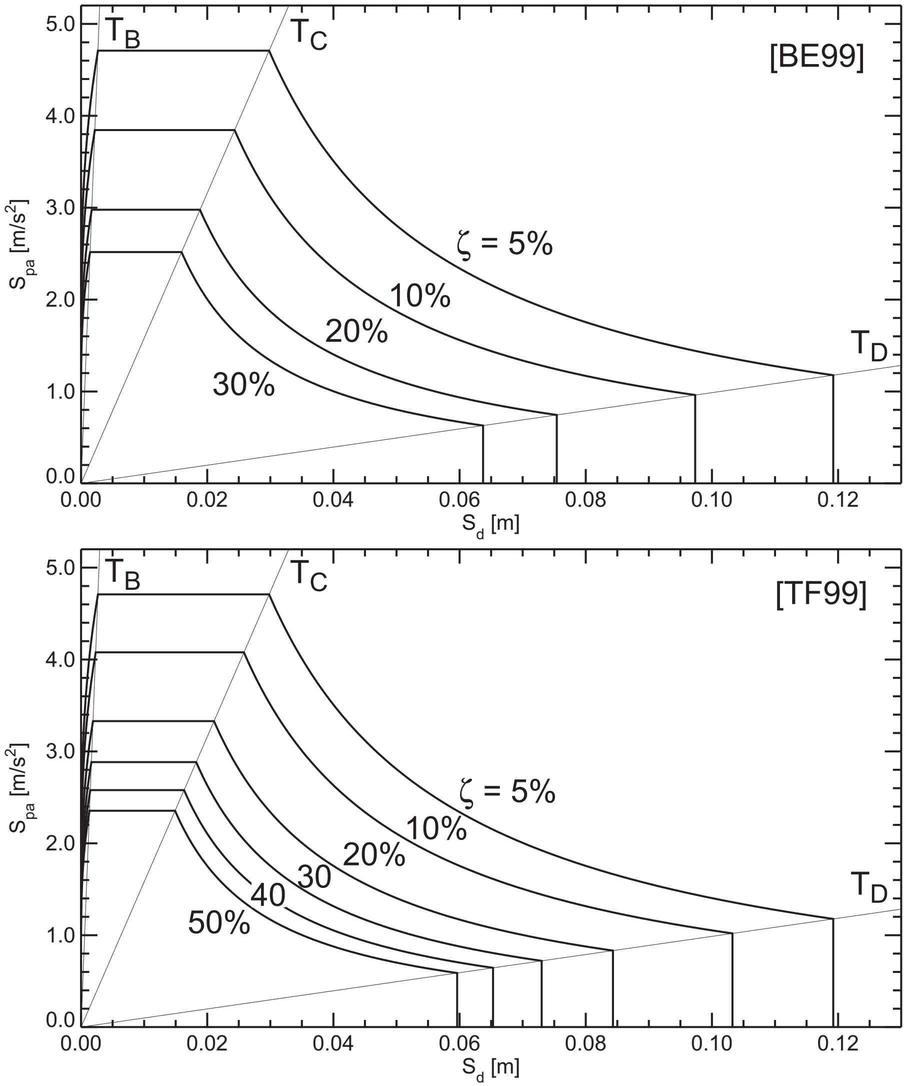
Comments:

- A discussion of the different approaches for computing the design spectra for high values of damping can be found in [PCK07].
- Equations (7.110) and (7.111) were derived for ground motions without near-field effects.
- Equations (7.110) and (7.111) were derived from the statistical analysis of several response spectra for different ground motions. For this reason Equ.s (7.110) and (7.111) should only be used in conjunction with smoothed response or design spectra.
- As for all statistical analyses the resulting design spectra correspond only in average with the true highly damped spectral ordinates. For single periods and ground motions the differences between the highly damped spectral ordinates obtained by Equ.s (7.110)/(7.111) and by time-history analyses of SDOF systems can be significant.

- Elastic design spectra according to [BE99]



- Elastic design spectra in ADRS-format



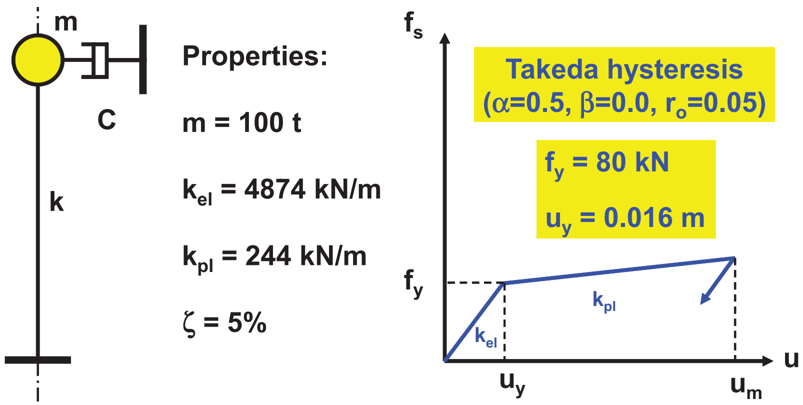
7.8.2 Determining the response of inelastic SDOF systems by means of a linear equivalent SDOF system and elastic design spectra with high damping

The computation of the seismic response of inelastic systems by means of linear equivalent systems was studied by Sozen and his co-workers in the seventies (see for example [GS74], [SS76] and [SS81]).

Today this approach gains new attention since the “Direct Displacement-Based Design (DDBD)” approach, which was developed by Priestley and his co-workers, is based on the idea of the linear equivalent system ([PCK07]).

This sections outlines the procedure for computing the response of an inelastic SDOF system by means of an linear equivalent SDOF system and elastic design spectra with high damping.

- Example: SDOF system with T_n= 0.9 s



For the example the spectra according to [BE99] will be used (Section 7.8.1).

- Response of the elastic SDOF system

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{100}{4874}} = 0.9\text{s}$$

$$S_{pa} = 2.62\text{m/s}^2$$

$$S_d = 0.054\text{m}$$

- Response of the inelastic SDOF system

  The maximum response of the inelastic SDOF system will be computed by means of the ADRS-spectra (page 7-106).

  Step 1: The capacity curve of the SDOF system is plotted on top of the ADRS spectra.

  Step 2: By means of Equ.s (7.102) and (7.104) the nonlinear scale, which represents the damping ζ_e as a function of the maximum response of the SDOF system, is plotted along the capacity curve.

  Step 3: Several spectra for different values of damping are plotted.

  Step 4: The “Performance Point” is the point where the spectrum with damping ζ_e intersects the capacity curve at the same value of ζ_e.

  For the considered example the maxmimum response of the inelastic SDOF system is:

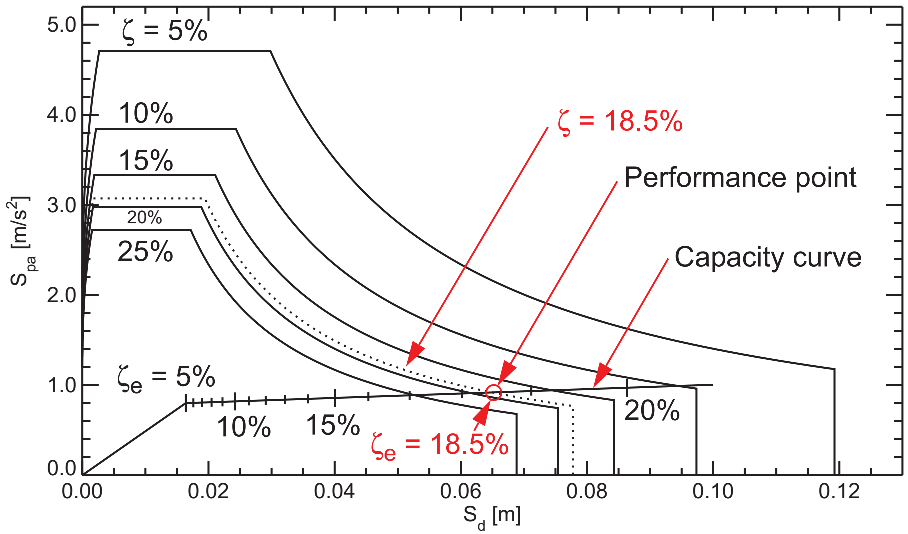
$$S_d = 0.065\text{m}$$

Comments regarding the example:

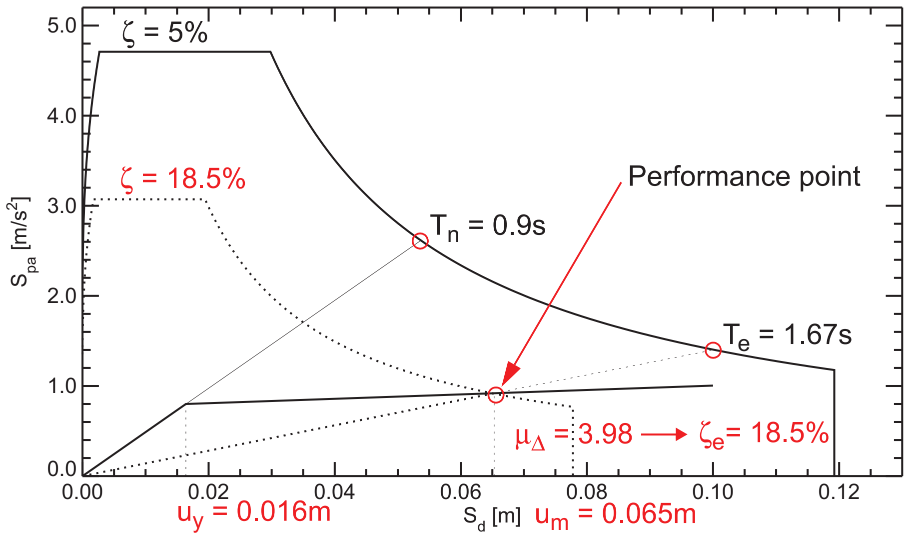
- To determine the “Performance Point” exactly, an iterativ approach is typically required.
- The linear equivalent SDOF system is fully defined by the period T_e and the damping ζ_e. The period T_e results from the slope of the line that connects the origin with the “Performance Point”.

- The damping values ζ_{eq} used in the figures on page 7-106 were determined according to Equation (7.104). In the figures on page 7-107 the damping ζ_{eq} was determined using Equation (7.108). The difference is, however, rather small.
- It should be noted that in both cases the computed maximum response of the inelastic SDOF system does not comply with the “equal displacement principle”.
- The linear equivalent SDOF system leads often to results that do not agree with the “equal displacement principle”. This applies in particular to SDOF systems with long periods or systems with large ductility demands.
- A second example is presented on page 7-107. It is a SDOF system with a shorter period and a smaller ductility demand than in Example 1. In this second example the “equal displacement principle” is approximately confirmed.

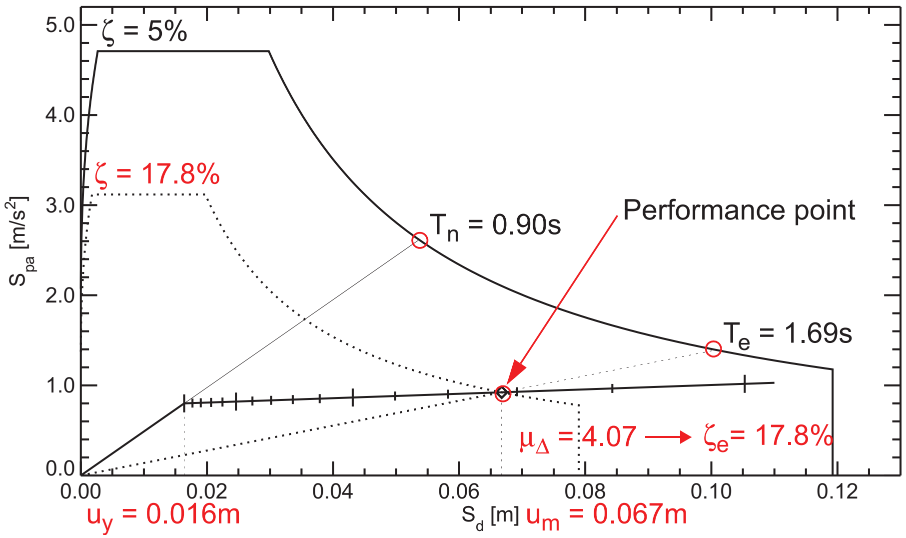
Determining the SDOF behaviour by means of elastic ADRS-spectra



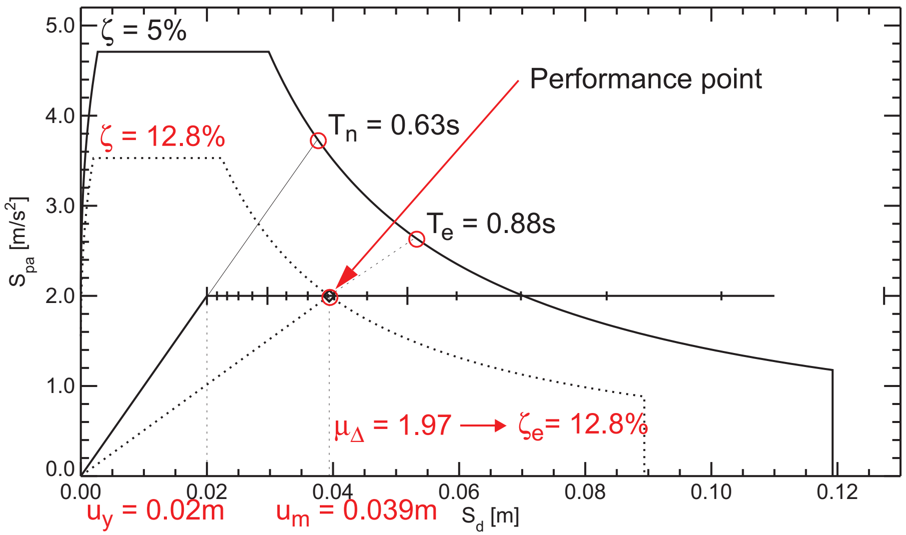
Alternative representation:



Recalculate the example with ζ_{eq} according to [GBP05]:



Second example with smaller ductility demand:



7.9 References

[AEM 86] Anderheggen E., Elmer H., Maag H.: "Nichtlineare Finite Element Methoden: Eine Einführung für Ingenieure". IBK Vorlesungsskript. Institut für Baustatik und Konstruktion (IBK), ETH Zürich. 1986.

[AP88] Allahabadi R., Powell G.: "Drain-2DX, User Guide". Report UBC/EERC-88/06. Earthquake Engineering Research Center, University of California, Berkeley 1988.

[Bat96] Bathe K-J.: Finite Element Procedures. ISBN 0-13-301458-4. Prentice-Hall, Upper Saddle River, New Jersey 1996.

[BE99] Bommer J.J., Elnashai A.S.: "Displacement spectra for seismic design". Journal of Earthquake Engineering vol. 3, No. 1, 1999.

[Cho11] Chopra A.K.: "Dynamics of Structures". Fourth Edition. Prentice Hall, 2011.

[DWB99] Dazio A., Wenk T., Bachmann H.: "Versuche an Stahlbetontragwänden unter zyklisch-statischer Einwirkung. (Tests on RC Structural Walls under Cyclic-Static Action)". IBK-Report No. 239, IBK-ETH Zurich. Birkhäuser Verlag, Basel 1999. (http://e-collection.ethbib.ethz.ch/view/eth:23296)

[Faj99] Fajfar P.: "Capacity Spectrum Method Based on Inelastic Demand Spectra". Earthquake Engineering and Structural Dynamics, 28, 979-993, 1999.

[HHT77] Hilber H.M., Hughes T.J.R, Taylor R.L.: "Improved Numerical Dissipation for Time Integration Algorithms in Structural Dynamics". Journal of Earthquake Engineering and Structural Dynamics, Vol. 5, 1977.

[HKS03] Hibbit Karlsson & Sorensen: "ABAQUS Version 6.4 - User's Manual". Pawtucket, RI, 2003.

[KN92] Krawinkler H., Nassar A.A.: "Seismic design based on ductility and cumulative damage demands and capacities". Nonlinear Seismic Analysis and Design of Reinforced Concrete Buildings, eds P. Fajfar and H. Krawinkler. New York: Elsevier Applied Science. 1992.

[GBP05] Grant D.N., Blandon C.A., Priestley M.J.N.: "Modelling Inelastic Response in Direct Displacement-Based Design". Rose School, Report 2005/03, IUSS Press, Pavia, 2005.

[GS74] Gulkan P., Sozen M.: "Inelastic Responses of Reinforced Concrete Structures to Earthquake Motions". ACI Journal, Title No. 71-41. December 1974.

[IG79] Iwan W.D., Gates N.C.: "Estimating Earthquake Response of Simple Hysteretic Structures". ASCE, Journal of the Engineering Mechanics Division, Vol. 105, EM3, June 1979.

[Jac60] Jacobsen L.S.: "Damping in composite structures". Proceedings of the 2nd World Conference in Earthquake Engineering. Vol. 2. Tokio and Kyoto, Japan, 1960.

[Jen68] Jennings P.C.: "Equivalent Viscous Damping for Yielding Structures". ASCE, Journal of the Engineering Mechanics Division, Vol. 94, No. EM1, February 1968.

[Mir01] Miranda E.: "Estimation of inelastic defromation demands of SDOF Systems". ASCE, Journal of structural engineering, Vol. 127, No. 9, 2001.

[New59] Newmark N.M.: "A Method of Computation for Structural Dynamics". ASCE, Journal of the Engineering Mechanics Division, Vol. 85, No. 3, July 1959.

[NH82] Newmark N.M., Hall W.J.: "Earthquake Spectra and Design". Earthquake Engineering Research Institute (www.eeri.org), 1982.

[PCK07] Priestley M.J.N., Calvi G.M., Kowalsky M.J.: "Displacement-Based Seismic Design of Structures". IUSS Press, 2007.

[Saa91] Saatcioglu M.: "Modeling Hysteretic Force-Deformation Relationships for Reinforced Concrete Elements". In Publication SP 127 "Earthquake-Resistant Concrete Structures, Inelastic Response and Design", ACI, Detroit 1991.

[SIA03] Swiss standard SIA 261: "Actions on structures". SIA, Zürich 2003.

[SS81] Saiidi A., Sozen M.: "Simple Nonlinear Seismic Analysis of R/C Structures". ASCE, Journal of the Structural Division, Vol. 107, No. 5, May 1981.

[SS76] Shibata A., Sozen M.: "Substitute Structure Method for Seismic Design in R/C". ASCE, Journal of the Structural Division, Vol. 102, No. 1, January 1976.

[TF99] Tolis S.V., Faccioli E.: "Displacement design spectra". Journal of Earthquake Engineering vol. 3, No. 1, 1999.

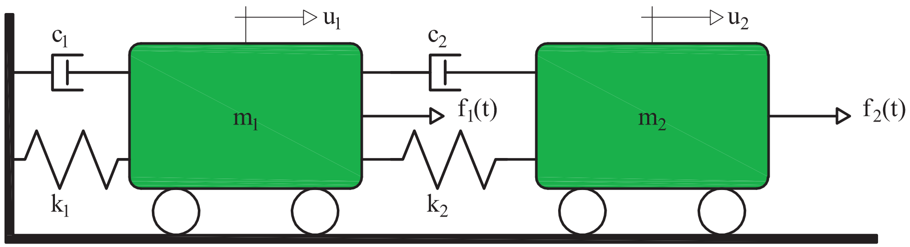
[TSN70] Takeda T., Sozen M.A., Nielsen N.N.: "Reinforced Concrete Response to Simulated Earthquakes". ASCE, Journal of Structural Engineering, Vol. 96, No. 12, December 1970.

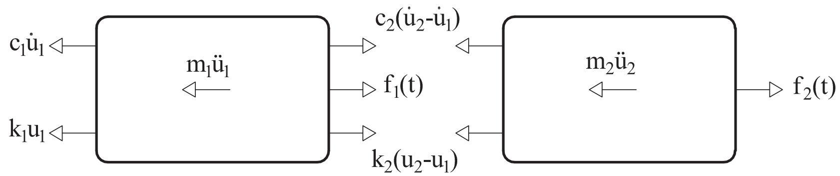
[VFF94] Vidic T., Fajfar P., Fischinger M.: "Consistent inelastic design spectra: strength and displacement". Earthquake Engineering and Structural Dynamics 23(5) 507-521. 1994.

8 Multi Degree of Freedom Systems

8.1 Formulation of the equation of motion

8.1.1 Equilibrium formulation





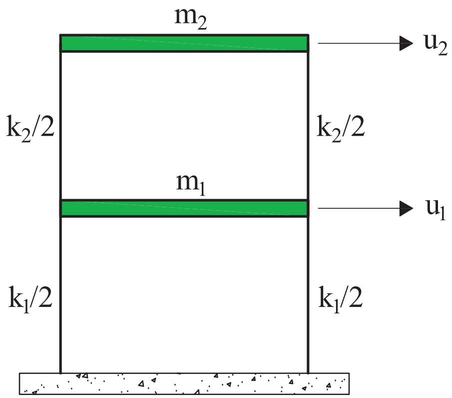
$$\begin{cases} m_1\ddot{u}_1 + c_1\dot{u}_1 + k_1u_1 = f_1(t) + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) \\ m_2\ddot{u}_2 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) = f_2(t) \end{cases} \quad (8.1)$$

$$\begin{cases} m_1\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 - c_2\dot{u}_2 + (k_1 + k_2)u_1 - k_2u_2 = f_1(t) \\ m_2\ddot{u}_2 - c_2\dot{u}_1 + c_2\dot{u}_2 - k_2u_1 + k_2u_2 = f_2(t) \end{cases} \quad (8.2)$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \quad (8.3)$$

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t) \quad (8.4)$$

8.1.2 Stiffness formulation

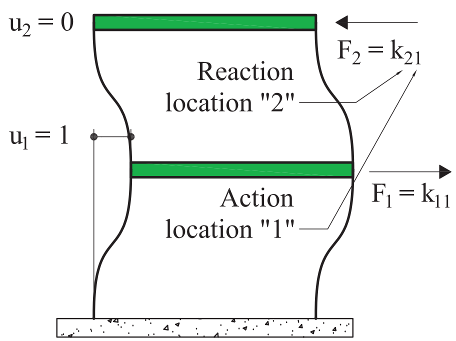


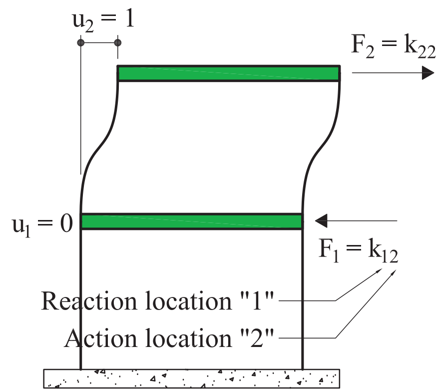
The degrees of freedoms are the horizontal displacements u_1 and u_2 at the level of the masses m_1 and m_2

- Stiffness matrix

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad (8.5)$$

Unit displacement $u_1 = 1$ Unit displacement $u_2 = 1$





- Mass matrix $\mathbf{M}$

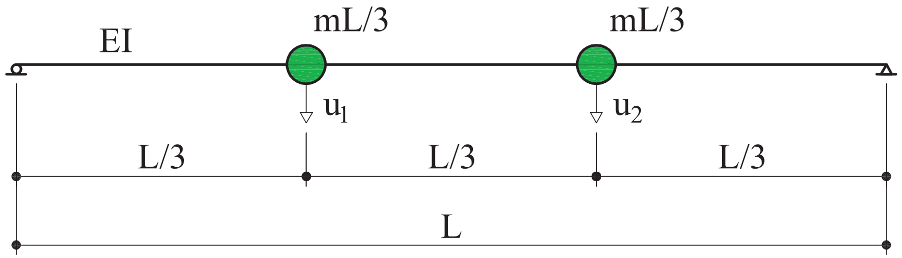
$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{8.6}$$

- Equation of motion

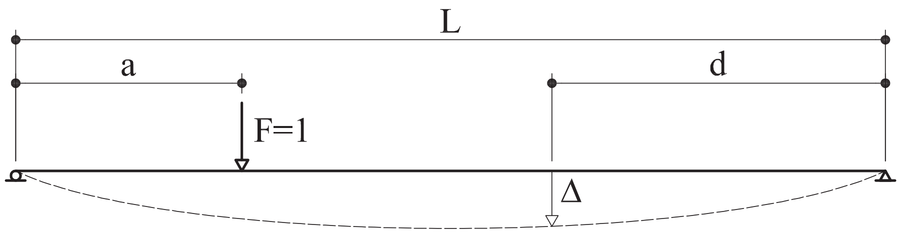
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.7}$$

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{8.8}$$

8.1.3 Flexibility formulation



- Flexibility matrix $\mathbf{D}$



By means of the principle of virtual forces the vertical displacement Δ at location d due to a unit force $F = 1$ acting at location a can be readily determined.

$$\Delta(\alpha,\delta) = -\alpha\delta(\alpha^2+\delta^2-1)\cdot\frac{FL^3}{6EI} \quad \text{with } \alpha = \frac{a}{L} \text{ and } \delta = \frac{d}{L} \tag{8.9}$$

The flexibilty matrix consists of the following elemnts:

$$\mathbf{u} = \mathbf{D}\mathbf{F} \tag{8.10}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\cdot\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{8.11}$$

The d_{ij} factors can be computed by means of Equation (8.9) as follows:

$$d_{11} = \Delta(1/3,2/3) = \frac{4}{243}\cdot\frac{L^3}{EI} \tag{8.12}$$

$$d_{12} = \Delta(2/3,2/3) = \frac{7}{486}\cdot\frac{L^3}{EI} \tag{8.13}$$

$$d_{21} = \Delta(1/3,1/3) = \frac{7}{486}\cdot\frac{L^3}{EI} \tag{8.14}$$

$$d_{22} = \Delta(2/3,1/3) = \frac{4}{243}\cdot\frac{L^3}{EI} \tag{8.15}$$

and the flexibilty matrix $\mathbf{D}$ becomes:

$$\mathbf{D} = \frac{L^3}{486EI}\cdot\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} \tag{8.16}$$

- Stiffness matrix $\mathbf{K}$

$$\mathbf{K} = \mathbf{D}^{-1} = \frac{162}{5} \cdot \frac{EI}{L^3} \cdot \begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix} \tag{8.17}$$

- Mass matrix $\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{8.18}$$

- Equation of motion

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \frac{162}{5} \cdot \frac{EI}{L^3} \cdot \begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.19}$$

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{8.20}$$

8.1.4 Principle of virtual work

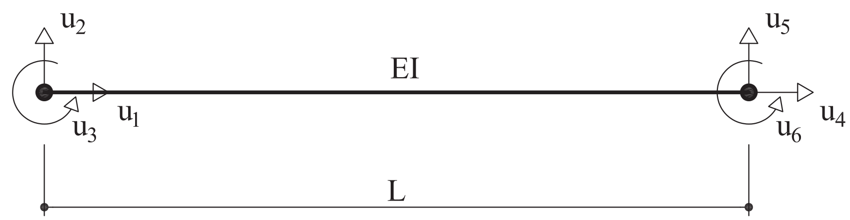
- See e.g. [Hum12]

8.1.5 Energie formulation

- See e.g. [Hum12]

8.1.6 “Direct Stiffness Method”

- Stiffness matrix of a beam element

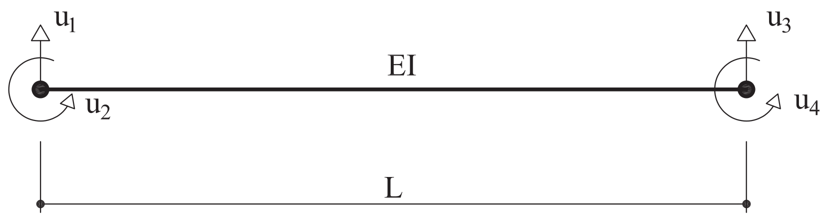


The stiffness matrix $\mathbf{K}$ of a beam element with constant flexural and axial stiffness is well known:

$$\mathbf{F} = \mathbf{K}\mathbf{u} \tag{8.21}$$

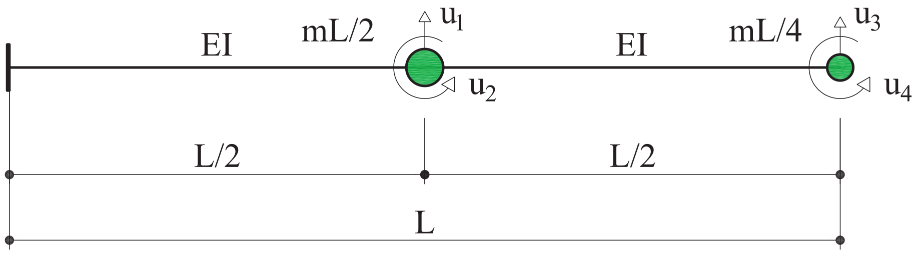
$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \tag{8.22}$$

If the axial elongation of the beam is not considered, the matrix can be further simplified as follows:



$$\mathbf{K} = \frac{EI}{L^3} \cdot \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (8.23)$$

- Example: Cantilever



Assemblage of the stiffness matrix

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} 12+12 & -6\bar{L}+6\bar{L} & -12 & 6\bar{L} \\ -6\bar{L}+6\bar{L} & 4\bar{L}^2+4\bar{L}^2 & -6\bar{L} & 2\bar{L}^2 \\ -12 & -6\bar{L} & 12 & -6\bar{L} \\ 6\bar{L} & 2\bar{L}^2 & -6\bar{L} & 4\bar{L}^2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad (8.24)$$

with $\bar{L} = L/2$

Equation of motion:

$$\begin{bmatrix} m\bar{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m\bar{L}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} 24 & 0 & -12 & 6\bar{L} \\ 0 & 8\bar{L}^2 & -6\bar{L} & 2\bar{L}^2 \\ -12 & -6\bar{L} & 12 & -6\bar{L} \\ 6\bar{L} & 2\bar{L}^2 & -6\bar{L} & 4\bar{L}^2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (8.25)$$

$$\begin{bmatrix} m\bar{L} & 0 & 0 & 0 \\ 0 & \frac{m\bar{L}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_3 \\ \ddot{u}_2 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} 24 & -12 & 0 & 6\bar{L} \\ -12 & 12 & -6\bar{L} & -6\bar{L} \\ 0 & -6\bar{L} & 8\bar{L}^2 & 2\bar{L}^2 \\ 6\bar{L} & -6\bar{L} & 2\bar{L}^2 & 4\bar{L}^2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_3 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (8.26)$$

Static condensation:

$$\left[\begin{array}{cc|cc} m\bar{L} & 0 & 0 & 0 \\ 0 & \frac{m\bar{L}}{2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cdot \left[\begin{array}{c} \ddot{u}_1 \\ \ddot{u}_3 \\ \hline \ddot{u}_2 \\ \ddot{u}_4 \end{array}\right] + \frac{EI}{\bar{L}^3} \cdot \left[\begin{array}{cc|cc} 24 & -12 & 0 & 6\bar{L} \\ -12 & 12 & -6\bar{L} & -6\bar{L} \\ \hline 0 & -6\bar{L} & 8\bar{L}^2 & 2\bar{L}^2 \\ 6\bar{L} & -6\bar{L} & 2\bar{L}^2 & 4\bar{L}^2 \end{array}\right] \cdot \left[\begin{array}{c} u_1 \\ u_3 \\ \hline u_2 \\ u_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \quad (8.27)$$

$$\begin{bmatrix} \mathbf{m}_{tt} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \ddot{\mathbf{u}}_t \\ \ddot{\mathbf{u}}_0 \end{bmatrix} + \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (8.28)$$

$$\begin{cases} \mathbf{m}_{tt}\ddot{\mathbf{u}}_t + \mathbf{k}_{tt}\mathbf{u}_t + \mathbf{k}_{t0}\mathbf{u}_0 = \mathbf{0} \\ \mathbf{k}_{0t}\mathbf{u}_t + \mathbf{k}_{00}\mathbf{u}_0 = \mathbf{0} \end{cases} \tag{8.29}$$

From the second row of Equation (8.29) the following expression can be derived:

$$\mathbf{u}_0 = -\mathbf{k}_{00}^{-1}\mathbf{k}_{0t}\mathbf{u}_t \tag{8.30}$$

Substituting Equation (8.30) in the first line of Equation (8.29) we obtain:

$$\mathbf{m}_{tt}\ddot{\mathbf{u}}_t + \mathbf{k}_{tt}\mathbf{u}_t - \mathbf{k}_{t0}\mathbf{k}_{00}^{-1}\mathbf{k}_{0t}\mathbf{u}_t = \mathbf{0} \tag{8.31}$$

$$\mathbf{m}_{tt}\ddot{\mathbf{u}}_t + (\mathbf{k}_{tt} - \mathbf{k}_{t0}\mathbf{k}_{00}^{-1}\mathbf{k}_{0t})\mathbf{u}_t = \mathbf{0} \tag{8.32}$$

and with $\mathbf{k}_{t0} = \mathbf{k}_{0t}^T$:

$$\mathbf{m}_{tt}\ddot{\mathbf{u}}_t + (\mathbf{k}_{tt} - \mathbf{k}_{0t}^T\mathbf{k}_{00}^{-1}\mathbf{k}_{0t})\mathbf{u}_t = \mathbf{0} \tag{8.33}$$

$$\mathbf{m}_{tt}\ddot{\mathbf{u}}_t + \hat{\mathbf{k}}_{tt}\mathbf{u}_t = \mathbf{0} \text{ with } \hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{0t}^T\mathbf{k}_{00}^{-1}\mathbf{k}_{0t} \tag{8.34}$$

Where $\hat{\mathbf{k}}_{tt}$ is the condensed stiffness matrix, and in our case it is equal to:

$$\hat{k}_{tt} = \frac{EI}{\bar{L}^3}\cdot\left(\begin{bmatrix} 24 & -12 \\ -12 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 6\bar{L} \\ -6\bar{L} & -6\bar{L} \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{7\bar{L}^2} & -\frac{1}{14\bar{L}^2} \\ -\frac{1}{14\bar{L}^2} & \frac{2}{7\bar{L}^2} \end{bmatrix}\cdot\begin{bmatrix} 0 & -6\bar{L} \\ 6\bar{L} & -6\bar{L} \end{bmatrix}\right) \tag{8.35}$$

$$\hat{k}_{tt} = \frac{EI}{\bar{L}^3}\cdot\frac{6}{7}\cdot\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix} \tag{8.36}$$

after substituting $\bar{L} = L/2$:

$$\hat{k}_{tt} = \frac{EI}{L^3}\cdot\frac{48}{7}\cdot\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix} \tag{8.37}$$

The final equation of motion of the cantilever is therefore:

$$\begin{bmatrix} \frac{mL}{2} & 0 \\ 0 & \frac{mL}{4} \end{bmatrix}\cdot\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_3 \end{bmatrix} + \frac{EI}{L^3}\cdot\frac{48}{7}\cdot\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.38}$$

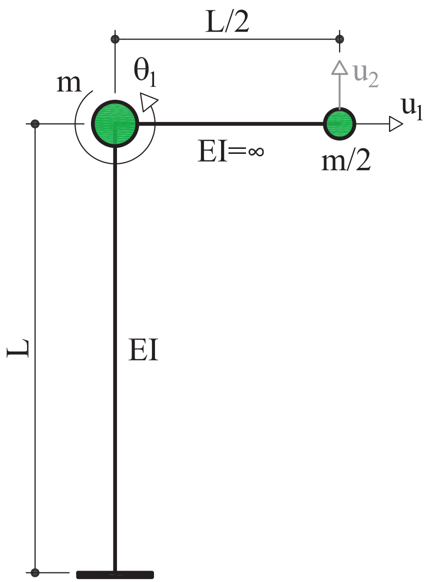
- Notes
  - The "Direct Stiffness Method" is often used in the Finite Element Method.
  - The derivation of the stiffness matrix $\mathbf{K}$ for a beam element and instructions for assembling the stiffness matrix of entire structures can be found e.g. in the following references:

[Prz85] Przemieniecki J.S.: "Theory of Matrix Structural Analysis". Dover Publications, New York 1985.

[Bat96] Bathe K-J.: "Finite Element Procedures". Prentice Hall, Upper Saddle River, 1996.

8.1.7 Change of degrees of freedom

The equation of motion for free vibration of the 2-DoF system depicted in the following can be immediately set up if the DoFs u_1 and θ_1 are considered.



Using Equation (8.23), the equation of motion for free vibrations of the system becomes

$$\begin{bmatrix} \frac{3m}{2} & 0 \\ 0 & \frac{mL^2}{8} \end{bmatrix} \cdot \begin{bmatrix} \ddot{u}_1 \\ \ddot{\theta}_1 \end{bmatrix} + \frac{EI}{L^3} \cdot \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.39}$$

or:

$$\bar{\mathbf{M}} \cdot \begin{bmatrix} \ddot{u}_1 \\ \ddot{\theta}_1 \end{bmatrix} + \bar{\mathbf{K}} \cdot \begin{bmatrix} u_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.40}$$

As an alternative, the motion of the system can be also expressed in terms of the DoFs u_1 and u_2. To this purpose, the relationship between the two sets of DoFs can be immediately written as:

$$\begin{cases} u_1 = u_1 \\ u_2 = \frac{L}{2} \cdot \theta_1 \end{cases} \tag{8.41}$$

which in matricial form yields the following system of equations:

$$\begin{bmatrix} u_1 \\ \theta_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2/L \end{bmatrix}}_{\mathbf{A}} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ or } \bar{\mathbf{u}} = \mathbf{A}\mathbf{u} \tag{8.42}$$

The matrix $\mathbf{A}$ is called coordinate transformation matrix and can be used to transform the mass matrix, the stiffness matrix and the load vector from one set of DoF to the other, i.e.

$$\mathbf{K} = \mathbf{A}^T \bar{\mathbf{K}} \mathbf{A} \tag{8.43}$$

$$\mathbf{M} = \mathbf{A}^T \bar{\mathbf{M}} \mathbf{A} \tag{8.44}$$

$$\mathbf{F} = \mathbf{A}^T \bar{\mathbf{F}} \tag{8.45}$$

For the example at hand, the stiffness matrix $\mathbf{K}$ expressed in the set of DoFs u_1 and u_2 becomes:

$$\mathbf{K} = \mathbf{A}^T\overline{\mathbf{K}}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2/L \end{bmatrix} \cdot \frac{EI}{L^3} \cdot \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2/L \end{bmatrix} \tag{8.46}$$

$$\mathbf{K} = \frac{EI}{L^3} \cdot \begin{bmatrix} 12 & -12 \\ -12 & 16 \end{bmatrix} \tag{8.47}$$

while the mass matrix $\mathbf{M}$ becomes:

$$\mathbf{M} = \mathbf{A}^T\overline{\mathbf{M}}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2/L \end{bmatrix} \cdot \begin{bmatrix} \frac{3m}{2} & 0 \\ 0 & \frac{mL^2}{8} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2/L \end{bmatrix} \tag{8.48}$$

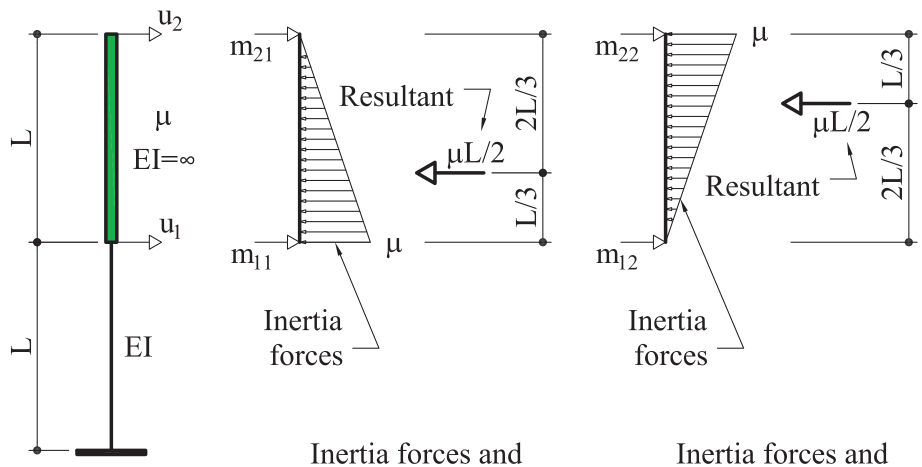
$$\mathbf{M} = \begin{bmatrix} \frac{3m}{2} & 0 \\ 0 & \frac{m}{2} \end{bmatrix} \tag{8.49}$$

which yields the equation of motion of the 2-DoF systems expressed in terms of the DoFs u_1 and u_2

$$\begin{bmatrix} \frac{3m}{2} & 0 \\ 0 & \frac{m}{2} \end{bmatrix} \cdot \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \frac{EI}{L^3} \cdot \begin{bmatrix} 12 & -12 \\ -12 & 16 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.50}$$

8.1.8 Systems incorporating rigid elements with distributed mass

The 2-DoF system depicted in the following incorporates a rigid element with distributed mass μ.



2-DoF system

Inertia forces and reactions for $\ddot{u}_1=1$ and $\ddot{u}_2=0$

Inertia forces and reactions for $\ddot{u}_1=0$ and $\ddot{u}_2=1$

The elements of the 2x2 mass matrix can be determined by imparting a unit acceleration $\ddot{u}_a=1$ to one degree of freedom while keeping the acceleration of the other degree of freedom equal to zero ($\ddot{u}_b=0$).

The resulting inertia forces are then applied as static forces acting onto the system, and the elements of the mass matrix are computed as the reactions to these static forces.

In the example at hand, if the DoFs u_1 and u_2 are considered, the elements of the mass matrix can be easily computed as follows:

$$m_{11} = \frac{2}{3}\mu L \quad (8.51)$$

$$m_{21} = \frac{1}{3}\mu L \quad (8.52)$$

$$m_{12} = \frac{1}{3}\mu L \quad (8.53)$$

$$m_{22} = \frac{2}{3}\mu L \quad (8.54)$$

Hence the mass matrix becomes:

$$\mathbf{M} = \frac{\mu L}{6} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (8.55)$$

Due to the fact that the mass is distributed, off-diagonal terms are present and therefore the mass matrix is coupled.

The stiffness matrix of the 2-DoF system can be easily computed by means of the methods discussed so far as:

$$\mathbf{K} = \frac{EI}{L^3} \cdot \begin{bmatrix} 28 & -10 \\ -10 & 4 \end{bmatrix} \quad (8.56)$$

and the equation of motion of the system for free vibration becomes:

$$\frac{\mu L}{6} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \frac{EI}{L^3} \cdot \begin{bmatrix} 28 & -10 \\ -10 & 4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (8.57)$$

9 Free Vibrations

9.1 Natural vibrations

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \quad (9.1)$$

Ansatz:

$$\mathbf{u}(t) = q_n(t)\boldsymbol{\phi}_\mathbf{n} \text{ where } q_n(t) = A_n\cos(\omega_n t) + B_n\sin(\omega_n t) \quad (9.2)$$

The double derivation of Equation (9.2) yields:

$$\ddot{q}_n(t) = -\omega_n^2[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)] = -\omega_n^2 q_n(t) \quad (9.3)$$

$$\ddot{\mathbf{u}}(t) = -\omega_n^2 q_n(t)\boldsymbol{\phi}_\mathbf{n} \quad (9.4)$$

and by substituting Equations (9.2) and (9.4) in (9.1) we obtain:

$$[-\omega_n^2\mathbf{M}\boldsymbol{\phi}_\mathbf{n} + \mathbf{K}\boldsymbol{\phi}_\mathbf{n}]q_n(t) = \mathbf{0} \quad (9.5)$$

Equation (9.5) is satisfied if $q_n(t) = 0$, which is a trivial solution meaning that there is no movement, because $\mathbf{u}(t) = q_n(t)\boldsymbol{\phi}_\mathbf{n} = 0$. To obtain a nontrivial solution the term in brackets in Equation (9.5) must be equal to zero, i.e.:

$$[-\omega_n^2\mathbf{M} + \mathbf{K}]\boldsymbol{\phi}_\mathbf{n} = \mathbf{0} \quad (9.6)$$

or:

$$\mathbf{A}\boldsymbol{\phi}_\mathbf{n} = \mathbf{0} \text{ with } \mathbf{A} = -\omega_n^2\mathbf{M} + \mathbf{K} \quad (9.7)$$

Also in the case of Equation (9.7), there is always the trivial solution $\boldsymbol{\phi}_\mathbf{n} = \mathbf{0}$, which corresponds to an absence of movement.

If the matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$, then Equation (9.7) can be rearranged as follows:

$$\mathbf{A}^{-1}\mathbf{A}\boldsymbol{\phi}_\mathbf{n} = \mathbf{A}^{-1}\mathbf{0} \quad (9.8)$$

and therefore

$$\boldsymbol{\phi}_\mathbf{n} = \mathbf{0} \quad (9.9)$$

This means that if matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$, then Equations (9.6) and (9.7) have only the trivial solution given by Equation (9.9).

The inverse of Matrix $\mathbf{A}$ has the form:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\hat{\mathbf{A}} \quad (9.10)$$

If the determinant $|\mathbf{A}|$ is equal to zero, then the matrix is singular and has no inverse.

Therefore, Equation (9.6) has a nontrivial solution only if:

$$\left|-\omega_n^2\mathbf{M} + \mathbf{K}\right| = 0 \quad (9.11)$$

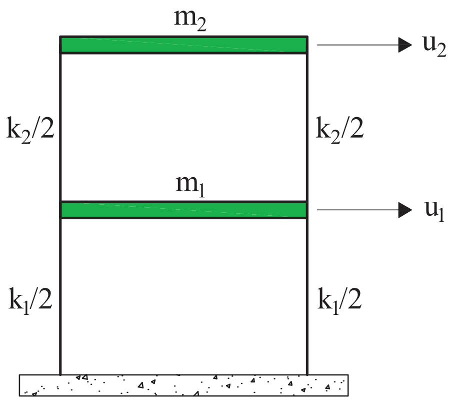
The determinant yields a polynomial of order N in ω_n^2 which is called **characteristic equation**. The N roots of the characteristic equation are called **eigenvalues** and allow the calculation of the N **natural circular frequencies** ω_n of the system.

As soon as the natural circular frequencies ω_n are computed, also the vectors $\boldsymbol{\phi}_n$ can be computed within a multiplicative constant by means of Equation (9.6). There are N independent Vectors which are called **eigenvectors** or **natural modes of vibration** of the system.

Summary

- A MDoF system with N degrees of freedom has N circular frequencies ω_n ($n = 1, 2, 3, \ldots, N$) and N eigenvectors. Each eigenvector has N elements. The circular frequencies are arranged in ascending order, i.e.: $\omega_1 < \omega_2 < \ldots < \omega_n$.
- Natural circular frequencies and eigenvectors are properties of the MDoF system and depends only from its mass and stiffness properties.
- The index n refers to the numbering of the eigenvectors and the first mode of vibration ($n = 1$) is commonly referred to as the **fundamental mode of vibration**.

9.2 Example: 2-DoF system



We consider a regular 2-DoF oscillator with

$$m_1 = m_2 = m$$

and

$$k_1 = k_2 = k$$

The equation of motion of the system corresponds to equation (8.7):

$$m\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\ddot{u}_1\\\ddot{u}_2\end{bmatrix} + k\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{9.12}$$

9.2.1 Eigenvalues

The eigenvalues are calculated from the determinant:

$$\left|\mathbf{K} - \omega_n^2\mathbf{M}\right| = \left|\begin{bmatrix}2k - \omega_n^2 m & -k\\-k & k-\omega_n^2 m\end{bmatrix}\right| = 0 \tag{9.13}$$

which gives a quadratic equation in ω_n^2

$$(2k - \omega_n^2 m)\cdot(k-\omega_n^2 m) - (-k)\cdot(-k) = m^2\omega_n^4 - 3km\omega_n^2 + k^2 = 0 \tag{9.14}$$

and both solutions yield the following eigenvalues:

$$\omega_n^2 = \frac{3km \pm \sqrt{9k^2m^2 - 4k^2m^2}}{2m^2} = \frac{3 \pm \sqrt{5}}{2} \cdot \frac{k}{m} \tag{9.15}$$

For each eigenvalue ω_n^2 we can now compute an eigenvector and a natural circular frequency.

9.2.2 Fundamental mode of vibration

With the smallest eigenvalue $\omega_1^2 = \frac{3-\sqrt{5}}{2} \cdot \frac{k}{m}$ we obtain the

1. circular frequency $\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2} \cdot \frac{k}{m}} = 0.618\sqrt{\frac{k}{m}}$ (9.16)

By substituting this eigenvalue ω_1^2 into the system of equations

$$[\mathbf{K} - \omega_1^2\mathbf{M}]\phi_1 = \begin{bmatrix} 2k - \left(\frac{3-\sqrt{5}}{2} \cdot \frac{k}{m}\right)m & -k \\ -k & k - \left(\frac{3-\sqrt{5}}{2} \cdot \frac{k}{m}\right)m \end{bmatrix} \cdot \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9.17}$$

we obtain two independent equations that can be used to determine the elements of the first eigenvector ϕ_1. The first row of the system yield the equation:

$$\frac{(1+\sqrt{5})k}{2}\phi_{11} - k\phi_{21} = 0 \quad \text{and} \quad \phi_{21} = \frac{(1+\sqrt{5})}{2}\phi_{11} \tag{9.18}$$

and by substituting this into the second row we obtain:

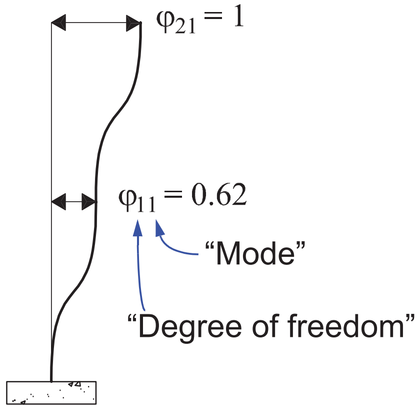
$$-k\phi_{11} + \frac{(-1+\sqrt{5})k}{2}\left(\frac{(1+\sqrt{5})}{2}\phi_{11}\right) = 0 \tag{9.19}$$

$$-k\phi_{11} + k\phi_{11} = 0$$

$$\phi_{11} = \phi_{11}$$

As expected, the eigenvector is determined within a multiplicative constant, and can therefore be arbitrarily normalized as follows:

- so that the largest element of the eigenvector is equal to 1
- so that one particular element of the eigenvector is equal to 1
- so that the norm of the eigenvector is equal to 1
- ...



Fundamental mode:

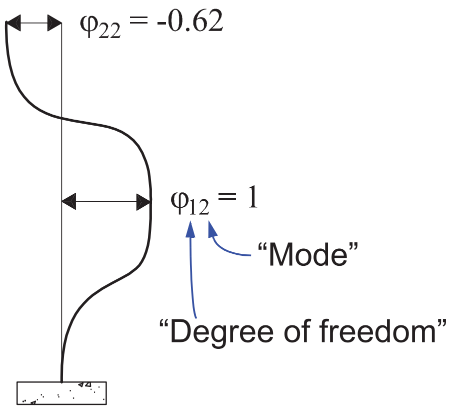
$$\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2} \cdot \frac{k}{m}} = 0.618\sqrt{\frac{k}{m}}$$

$$\phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} \frac{2}{1+\sqrt{5}} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.618 \\ 1 \end{bmatrix}$$

9.2.3 Higher modes of vibration

Additionally to the fundamental mode of vibration, the considered 2-DoF system has a second mode of vibration.

The properties of this second mode of vibration can be computed in analogy to the fundamental mode and the following results are obtained:



Second mode

$$\omega_2 = \sqrt{\frac{3+\sqrt{5}}{2} \cdot \frac{k}{m}} = 1.618\sqrt{\frac{k}{m}}$$

$$\phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.618 \end{bmatrix}$$

9.2.4 Free vibrations of the 2-DoF system

According to Equation (9.2), the free vibration of the 2-DoF system is:

$$\mathbf{u} = [C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)]\boldsymbol{\phi}_1 + [C_3\cos(\omega_2 t) + C_4\sin(\omega_2 t)]\boldsymbol{\phi}_2 \tag{9.20}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = [C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)]\begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} + [C_3\cos(\omega_2 t) + C_4\sin(\omega_2 t)]\begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} \tag{9.21}$$

The still unknown constants C_1 to C_4 can be computed using the initial conditions given by Equation (9.24) and become:

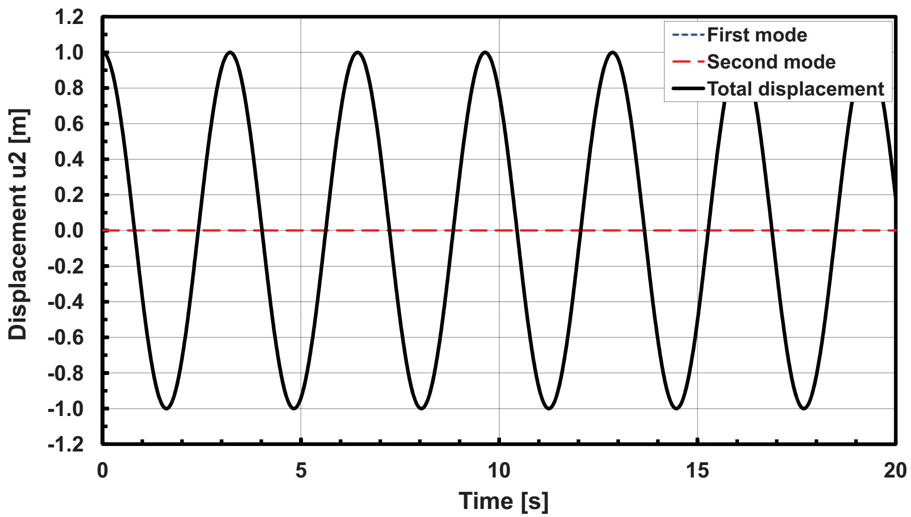
$$C_1 = \frac{\phi_{22}u1 - \phi_{12}u2}{\phi_{11}\phi_{22} - \phi_{21}\phi_{12}} \quad , \quad C_2 = \frac{\phi_{22}v1 - \phi_{12}v2}{(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})\omega_1} \tag{9.22}$$

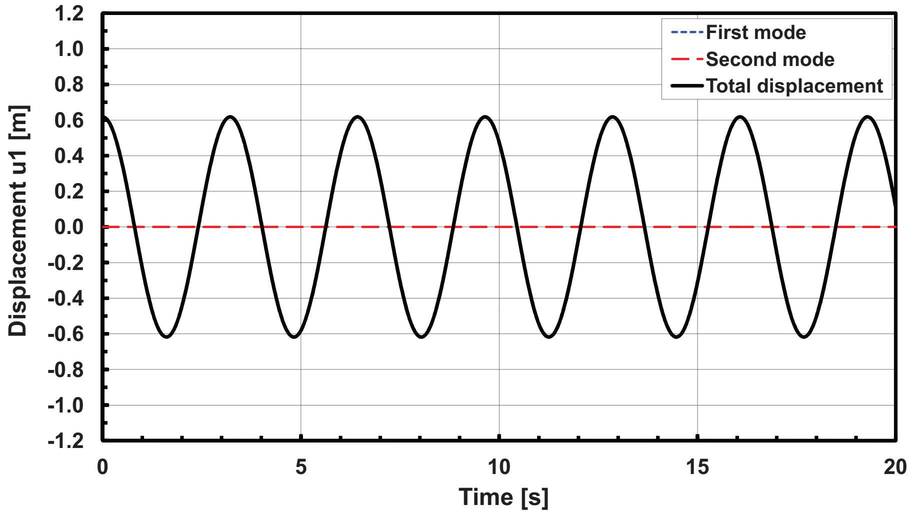
$$C_3 = \frac{\phi_{11}u2 - \phi_{21}u1}{\phi_{11}\phi_{22} - \phi_{21}\phi_{12}} \quad , \quad C_4 = \frac{\phi_{11}v2 - \phi_{21}v1}{(\phi_{11}\phi_{22} - \phi_{21}\phi_{12})\omega_2} \tag{9.23}$$

$$\text{Initial conditions:} \begin{cases} u_1(0) = u1 \\ u_2(0) = u2 \\ \dot{u}_1(0) = v1 \\ \dot{u}_2(0) = v2 \end{cases} \tag{9.24}$$

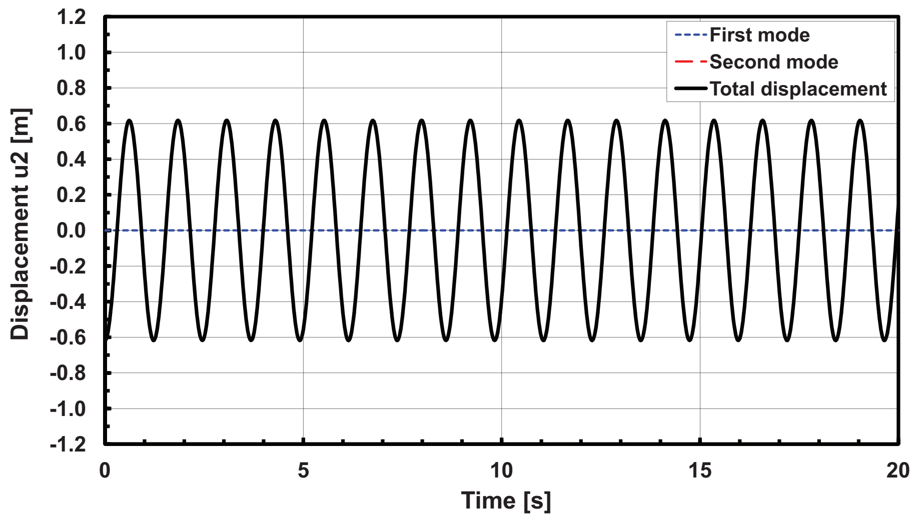
For an alternative methodology to compute the constants C_1 to C_4 see Section 9.6.

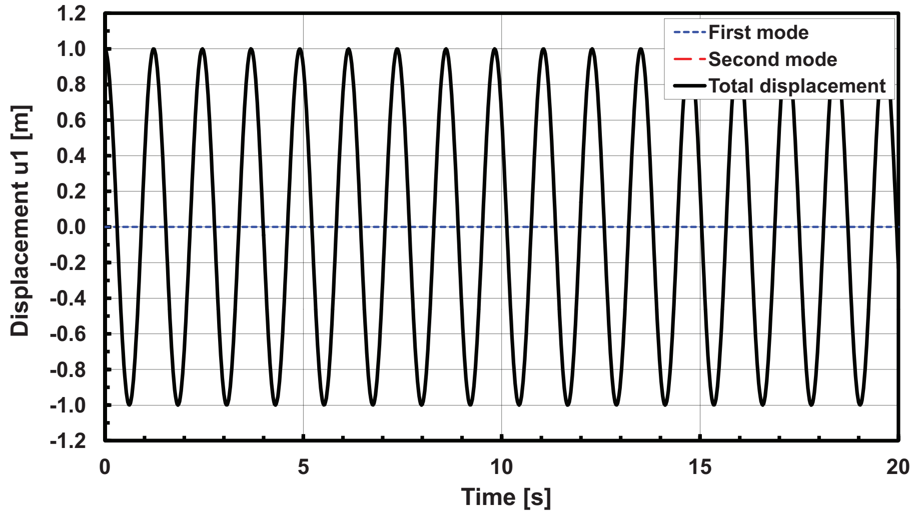
- Case 1: u1 = 0.618, u2 = 1.000, v1 = v2 = 0



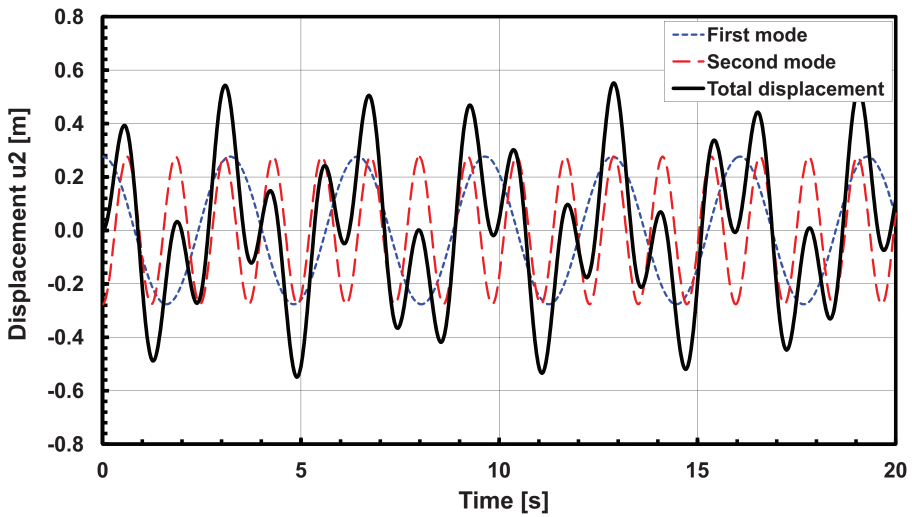


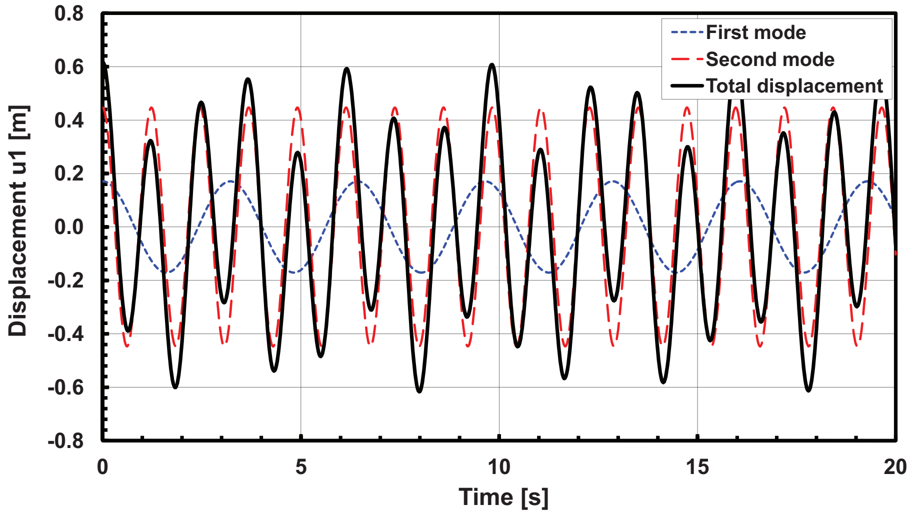
- Case 2: u1 = 1.000, u2 = –0.618, v1 = v2 = 0





- Case 3: $u1 = 0.618, u2 = 0.000, v1 = v2 = 0$





9.3 Modal matrix and Spectral matrix

All N eigenvalues and all N eigenvectors can be compactly represented in matricial form:

- Modal matrix

$$\mathbf{\Phi} = [\phi_{jn}] = \begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2N} \\ \dots & \dots & \dots & \dots \\ \phi_{N1} & \phi_{N2} & \dots & \phi_{NN} \end{bmatrix} \tag{9.25}$$

- Spectral matrix

$$\mathbf{\Omega}^2 = \begin{bmatrix} \omega_1^2 & 0 & \dots & 0 \\ 0 & \omega_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \omega_N^2 \end{bmatrix} \tag{9.26}$$

Equation (9.6) can therefore be rearranged as follows:

$$\mathbf{K}\boldsymbol{\phi}_\mathbf{n} = \mathbf{M}\boldsymbol{\phi}_\mathbf{n}\omega_n^2 \tag{9.27}$$

and it is immediately apparent that the equation for all eigenvalues and all eigenvectors can be expressed in terms of modal and the spectral matrices, as follows:

$$\mathbf{K\Phi} = \mathbf{M\Phi\Omega}^2 \tag{9.28}$$

9.4 Properties of the eigenvectors

9.4.1 Orthogonality of eigenvectors

The orthogonality conditions of the eigenvectors are:

$$\phi_n^T K \phi_r = 0 \text{ and } \phi_n^T M \phi_r = 0 \text{ for } n \neq r \tag{9.29}$$

and can be proven by means of Equation (9.27). Equation (9.27) is first to be set up for the eigenvector vector n, and then pre-multiplied with ϕ_r^T on both sides:

$$\phi_r^T K \phi_n = \omega_n^2 \phi_r^T M \phi_n \tag{9.30}$$

Afterwards, Equation (9.30) shall be transposed making use of the symmetry properties of the matrices $K^T = K$ and $M^T = M$:

$$\phi_n^T K \phi_r = \omega_n^2 \phi_n^T M \phi_r \tag{9.31}$$

Now, Equation (9.27) shall be set up for the eigenvector vector r, and then pre-multiplied with ϕ_n^T on both sides:

$$\phi_n^T K \phi_r = \omega_r^2 \phi_n^T M \phi_r \tag{9.32}$$

Equation (9.32) can now be subtracted from Equation (9.31) yielding the following equation:

$$(\omega_n^2 - \omega_r^2)\phi_n^T M \phi_r = 0 \tag{9.33}$$

In the case that the eigenvalues are different, then for $n \neq r$ we have $(\omega_n^2 - \omega_r^2) \neq 0$ and the expression $\phi_n^T M \phi_r$ must be zero. In the case that an eigenvalue occurs more than once, the eigen-

vectors are linearly independent and can be chosen so that they are orthogonal (proof complicated).

So far we have shown that $\phi_n^T M \phi_r = 0$ for $n \neq r$. By means of Equation (9.32) we can prove also the $\phi_n^T K \phi_r = 0$ for $n \neq r$. We have already seen that for $n \neq r$ the right hand side of Equation (9.32) is equal to zero. For this reason also the left hand side of Equation (9.32) must be equal to zero, which conclude the verification.

Example: 2-DoF system

In the following the orthogonality of the eigenvectors of the 2-DoF system presented in Section 9.2 is checked:

- Relative to the mass matrix

$$\phi_1^T M \phi_1 = \begin{bmatrix} \frac{2}{1+\sqrt{5}} & 1 \end{bmatrix} \cdot \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{1+\sqrt{5}} \\ 1 \end{bmatrix} = \frac{2m(5+\sqrt{5})}{(1+\sqrt{5})^2} \cong 1.382m \tag{9.34}$$

$$\phi_1^T M \phi_2 = \begin{bmatrix} \frac{2}{1+\sqrt{5}} & 1 \end{bmatrix} \cdot \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = 0 \tag{9.35}$$

$$\phi_2^T M \phi_1 = \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \cdot \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{1+\sqrt{5}} \\ 1 \end{bmatrix} = 0 \tag{9.36}$$

$$\phi_2^{\mathbf{T}}\mathbf{M}\phi_2 = \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \cdot \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \frac{m}{2}(5-\sqrt{5}) \approx 1.382m \quad (9.37)$$

• Relative to the stiffness matrix

$$\phi_1^{\mathbf{T}}\mathbf{K}\phi_1 = \begin{bmatrix} \frac{2}{1+\sqrt{5}} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{1+\sqrt{5}} \\ 1 \end{bmatrix} = \frac{2k(5-\sqrt{5})}{(1+\sqrt{5})^2} \cong 0.528k \quad (9.38)$$

$$\phi_1^{\mathbf{T}}\mathbf{K}\phi_2 = \begin{bmatrix} \frac{2}{1+\sqrt{5}} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = 0 \quad (9.39)$$

$$\phi_2^{\mathbf{T}}\mathbf{K}\phi_1 = \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \cdot \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{1+\sqrt{5}} \\ 1 \end{bmatrix} = 0 \quad (9.40)$$

$$\phi_2^{\mathbf{T}}\mathbf{K}\phi_2 = \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \cdot \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \frac{k}{2}(5+\sqrt{5}) \approx 3.618k \quad (9.41)$$

9.4.2 Linear independence of the eigenvectors

The eigenvectors are linearly independent. To prove this, it needs to be shown that if

$$\alpha_1\phi_1 + \alpha_2\phi_2 + \ldots + \alpha_n\phi_n = 0 \quad (9.42)$$

then all scalars α_i must be equal to zero.

To this purpose, we left-multiply Equation (9.42) by $\phi_{\mathbf{i}}^{\mathbf{T}}\mathbf{M}$ and we obtain:

$$\phi_{\mathbf{i}}^{\mathbf{T}}\mathbf{M}(\alpha_1\phi_1 + \alpha_2\phi_2 + \ldots + \alpha_n\phi_n) = \phi_{\mathbf{i}}^{\mathbf{T}}\mathbf{M}\phi_{\mathbf{i}}\alpha_i = 0 \quad (9.43)$$

In Section 9.4.1 we have shown that $\phi_{\mathbf{i}}^{\mathbf{T}}\mathbf{M}\phi_{\mathbf{i}} \neq 0$, therefore $\alpha_i = 0$ meaning that the eigenvectors are linearly independent.

The property that the eigenvectors are linearly independent, is very important because it allows to represent any displacement vector as a linear combination of the eigenvectors.

9.5 Decoupling of the equation of motion

The equation of motion for free vibrations is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{9.44}$$

and as a possible solution the displacement vector

$$\mathbf{u}(t) = \sum_i q_i(t)\phi_i \tag{9.45}$$

can be assumed, where:

ϕ_i: linearly independent eigenvectors of the system

q_i: modal coordinates

The displacement vector $\mathbf{u}(t)$ and its double derivative

$$\ddot{\mathbf{u}}(t) = \sum_i \ddot{q}_i(t)\boldsymbol{\phi}_i \tag{9.46}$$

can be substituted into Equation (9.44), and the latter can be left-multiplied by $\phi_n^{\mathbf{T}}$ yielding the following equation:

$$\phi_n^{\mathbf{T}}\mathbf{M}\left(\sum_i \ddot{q}_i(t)\boldsymbol{\phi}_i\right) + \phi_n^{\mathbf{T}}\mathbf{K}\left(\sum_i q_i(t)\boldsymbol{\phi}_i\right) = \mathbf{0} \tag{9.47}$$

Because of the orthogonality properties of the eigenvectors only one term of the summations remains, i.e.:

$$\phi_n^{\mathbf{T}}\mathbf{M}\phi_n\ddot{q}_n(t) + \phi_n^{\mathbf{T}}\mathbf{K}\phi_n q_n(t) = 0 \tag{9.48}$$

where:

Modal mass: $$m_n^* = \phi_n^{\mathbf{T}}\mathbf{M}\phi_n \tag{9.49}$$

Modal stiffness: $$k_n^* = \phi_n^{\mathbf{T}}\mathbf{K}\phi_n \tag{9.50}$$

and Equation (9.48) can be rewritten as follows:

$$m_n^*\ddot{q}_n(t) + k_n^* q_n(t) = 0 \tag{9.51}$$

For each n we can set up such an equation, which yields to N decoupled Single Degree of Freedom systems. The total displacement of the system can then be computes as the sum of the contribution of all decoupled SDoF systems, i.e.:

$$\mathbf{u}(t) = \sum_{i=1}^{N} q_i(t)\phi_i \tag{9.52}$$

in matricial form:

$$\mathbf{u}(t) = \mathbf{\Phi}\mathbf{q}(t) \text{ with } q = \begin{bmatrix} q_1(t) \\ \dots \\ q_N(t) \end{bmatrix} \tag{9.53}$$

$$\mathbf{M}^*\ddot{\mathbf{q}} + \mathbf{K}^*\mathbf{q} = \mathbf{0} \tag{9.54}$$

with

$$\mathbf{M}^* = \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \begin{bmatrix} m_1^* & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & m_n^* \end{bmatrix} \text{ and } \mathbf{K}^* = \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \begin{bmatrix} k_1^* & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & k_n^* \end{bmatrix} \tag{9.55}$$

In Equations (9.52) and (9.53) are rewritten

$$\mathbf{u}(t) = \sum_{i=1}^{N} q_i(t)\phi_i = \mathbf{\Phi q}(t) \tag{9.56}$$

the LHS and the RHS of the resulting equation can be pre-multiplied by $\phi_n^{\mathbf{T}}\mathbf{M}$ and we obtain:

$$\phi_n^{\mathbf{T}}\mathbf{Mu}(t) = \sum_{i=1}^{N} \phi_n^{\mathbf{T}}\mathbf{M}\phi_i q_i(t) \tag{9.57}$$

Because of the orthogonality of the eigenvectors (see Section 9.4.1), Equation can be further simplified to:

$$\phi_n^{\mathbf{T}}\mathbf{Mu}(t) = \phi_n^{\mathbf{T}}\mathbf{M}\phi_n q_n(t) \tag{9.58}$$

which yields the following relationship between $q_n(t)$ and $\mathbf{u}(t)$

$$q_n(t) = \frac{\phi_n^{\mathbf{T}}\mathbf{Mu}(t)}{\phi_n^{\mathbf{T}}\mathbf{M}\phi_n} \tag{9.59}$$

or introducing the definition of the modal mass given by Equation (9.49) we obtain the equivalent expression

$$q_n(t) = \frac{\phi_n^{\mathbf{T}}\mathbf{Mu}(t)}{m_n^*} \tag{9.60}$$

Equations (9.59) and (9.60) will be later used to compute the response of MDoF systems.

Example: 2-DoF system

The modal masses and modal stiffness of the 2-DoF system of Section 9.2 were already checked during the verification of the orthogonality of the eigenvectors (See Equations (9.34), (9.37), (9.38) and (9.41)). They are:

$$m_1^* = \phi_1^{\mathbf{T}}\mathbf{M}\phi_1 = \frac{2m(5+\sqrt{5})}{(1-\sqrt{5})^2} \approx 1.382m \tag{9.61}$$

$$m_2^* = \phi_2^{\mathbf{T}}\mathbf{M}\phi_2 = \frac{m}{2}(5-\sqrt{5}) \approx 1.382m \tag{9.62}$$

$$k_1^* = \phi_1^{\mathbf{T}}\mathbf{K}\phi_1 = \frac{2k(5-\sqrt{5})}{(1-\sqrt{5})^2} \cong 0.528k \tag{9.63}$$

$$k_2^* = \phi_2^{\mathbf{T}}\mathbf{K}\phi_2 = \frac{k}{2}(5+\sqrt{5}) \approx 3.618k \tag{9.64}$$

- First modal SDoF system:

$$m_1^*\ddot{q}_1(t) + k_1^*q_1(t) = 0 \tag{9.65}$$

$$1.382m\ddot{q}_1(t) + 0.528kq_1(t) = 0 \tag{9.66}$$

$$\omega_1 = \sqrt{\frac{k_1^*}{m_1^*}} = \sqrt{\frac{0.528k}{1.382m}} = 0.618\sqrt{\frac{k}{m}} \tag{9.67}$$

The natural frequency corresponds to equation (9.16) of this chapter

- Second modal SDoF system:

$$m_2^*\ddot{q}_2(t) + k_2^* q_2(t) = 0 \tag{9.68}$$

$$1.382m\ddot{q}_2(t) + 3.618kq_2(t) = 0 \tag{9.69}$$

$$\omega_2 = \sqrt{\frac{k_2^*}{m_2^*}} = \sqrt{\frac{3.618k}{1.382m}} = 1.618\sqrt{\frac{k}{m}} \tag{9.70}$$

The natural frequency corresponds to the result shown on page 9-7.

9.6 Free vibration response

9.6.1 Systems without damping

The equation of motion for free vibration of a system without damping is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{9.71}$$

and making use of the possibility of decoupling of the equation of motion, the total deformation $\mathbf{u}(t)$ under free vibration can be computed as the sum of the contribution of all modes. The equation of motion of the n^{th} decoupled SDoF system is:

$$m_n^*\ddot{q}_n(t) + k_n^* q_n(t) = 0 \tag{9.72}$$

and its solution can be computed as discussed in Chapter 3 for SDoF systems. If we make use of the second formulation with “trigonometric functions” (see Section 3.1.2), the solution is:

$$q_n(t) = A_n\cos(\omega_n t) + B_n\sin(\omega_n t) \tag{9.73}$$

The the total deformation $\mathbf{u}(t)$ under free vibration is hence

$$\mathbf{u}(t) = \sum_{i=1}^{N}\phi_i q_i(t) = \sum_{i=1}^{N}\phi_i[A_i\cos(\omega_i t) + B_i\sin(\omega_i t)] \tag{9.74}$$

The $2 \cdot N$ constants A_i and B_i can be computed by means of the initial conditions $\mathbf{u}(0) = \mathbf{u_0}$ and $\dot{\mathbf{u}}(0) = \mathbf{v_0}$.

To this purpose, the vector of the velocity is needed and can easily be computed by deriving Equation (9.74), i.e.:

$$\dot{\mathbf{u}}(t) = \sum_{i=1}^{N} \phi_i \dot{q}_i(t) = \sum_{i=1}^{N} \phi_i \omega_i [-A_i \sin(\omega_i t) + B_i \cos(\omega_i t)] \quad (9.75)$$

Considering Equations (9.74) and (9.75) at the time $t = 0$, we have.

$$\mathbf{u}(0) = \sum_{i=1}^{N} \phi_i q_i(0) \text{ and } \dot{\mathbf{u}}(0) = \sum_{i=1}^{N} \phi_i \dot{q}_i(0) \quad (9.76)$$

Making use of Equation (9.59) we can now write the equations to compute the initial conditions of the n^{th} decoupled SDoF system as:

$$q_n(0) = \frac{\phi_n^T \mathbf{M} \mathbf{u}(0)}{\phi_n^T \mathbf{M} \phi_n} \quad (9.77)$$

$$\dot{q}_n(0) = \frac{\phi_n^T \mathbf{M} \dot{\mathbf{u}}(0)}{\phi_n^T \mathbf{M} \phi_n} \quad (9.78)$$

In Section 3.1.2 (see Equation 3.18) it as been shown that the constants A_n and B_n are equal to $q_n(0)$ and $\dot{q}_n(0)/\omega_n$, respectively, hence Equation (9.74) can be rewritten as:

$$\mathbf{u}(t) = \sum_{i=1}^{N} \phi_i \left[q_i(0) \cos(\omega_i t) + \frac{\dot{q}_i(0)}{\omega_i} \sin(\omega_i t) \right] \quad (9.79)$$

9.6.2 Classically damped systems

The equation of motion for free vibration of a system with damping is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \quad (9.80)$$

As it will be shown in Chapter 10, if the MDoF system is classically damped, the equation of motion can be decoupled analogously to system without damping, and the total deformation $\mathbf{u}(t)$ under free vibration can be computed again as the sum of the contribution of all modes. The equation of motion of the n^{th} decoupled SDoF system is:

$$m_n^* \ddot{q}_n(t) + c_n^* \dot{q}_n(t) + k_n^* q_n(t) = 0 \quad (9.81)$$

or

$$\ddot{q}_n(t) + 2\omega_n \zeta_n q_n(t) + \omega_n^2 q_n(t) = 0 \quad (9.82)$$

where

$$c_n^* = \phi_n^T \mathbf{C} \phi_n \text{ and } \zeta_n = \frac{c_n^*}{2 m_n^* \omega_n} \text{, respectively.} \quad (9.83)$$

The solution of Equation (9.82) can be computed as discussed in Chapter 3 for SDoF systems. According to Equation (3.50) we have:

$$q_n(t) = e^{-\zeta \omega_n t} [A_n \cos(\omega_{nd} t) + B_n \sin(\omega_{nd} t)] \quad (9.84)$$

where:

$$\omega_{nd} = \omega_n\sqrt{1-\zeta^2}$$ **"damped circular frequency of the n^th mode"**

(9.85)

The the total deformation $\mathbf{u}(t)$ under free vibration is hence

$$\mathbf{u}(t) = \sum_{i=1}^{N} \phi_i q_i(t) = \sum_{i=1}^{N} \phi_i e^{-\zeta\omega_n t}[A_i\cos(\omega_{id}t) + B_i\sin(\omega_{id}t)] \tag{9.86}$$

As in the case of the undamped systems, the $2 \cdot N$ constants A_i and B_i can be computed by means of the initial conditions $\mathbf{u}(0) = \mathbf{u_0}$ and $\dot{\mathbf{u}}(0) = \mathbf{v_0}$.

For the n^th decoupled SDoF system according to Equation (3.51) the constants A_n and B_n can be expressed in function of the initial conditions of the modal coordinate q as follows:

$$A_n = q_n(0) \tag{9.87}$$

$$B_n = \frac{\dot{q}_n(0) + \zeta\omega_n q_n(0)}{\omega_{nd}} \tag{9.88}$$

where the initial displacement $q_n(0)$ and the initial velocity $\dot{q}_n(0)$ can be computed by means of Equations (9.77) and (9.78).

Hence, the total displacement of a classically damped MDoF system under free vibration can be computed as:

$$\mathbf{u}(t) = \sum_{i=1}^{N} \phi_i e^{-\zeta\omega_n t}\left[q_i(0)\cos(\omega_{id}t) + \frac{\dot{q}_i(0) + \zeta\omega_i q_i(0)}{\omega_{id}}\sin(\omega_{id}t)\right] \tag{9.89}$$

For nonclassically damped system see e.g. [Cho11], Chapter 14.

Blank Page

10 Damping

10.1 Free vibrations with damping

The differential equation to compute the free vibrations of a MDoF system is:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{10.1}$$

with the initial conditions:

$$\mathbf{u}(0) = \mathbf{u_0} \qquad \text{and} \qquad \dot{\mathbf{u}}(0) = \mathbf{v_0} \tag{10.2}$$

The displacement vector $\mathbf{u}(t)$ may be expressed as linear combination of the eigenvectors, i.e. $\mathbf{u}(t) = \mathbf{\Phi q}(t)$, and Equation (10.1) becomes:

$$\mathbf{M\Phi}\ddot{\mathbf{q}} + \mathbf{C\Phi}\dot{\mathbf{q}} + \mathbf{K\Phi q} = \mathbf{0} \tag{10.3}$$

Equation (10.3) may be further multiplied by $\mathbf{\Phi}^T$ yielding the following equations:

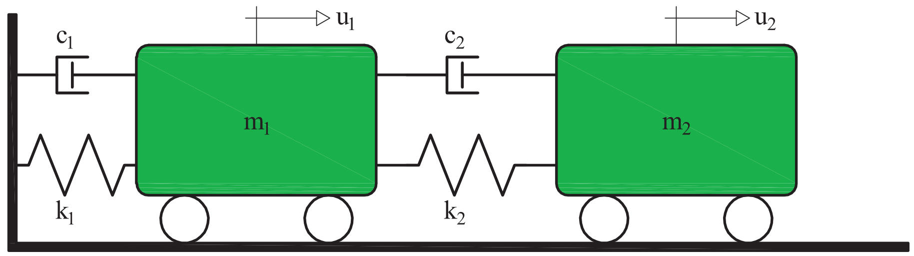
$$\mathbf{\Phi}^T\mathbf{M\Phi}\ddot{\mathbf{q}} + \mathbf{\Phi}^T\mathbf{C\Phi}\dot{\mathbf{q}} + \mathbf{\Phi}^T\mathbf{K\Phi q} = \mathbf{0} \tag{10.4}$$

$$\mathbf{M}^*\ddot{\mathbf{q}} + \mathbf{C}^*\dot{\mathbf{q}} + \mathbf{K}^*\mathbf{q} = \mathbf{0} \tag{10.5}$$

Definition:

- A system is classically damped if the matrix $\mathbf{C}^*$ is diagonal
- A system is non-classically damped if the matrix $\mathbf{C}^*$ is not diagonal

10.2 Example



The properties of the 2-DoF system are:

$$m_1 = 2m \quad , \quad m_2 = m \tag{10.6}$$

$$k_1 = 2k \quad , \quad k_2 = k \tag{10.7}$$

while the damping characteristics will be defined later.

- **Natural frequencies and eigenvectors**

The natural frequencies and the eigenvectors of the 2-DoF system can be easily computed as:

$$\text{Natural frequencies:} \quad \omega_1 = \sqrt{\frac{k}{2m}}, \quad \omega_2 = \sqrt{\frac{2k}{m}} \tag{10.8}$$

$$\text{Eigenvectors:} \quad \phi_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{10.9}$$

10.2.1 Non-classical damping

The damping characteristics of the 2-DoF system are chosen as:

$$c_1 = c \quad , \quad c_2 = 4c \tag{10.10}$$

The equation of motion of the system can be easily assembled by means of the equilibrium formulation:

$$m\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\ddot{u}_1\\\ddot{u}_2\end{bmatrix} + c\begin{bmatrix}5 & -4\\-4 & 4\end{bmatrix}\begin{bmatrix}\dot{u}_1\\\dot{u}_2\end{bmatrix} + \begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{10.11}$$

It is now attempted to decouple the equations by computing the modal properties of the 2-DoF system:

$$\mathbf{M}^* = \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \begin{bmatrix}\frac{1}{2} & 1\\-1 & 1\end{bmatrix}\cdot\begin{bmatrix}2m & 0\\0 & m\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2} & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}m & 0\\0 & 3m\end{bmatrix} \tag{10.12}$$

$$\mathbf{K}^* = \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \begin{bmatrix}\frac{1}{2} & 1\\-1 & 1\end{bmatrix}\cdot\begin{bmatrix}3k & -k\\-k & k\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2} & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}\frac{3}{4}k & 0\\0 & 6k\end{bmatrix} \tag{10.13}$$

$$\mathbf{C}^* = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = \begin{bmatrix}\frac{1}{2} & 1\\-1 & 1\end{bmatrix}\cdot\begin{bmatrix}5c & -4c\\-4c & 4c\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2} & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}\frac{5}{4}c & \frac{7}{2}c\\\frac{7}{2}c & 17c\end{bmatrix} \tag{10.14}$$

The Matrix $\mathbf{C}^*$ is **not** diagonal, hence it is **not** possible to decouple the equations!

10.2.2 Classical damping

The damping characteristics of the 2-DoF system are chosen as:

$$c_1 = 4c \quad , \quad c_2 = 2c \tag{10.15}$$

The equation of motion of the system can be easily assembled by means of the equilibrium formulation:

$$m\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\ddot{u}_1\\\ddot{u}_2\end{bmatrix} + c\begin{bmatrix}6 & -2\\-2 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_1\\\dot{u}_2\end{bmatrix} + \begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{10.16}$$

It is now attempted to decouple the equations by computing the modal properties of the 2-DoF system:

$$\mathbf{M}^* = \begin{bmatrix}\frac{3}{2}m & 0\\0 & 3m\end{bmatrix} \quad , \quad \mathbf{K}^* = \begin{bmatrix}\frac{3k}{4} & 0\\0 & 6k\end{bmatrix} \tag{10.17}$$

$$\mathbf{C}^* = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = \begin{bmatrix}\frac{1}{2} & 1\\-1 & 1\end{bmatrix}\cdot\begin{bmatrix}6c & -2c\\-2c & 2c\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2} & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}c & 0\\0 & 12c\end{bmatrix} \tag{10.18}$$

The Matrix $\mathbf{C}^*$ is diagonal, hence it is possible to decouple the equations!

10.3 Classical damping matrices

10.3.1 Mass proportional damping (MpD)

$$\mathbf{C} = a_0\mathbf{M} \quad (10.19)$$

The damping constant of each mode of vibration is therefore:

$$c_n^* = a_0 m_n^* \quad (10.20)$$

and the corresponding damping ratio ζ_n becomes (see Section 3.2):

$$\zeta_n = \frac{c_n^*}{2\omega_n m_n^*} = \frac{a_0 m_n^*}{2\omega_n m_n^*} = \frac{a_0}{2\omega_n} \quad (10.21)$$

10.3.2 Stiffness proportional damping (SpD)

$$\mathbf{C} = a_1\mathbf{K} \quad (10.22)$$

The damping constant of each mode of vibration is therefore:

$$c_n^* = a_1 k_n^* = a_1 \omega_n^2 m_n^* \quad (10.23)$$

and the corresponding damping ratio ζ_n becomes:

$$\zeta_n = \frac{c_n^*}{2\omega_n m_n^*} = \frac{a_1 \omega_n^2 m_n^*}{2\omega_n m_n^*} = \frac{a_1}{2}\omega_n \quad (10.24)$$

Remark

Both **MpD** and **SpD**, taken alone, are not a good approximation of the behaviour of real structures. Studies have shown that different modes of vibration exhibit similar damping ratios.

10.3.3 Rayleigh damping

$$\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K} \quad (10.25)$$

The damping constant of each mode of vibration is therefore:

$$c_n^* = a_0 m_n^* + a_1 k_n^* = (a_0 + a_1\omega_n^2) m_n^* \quad (10.26)$$

and using the results for **MpD** and **SpD** damping ratio ζ_n becomes:

$$\zeta_n = \frac{a_0}{2\omega_n} + \frac{a_1}{2}\omega_n \quad (10.27)$$

The coefficients a_0 and a_1 may be computed for vibration modes i and j by means of equation (10.28):

$$\begin{cases} \frac{a_0}{2} \cdot \frac{1}{\omega_i} + \frac{a_1}{2} \cdot \omega_i = \zeta_i \\ \frac{a_0}{2} \cdot \frac{1}{\omega_j} + \frac{a_1}{2} \cdot \omega_j = \zeta_j \end{cases} \quad (10.28)$$

In the case that $\zeta_i = \zeta_j = \zeta$, coefficients a_0 and a_1 can be computed as follows:

$$a_0 = \zeta \cdot \frac{2\omega_i\omega_j}{\omega_i + \omega_j} \qquad a_1 = \zeta \cdot \frac{2}{\omega_i + \omega_j} \quad (10.29)$$

10.3.4 Example

A damping matrix shall be assembled so that in the case of the 2-DoF system shown on page 10-2 both modes of vibration are characterised by the same damping ratio ζ.

The natural frequencies are:

$$\omega_1 = \sqrt{\frac{k}{2m}}, \qquad \omega_2 = \sqrt{\frac{2k}{m}} \tag{10.30}$$

hence the coefficients a_0 and a_1 become:

$$a_0 = \frac{4\zeta}{3} \cdot \sqrt{\frac{m}{2k}} \cdot \frac{k}{m}, \qquad a_1 = \frac{4\zeta}{3} \cdot \sqrt{\frac{m}{2k}} \tag{10.31}$$

yielding the damping matrix $\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}$ equal to:

$$C = \frac{4\zeta}{3} \cdot \sqrt{\frac{m}{2k}} \cdot \begin{bmatrix} \frac{k}{m} \cdot 2m + 3k & 0 - k \\ 0 - k & \frac{k}{m} \cdot m + k \end{bmatrix} = \frac{4\zeta}{3} \cdot \sqrt{\frac{mk}{2}} \cdot \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \tag{10.32}$$

Check:

$$\mathbf{C}^* = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \cdot \frac{4\zeta}{3} \cdot \sqrt{\frac{mk}{2}} \cdot \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{bmatrix} = \zeta \cdot \sqrt{\frac{mk}{2}} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 12 \end{bmatrix} \tag{10.33}$$

The damping matrix is indeed diagonal.

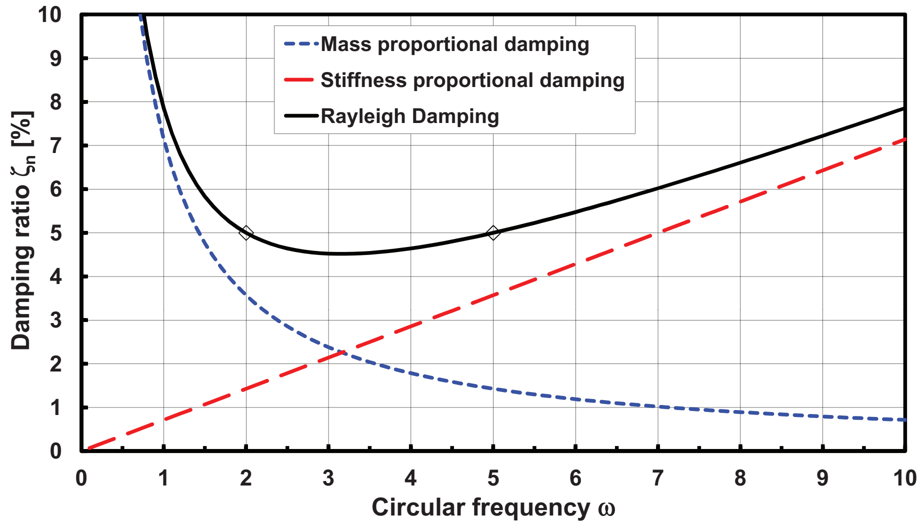
If we choose values for m, k and ζ so that

$$\omega_1 = 2\text{rad/s}, \; \omega_2 = 5\text{rad/s}, \; \zeta = 5\% \tag{10.34}$$

the coefficients a_0 and a_1 become:

$$a_0 = 14.287, \qquad a_1 = 1.429 \tag{10.35}$$

and the representation of the damping ratio in function of the natural circular frequency is:



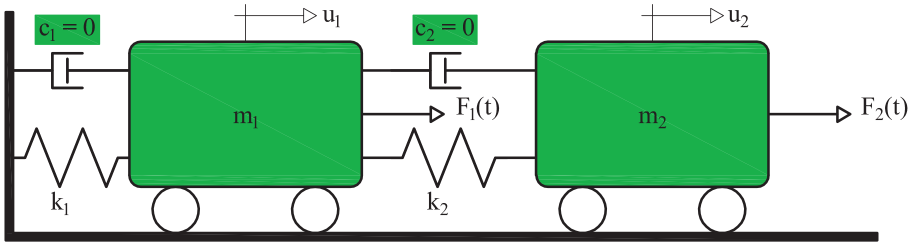
Remarks

- If there are more than 2 modes of vibrations, then not all of them will have the same damping ratio.
- If more than 2 modes of vibrations should have the same damping, then a different damping modal shall be used. To this purpose see e.g. “Caughey-Damping” in [Cho11].

11 Forced Vibrations

11.1 Forced vibrations without damping

11.1.1 Introduction



Sought is the response of the 2-DoF system as a result of the external excitation force $\mathbf{F}(t)$ given by Equation (11.1)

$$\mathbf{F}(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} \tag{11.1}$$

The equation of motion of the system is:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \tag{11.2}$$

The displacement vector $\mathbf{u}(t)$ can be represented as a linear combination of the eigenvectors of the 2-DoF system, $\mathbf{u}(t) = \mathbf{\Phi}\mathbf{q}(t)$, and Equation (11.2) becomes:

$$\mathbf{M}\mathbf{\Phi}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{\Phi}\mathbf{q} = \mathbf{F}(t) \tag{11.3}$$

We can now multiply Equation (11.3) by $\mathbf{\Phi}^{\mathbf{T}}$ obtaining:

$$\mathbf{\Phi}^{\mathbf{T}}\mathbf{M}\mathbf{\Phi}\ddot{\mathbf{q}} + \mathbf{\Phi}^{\mathbf{T}}\mathbf{K}\mathbf{\Phi}\mathbf{q} = \mathbf{\Phi}^{\mathbf{T}}\mathbf{F}(t) \tag{11.4}$$

$$\mathbf{M}^*\ddot{\mathbf{q}} + \mathbf{K}^*\mathbf{q} = \mathbf{F}^*(t) \tag{11.5}$$

Where

- $\mathbf{M}^*$: Diagonal matrix of the modal masses m_n^*
- $\mathbf{K}^*$: Diagonal matrix of the modal stiffnesses k_n^*
- $\mathbf{F}^*$: Vector of the modal forces F_n^*

For the considered 2-DoF system, Equation (11.5) can be rearranged as:

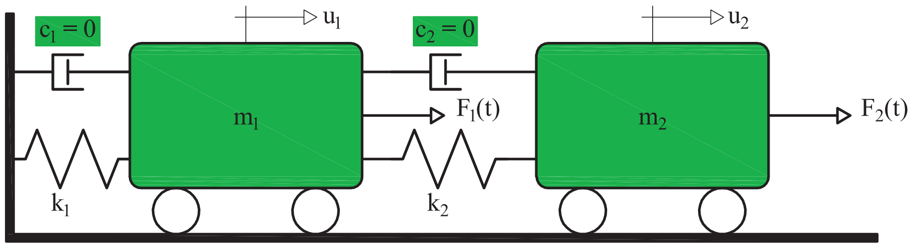
$$\begin{cases} m_1^*\ddot{q}_1 + k_1^*q_1 = F_1^* \\ m_2^*\ddot{q}_2 + k_2^*q_2 = F_2^* \end{cases} \tag{11.6}$$

or as alternative:

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = \dfrac{F_1^*}{m_1^*} \\ \ddot{q}_2 + \omega_2^2 q_2 = \dfrac{F_2^*}{m_2^*} \end{cases} \tag{11.7}$$

The two equations of the system (11.7) are decoupled and can be solved independently. The constants resulting from the solution of the system can be determined by means of the initial conditions $\mathbf{u}(0) = \mathbf{u_0}$ and $\dot{\mathbf{u}}(0) = \mathbf{v_0}$.

11.1.2 Example 1: 2-DoF system



The properties of the 2-DoF system are:

$$m_1 = 2m \quad , \quad m_2 = m \tag{11.8}$$

$$k_1 = 2k \quad , \quad k_2 = k \tag{11.9}$$

$$c_1 = 0 \quad , \quad c_2 = 0 \tag{11.10}$$

The external excitation is:

$$\mathbf{F}(t) = \begin{bmatrix} F_0 \sin(\omega t) \\ 0 \end{bmatrix} \tag{11.11}$$

and the modal excitation force is calculated using the modal matrix:

$$\mathbf{F}^*(t) = \mathbf{\Phi}^T \mathbf{F}(t) = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} F_0 \sin(\omega t) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{F_0 \sin(\omega t)}{2} \\ -F_0 \sin(\omega t) \end{bmatrix} \tag{11.12}$$

The system of equations becomes:

$$\begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = \dfrac{F_0 \sin(\omega t)}{2 \cdot (3/2)m} = \dfrac{F_0 \sin(\omega t)}{3m} = f_1 \sin(\omega t) \\ \ddot{q}_2 + \omega_2^2 q_2 = \dfrac{-F_0 \sin(\omega t)}{3m} = \dfrac{-F_0 \sin(\omega t)}{3m} = f_2 \sin(\omega t) \end{cases} \tag{11.13}$$

with

$$f_1 = \frac{F_0}{3m} \quad \text{and} \quad f_2 = -\frac{F_0}{3m} \tag{11.14}$$

Each equation of the system (11.13) corresponds to the equation of motion of an undamped SDoF system under an harmonic sine excitation. The complete solution of these differential equations has been discussed in Chapter 4 and it is:

$$q_n = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) + \frac{f_n}{\omega_n^2 - \omega^2} \sin(\omega t) \tag{11.15}$$

The two equations have the following solutions:

$$\begin{cases} q_1 = A_1 \cos(\omega_1 t) + A_2 \sin(\omega_1 t) + \dfrac{f_1}{\omega_1^2 - \omega^2} \sin(\omega t) \\ q_2 = A_3 \cos(\omega_2 t) + A_4 \sin(\omega_2 t) + \dfrac{f_2}{\omega_2^2 - \omega^2} \sin(\omega t) \end{cases} \tag{11.16}$$

The 4 constants A_1 to A_4 can be easily computed for the initial conditions $\mathbf{u}(0) = \dot{\mathbf{u}}(0) = \mathbf{0}$ by means of the mathematical software “Maple”. They are:

$$A_1 = 0 \tag{11.17}$$

$$A_2 = -\frac{\omega/\omega_1}{\omega_1^2-\omega^2}\cdot\frac{F_0}{3m} = -\frac{\omega/\omega_1}{\omega_1^2-\omega^2}\cdot f_1 \tag{11.18}$$

$$A_3 = 0 \tag{11.19}$$

$$A_4 = \frac{\omega/\omega_2}{\omega_2^2-\omega^2}\cdot\frac{F_0}{3m} = -\frac{\omega/\omega_2}{\omega_2^2-\omega^2}\cdot f_2 \tag{11.20}$$

The displacements q_n becomes:

$$\begin{cases} q_1 = \left(-\frac{\omega/\omega_1}{\omega_1^2-\omega^2}\cdot f_1\right)\sin(\omega_1 t) + \frac{f_1}{\omega_1^2-\omega^2}\sin(\omega t) \\ q_2 = \left(-\frac{\omega/\omega_2}{\omega_2^2-\omega^2}\cdot f_2\right)\sin(\omega_2 t) + \frac{f_2}{\omega_2^2-\omega^2}\sin(\omega t) \end{cases} \tag{11.21}$$

or

$$\begin{cases} q_1 = f_1\left[\frac{\sin(\omega t)-(\omega/\omega_1)\sin(\omega_1 t)}{\omega_1^2-\omega^2}\right] \\ q_2 = f_2\left[\frac{\sin(\omega t)-(\omega/\omega_2)\sin(\omega_2 t)}{\omega_2^2-\omega^2}\right] \end{cases} \tag{11.22}$$

Therefore, the total displacement $\mathbf{u}(t)$ becomes:

$$\mathbf{u}(t) = \mathbf{\Phi}\mathbf{q}(t) = \sum_n \phi_n q_n(t) = \phi_1 q_1(t) + \phi_2 q_2(t) \tag{11.23}$$

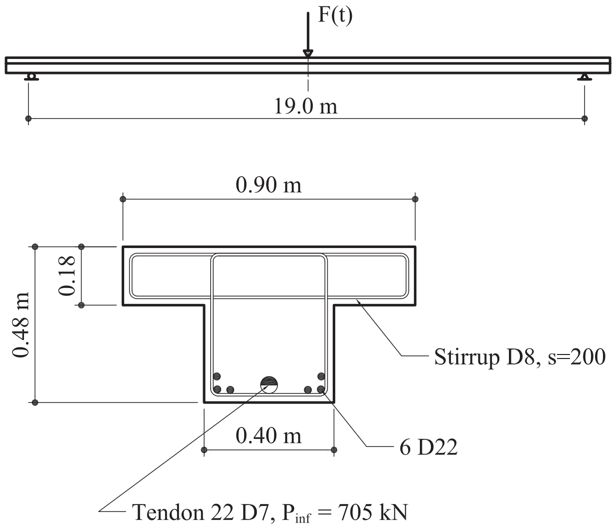
$$\mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\left(f_1\left[\frac{\sin(\omega t)-(\omega/\omega_1)\sin(\omega_1 t)}{\omega_1^2-\omega^2}\right]\right) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\left(f_2\left[\frac{\sin(\omega t)-(\omega/\omega_2)\sin(\omega_1 t)}{\omega_2^2-\omega^2}\right]\right) \tag{11.24}$$

where:

$$f_1 = \frac{F_0}{3m},\ f_2 = -\frac{F_0}{3m},\ \omega_1 = \sqrt{\frac{k}{2m}},\ \omega_2 = \sqrt{\frac{2k}{m}} \tag{11.25}$$

11.1.3 Example 2: RC beam with Tuned Mass Damper (TMD) without damping

• RC Beam



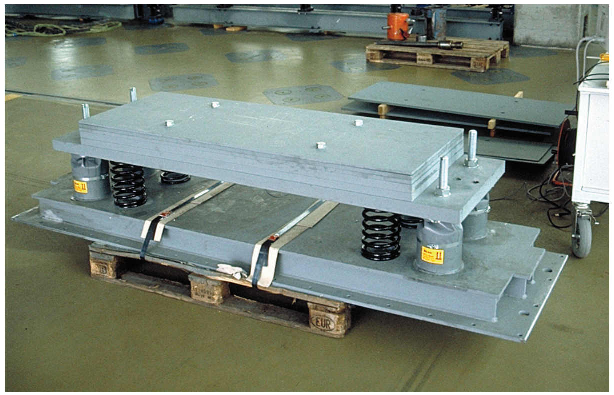
• Damping ratio
$\zeta_n = 0.0$

• Modal mass
$M_n = 5.626t$

• Modal stiffness
$K_n = 886kN/m$

• Natural frequency
$f_n = 2Hz$

• TMD (In this case damping is neglected)



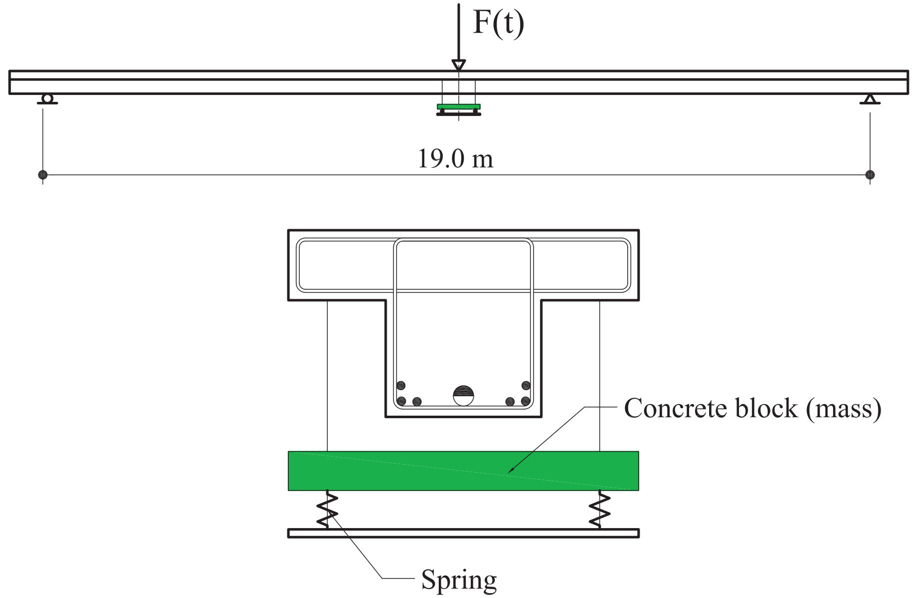
• Damping ratio
$\zeta_T = 0.0$

• Mass
$M_T = 0.310t$

• Stiffness
$K_T = 44kN/m$

• Natural frequency
$f_T = 1.90Hz$

• RC beam with TMD



• Excitation

As excitation a vertical harmonic sine force acting only on the beam is assumed.

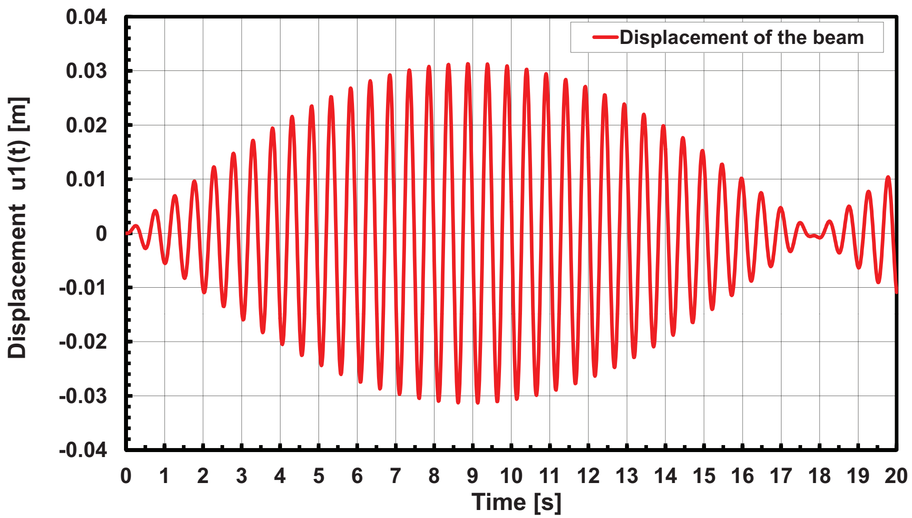
$$F(t) = F_o \sin(\omega t) \quad (11.26)$$

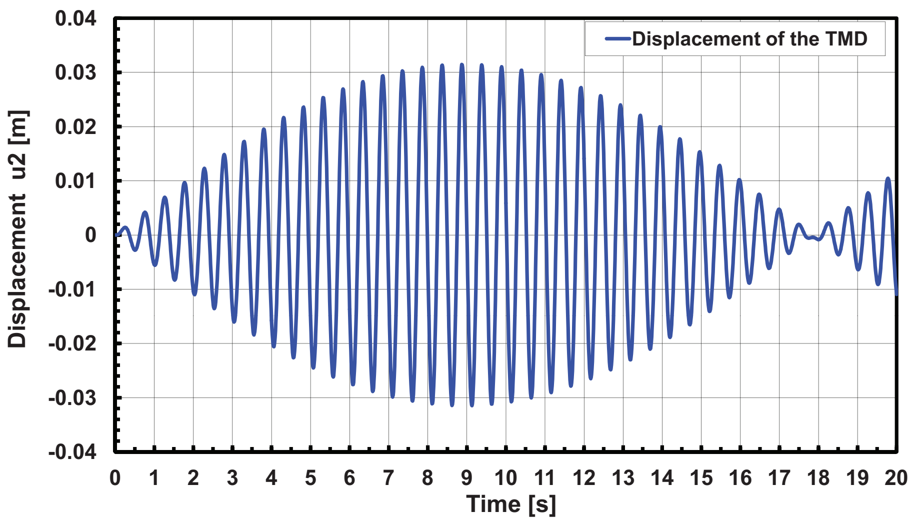
with: ω: excitation frequency
F_o: static excitation force: $F_o = 0.8kN$

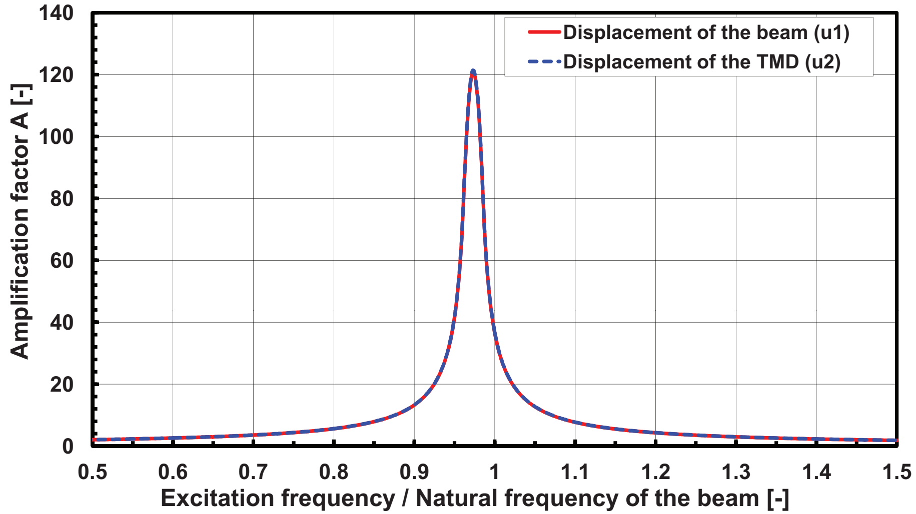
• Solution

Both the transient and the steady-state part of the solution are considered.

- Case 1: $K_T = 10000\text{kN/m}$, excitation frequency $f = 2\text{Hz}$





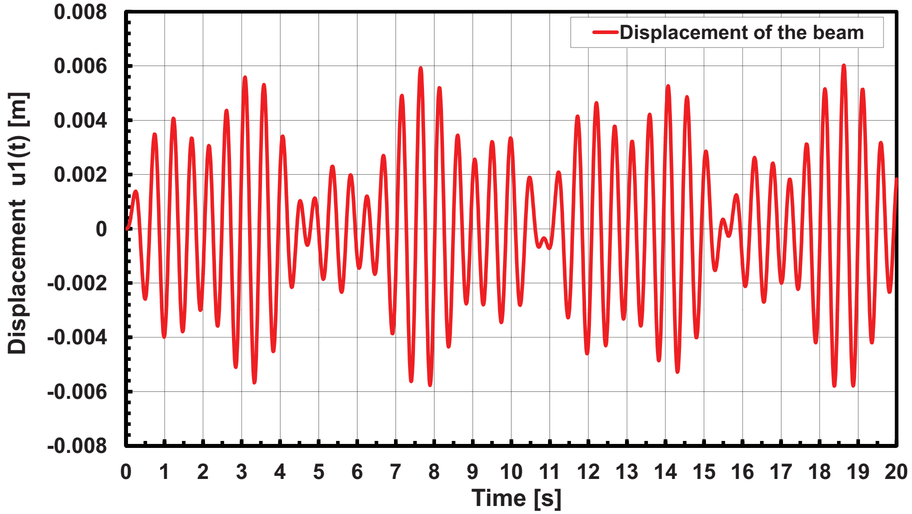


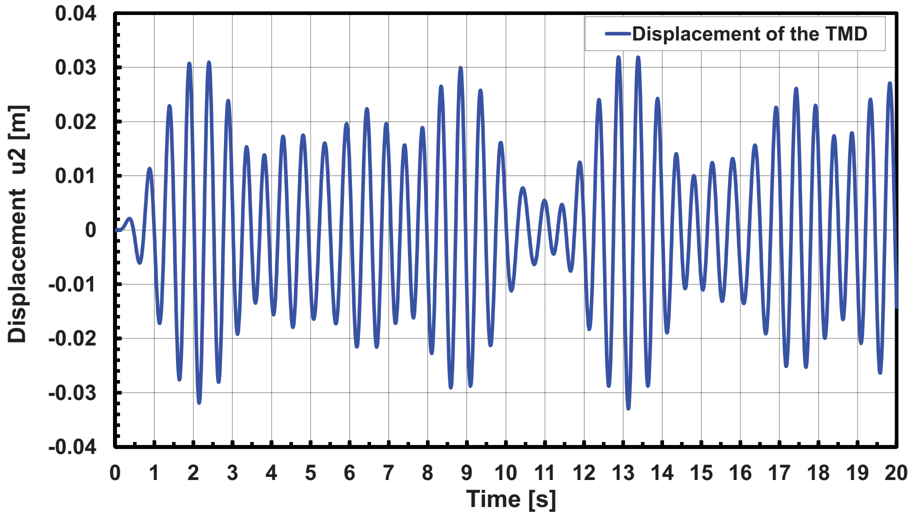
- Remarks
  - The amplification factor A is defined as:

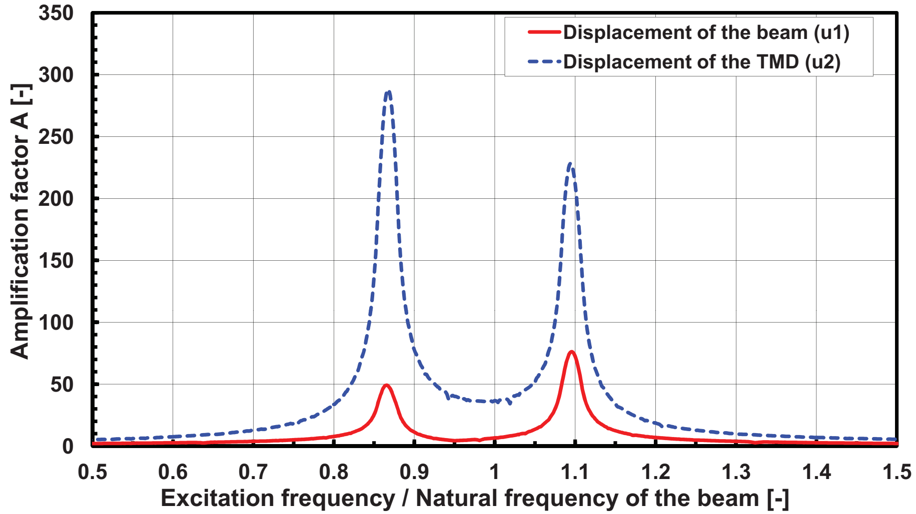
    $$A_{TMD} = u_2/u_{st} \quad , \quad A_{Beam} = u_1/u_{st}$$

    where $u_{st} = F_o/K_n$
  - The solution was computed by means of the Excel file given on the web page of the course (SD_MDOF_TMD.xlsx)
  - The Tuned Mass Damper (TMD) is blocked
  - The natural frequency of the beam with TMD is: $f_n = 1.94\text{Hz}$
  - At $f = f_n$ resonance occurs. In the diagram above the amplification factor is limited, because the response of the system was only calculated during 60 seconds.

- Case 2: $K_T = 44kN/m$, excitation frequency $f = 2Hz$







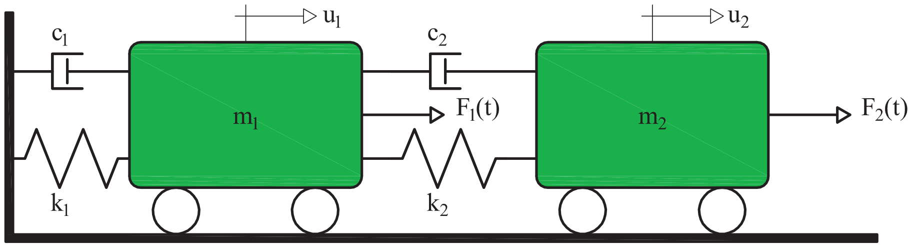
- Remarks
  - The amplification factor A is defined as:

    $$A_{TMD} = u_2/u_{st} \quad , \quad A_{Beam} = u_1/u_{st}$$

    where $u_{st} = F_o/K_n$
  - The solution was computed by means of the Excel file given on the web page of the course (SD_MDOF_TMD.xlsx)
  - The Tuned Mass Damper (TMD) is free to move
  - No resonance at $f = f_n$ occurs. Resonance occurs in correspondence of the first and of the second natural frequencies of the 2-DoF system.
    In the diagram above the factor A is limited, because the response of the system was only calculated during 60s.

11.2 Forced vibrations with damping

11.2.1 Introduction



Sought is the response of the 2-DoF system as a result of the external excitation force $\mathbf{F}(t)$ given by Equation (11.27)

$$\mathbf{F}(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} \tag{11.27}$$

The equation of motion of the system is:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \tag{11.28}$$

The displacement vector $\mathbf{u}(t)$ can be represented as a linear combination of the eigenvectors of the 2-DoF system, $\mathbf{u}(t) = \mathbf{\Phi}\mathbf{q}(t)$, and Equation (11.28) becomes:

$$\mathbf{M\Phi}\ddot{\mathbf{q}} + \mathbf{C\Phi}\dot{\mathbf{q}} + \mathbf{K\Phi q} = \mathbf{F}(t) \tag{11.29}$$

We can now multiply Equation (11.29) by $\mathbf{\Phi}^T$ obtaining:

$$\mathbf{\Phi}^T\mathbf{M\Phi}\ddot{\mathbf{q}} + \mathbf{\Phi}^T\mathbf{C\Phi}\dot{\mathbf{q}} + \mathbf{\Phi}^T\mathbf{K\Phi q} = \mathbf{\Phi}^T\mathbf{F}(t) \tag{11.30}$$

$$\mathbf{M}^*\ddot{\mathbf{q}} + \mathbf{C}^*\dot{\mathbf{q}} + \mathbf{K}^*\mathbf{q} = \mathbf{F}^*(t) \tag{11.31}$$

Where:

- $\mathbf{M}^*$: Diagonal matrix of the modal masses m_n^*
- $\mathbf{K}^*$: Diagonal matrix of the modal stiffnesses k_n^*
- $\mathbf{F}^*$: Vector of the modal forces F_n^*
- $\mathbf{C}^*$: Matrix of the modal damping constants. It is diagonal only if the system is classically damped (see Chapter 10).

For the considered classically damped 2-DoF system, Equation (11.31) can be rearranged as:

$$\begin{cases} m_1^*\ddot{q}_1 + c_1^*\dot{q}_1 + k_1^*q_1 = F_1^* \\ m_2^*\ddot{q}_2 + c_2^*\dot{q}_2 + k_2^*q_2 = F_2^* \end{cases} \tag{11.32}$$

or as alternative:

$$\begin{cases} \ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 = \dfrac{F_1^*}{m_1^*} \\ \ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2 q_2 = \dfrac{F_2^*}{m_2^*} \end{cases} \tag{11.33}$$

The two equations of the system (11.33) are decoupled and can be solved independently. The constants resulting from the solution of the system can be determined by means of the initial conditions $\mathbf{u}(0) = \mathbf{u_0}$ and $\dot{\mathbf{u}}(0) = \mathbf{v_0}$.

11.3 Modal analysis: A summary

The dynamic response of a Multi-Degree of Freedom (MDoF) system due to an external force $\mathbf{F}(t)$ can be computed by means of modal analysis. The required steps are:

1) Compute the properties of the MDoF system
   - Compute the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$.
   - Estimate the modal damping ratios ζ_n^*
2) Compute the natural circular frequencies ω_n and the eigenvectors ϕ_n
   - Compute the modal properties of the MDoF system ($\mathbf{M}^*$, $\mathbf{K}^*$)
3) Compute the response of every mode of vibration
   - Set up the equation of motion of the modal SDoF systems $\ddot{q}_n + 2\zeta_n^*\omega_n\dot{q}_n + \omega_n^2 q_n = \frac{F_n^*}{m_n^*}$ and solve it for q_n
   - Compute the modal displacements $\mathbf{u_n}(t) = \boldsymbol{\phi}_\mathbf{n} q_n$
   - Compute the sectional forces by means of the static equivalent forces $\mathbf{F_n}(t) = \mathbf{K}\mathbf{u_n}(t) = \mathbf{K}\boldsymbol{\phi}_\mathbf{n} q_n = \omega_n^2\mathbf{M}\boldsymbol{\phi}_\mathbf{n} q_n(t)$
4) Sum up (respectively combine) the contribution from all modes of vibration to obtain the total response of the system.

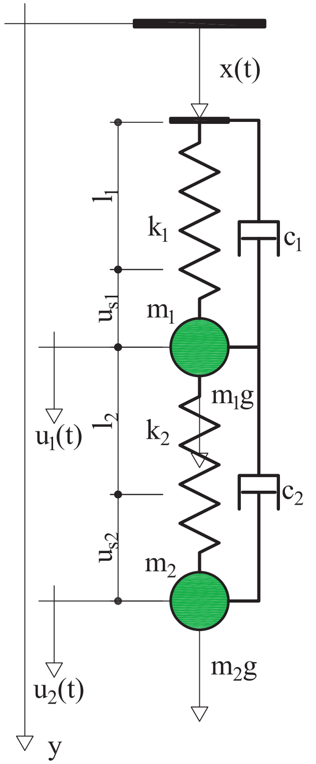
Blank Page

12 Seismic Excitation

12.1 Equation of motion

12.1.1 Introduction

In analogy to Section 2.1.1, the equation of motion of the system depicted here can be formulated by means of the d'Alembert principle $F + T = 0$ applied to each one of the masses.



$$y_1 = x(t) + l_1 + u_{s1} + u_1(t)$$

$$\ddot{y}_1 = \ddot{x}(t) + \ddot{u}_1(t)$$

$$T_1 = -m_1\ddot{y}_1 = -m_1(\ddot{x} + \ddot{u}_1)$$

$$F_1 = -k_1(u_{s1} + u_1) - c_1\dot{u}_1 + m_1 g + k_2(u_{s2} + u_2 - u_1) + c_2(\dot{u}_2 - \dot{u}_1)$$

$$F_1 = -(k_1 + k_2)u_1 + k_2 u_2 - (c_1 + c_2)\dot{u}_1 + c_2\dot{u}_2$$

$$y_2 = x(t) + l_1 + u_{s1} + l_2 + u_{s2} + u_2(t)$$

$$\ddot{y}_2 = \ddot{x}(t) + \ddot{u}_2(t)$$

$$T_2 = -m_2\ddot{y}_2 = -m_2(\ddot{x} + \ddot{u}_2)$$

$$F_2 = -k_2(u_{s2} + u_2 - u_1) - c_2(\dot{u}_2 - \dot{u}_1) + m_1 g$$

$$F_2 = k_2 u_1 - k_2 u_2 + c_2\dot{u}_1 - c_2\dot{u}_2$$

The system of equations governing the motion of the system is

$$\begin{cases} F_1 + T_1 = 0 \\ F_2 + T_2 = 0 \end{cases} \quad (12.1)$$

$$\begin{cases} -m_1(\ddot{x} + \ddot{u}_1) - (c_1 + c_2)\dot{u}_1 + c_2\dot{u}_2 - (k_1 + k_2)u_1 + k_2 u_2 = 0 \\ -m_2(\ddot{x} + \ddot{u}_2) + c_2\dot{u}_1 - c_2\dot{u}_2 + k_2 u_1 - k_2 u_2 = 0 \end{cases} \quad (12.2)$$

and in matricial form:

$$-\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x} + \ddot{u}_1 \\ \ddot{x} + \ddot{u}_2 \end{bmatrix} - \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} - \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (12.3)$$

or:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x} + \ddot{u}_1 \\ \ddot{x} + \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (12.4)$$

or:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{x} \end{bmatrix} \quad (12.5)$$

which is similar to Equation (8.3) meaning that the base point excitation $x(t)$ can be considered equivalent to two external forces $f_1(t) = m_1\ddot{x}(t)$ and $f_2(t) = m_2\ddot{x}(t)$ acting on the masses m_1 and m_2. This is the same interpretation given in Section 2.1.1 for SDoF systems.

12.1.2 Synchronous Ground motion

As shown in the previous section, the equation of motion of a system subjected to a base excitation is:

$$\mathbf{M}\ddot{\mathbf{u}}_a + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \qquad (12.6)$$

where $\ddot{\mathbf{u}}_a$ is vector of the absolute accelerations of the DoFs of the system while $\dot{\mathbf{u}}$ and $\mathbf{u}$ are the vectors of the relative velocities and of the relative displacements of the DoFs of the system, respectively.

The absolute displacement $\mathbf{u}_a$ of the system can be expressed as:

$$\mathbf{u}_a = \mathbf{u}_s + \mathbf{u} \qquad (12.7)$$

where $\mathbf{u}_s$ is displacement of the DoFs due to the static application (i.e. very slow so that no inertia and damping forces are generated) of the ground motion, and $\mathbf{u}$ is again the vector of the relative displacements of the DoFs of the system.

The “static displacements” $\mathbf{u}_s(t)$ can now be expressed in function of the ground displacement $u_g(t)$ as follows:

$$\mathbf{u}_s(t) = \mathbf{\iota} u_g(t) \qquad (12.8)$$

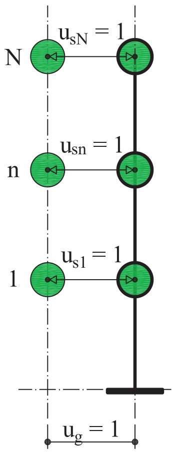
where $\mathbf{\iota}$ is the so-called **influence vector**. Equation (12.6) can now be rewritten as:

$$\mathbf{M}(\mathbf{\iota}\ddot{u}_g + \ddot{\mathbf{u}}) + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \qquad (12.9)$$

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = -\mathbf{M}\mathbf{\iota}\ddot{u}_g(t) \qquad (12.10)$$

Influence vector for some typical cases

- Planar system with translational ground motion (Case 1)

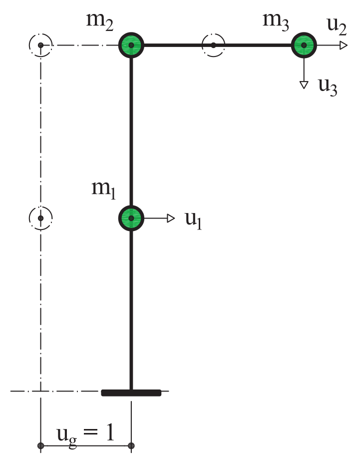


In this case all DoFs of the system undergo static displacements $\mathbf{u}_s(t)$ which are equal to the ground displacement $u_g(t)$, hence:

$$\mathbf{\iota} = \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix} = \mathbf{1} \qquad (12.11)$$

where $\mathbf{1}$ is a vector of order N, i.e. the number of DoFs, with all elements equal to 1.

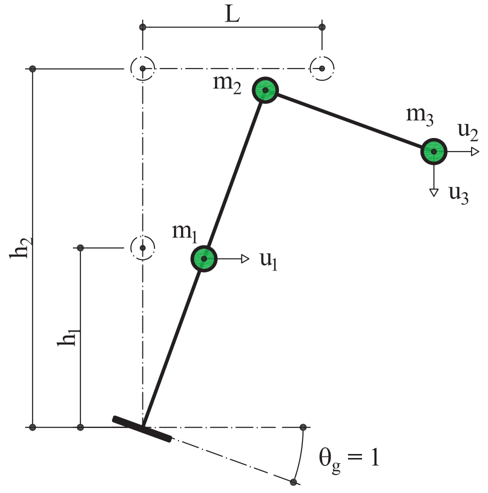
- Planar system with translational ground motion (Case 2)



The axial flexibility of the elements of the depicted system can be neglected, hence 3 DoFs are defined. In this case DoFs 1 and 2 undergo static displacements which are equal to the ground displacement, while the static displacement of DoF 3 is equal to 0, i.e.:

$$\mathbf{\iota} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad (12.12)$$

- Planar system with rotational ground motion.



The depicted system is subjected to a rotational ground motion θ_g which generates the following static displacements of the DoFs:

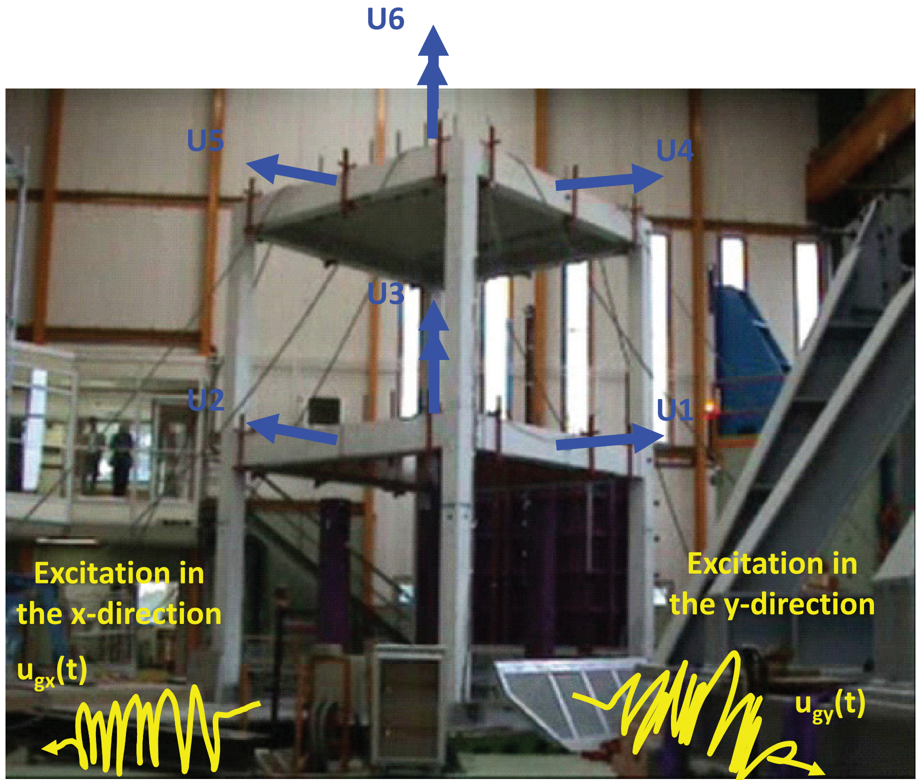
$$\mathbf{u}_s(t) = \begin{bmatrix} h_1 \\ h_2 \\ L \end{bmatrix} \theta_g(t) \text{ hence } \boldsymbol{\iota} = \begin{bmatrix} h_1 \\ h_2 \\ L \end{bmatrix} \tag{12.13}$$

Remark

If the planar system with rotational ground motion has more than one support and every support is subjected only to the base rotation θ_g, then the static application of the base rotations typically create stresses within the system. Such a case must be considered like a multiple support excitation (see Section 12.1.3).

- Spatial system with multiple translational ground motion

Consider the spatial frame depicted here:



Picture from: Chaudat T., Pilakoutas K., Papastergiou P., Ciupala M. A. (2006) "Shaking Table Tests on Reinforced Concrete Retrofitted Frame With Carbon Fibre Reinforced Polymers (CFRP)," Proceedings of the First European Conference on Earthquake Engineering and Seismology, Geneva, Switzerland, 3-8 September 2006

The equation of motion of the frame structure for the ground motions $u_{gx}(t)$ and $u_{gy}(t)$ neglecting damping is:

$$\mathbf{M}\begin{bmatrix}\ddot{u}_1\\ \ddot{u}_2\\ \ddot{u}_3\\ \ddot{u}_4\\ \ddot{u}_5\\ \ddot{u}_6\end{bmatrix} + \mathbf{K}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{bmatrix} = -\mathbf{M}\boldsymbol{\iota}_x\ddot{u}_{gx}(t) - \mathbf{M}\boldsymbol{\iota}_y\ddot{u}_{gy}(t) \qquad (12.14)$$

$$\mathbf{M}\begin{bmatrix}\ddot{u}_1\\ \ddot{u}_2\\ \ddot{u}_3\\ \ddot{u}_4\\ \ddot{u}_5\\ \ddot{u}_6\end{bmatrix} + \mathbf{K}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{bmatrix} = -\mathbf{M}\left(\begin{bmatrix}0\\ 1\\ 0\\ 0\\ 1\\ 0\end{bmatrix}\ddot{u}_{gx}(t) + \begin{bmatrix}1\\ 0\\ 0\\ 1\\ 0\\ 0\end{bmatrix}\ddot{u}_{gy}(t)\right) \qquad (12.15)$$

and with

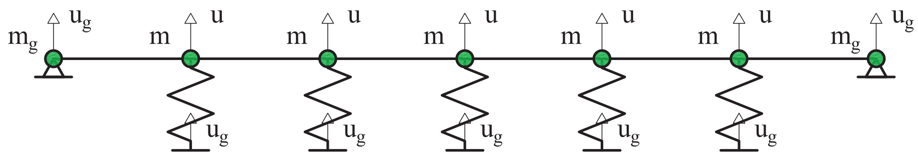
$$\mathbf{M} = \begin{bmatrix}m_1 & & & & & \\ & m_2 & & 0 & & \\ & & I_3 & & & \\ & & & m_4 & & \\ & 0 & & & m_5 & \\ & & & & & I_6\end{bmatrix} \text{ we obtain } \mathbf{M}\begin{bmatrix}\ddot{u}_1\\ \ddot{u}_2\\ \ddot{u}_3\\ \ddot{u}_4\\ \ddot{u}_5\\ \ddot{u}_6\end{bmatrix} + \mathbf{K}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{bmatrix} = \begin{bmatrix}-m_1\ddot{u}_{gy}\\ -m_2\ddot{u}_{gx}\\ 0\\ -m_4\ddot{u}_{gy}\\ -m_5\ddot{u}_{gx}\\ 0\end{bmatrix} \qquad (12.16)$$

Remarks

- For other cases see [Cho11] Sections 9.4 to 9.6.

12.1.3 Multiple support ground motion

Structures with a significative spatial extension may be subjected to ground motion time-histories that are different from support to support. A typical example for such structures is the bridge shown in the following figure.



Example of structure where often multiple support excitation is applied: Plan view of the dynamic model for the seismic analysis of a bridge in the transverse direction. The springs represent the piers.

In this case it is distinguished between the DoFs of the structure $\mathbf{u}_a$, which are free to move and whose displacements are expressed in absolute coordinates, and those of the ground $\mathbf{u}_g$, which undergo the displacements imposed to the support. The vector containing the displacements of all DoFs is hence:

$$\bar{\mathbf{u}} = \begin{bmatrix}\mathbf{u}_a\\ \mathbf{u}_g\end{bmatrix} \qquad (12.17)$$

The equation of motion of the system can hence be expressed as (see [Cho11]):

$$\begin{bmatrix}\mathbf{m} & \mathbf{m}_g\\ \mathbf{m}_g^T & \mathbf{m}_{gg}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{u}}_a\\ \ddot{\mathbf{u}}_g\end{bmatrix} + \begin{bmatrix}\mathbf{c} & \mathbf{c}_g\\ \mathbf{c}_g^T & \mathbf{c}_{gg}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{u}}_a\\ \dot{\mathbf{u}}_g\end{bmatrix} + \begin{bmatrix}\mathbf{k} & \mathbf{k}_g\\ \mathbf{k}_g^T & \mathbf{k}_{gg}\end{bmatrix}\begin{bmatrix}\mathbf{u}_a\\ \mathbf{u}_g\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{p}_g(t)\end{bmatrix} \qquad (12.18)$$

Where $\mathbf{p}_g(t)$ are the forces resulting at the supports when the supports undergo the displacements $\mathbf{u}_g(t)$.

In Equation (12.18) the different matrices $\ldots_g$, $\ldots_{gg}$ are not computed separately, they just result form the partition of the overall system of equations when the DoFs of the structure and of the ground are collected as it is shown in the example of page 12-12.

The vector of Equation (12.17) can be rewritten as:

$$\bar{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_a \\ \mathbf{u}_g \end{bmatrix} = \begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_g \end{bmatrix} + \begin{bmatrix} \mathbf{u} \\ \mathbf{0} \end{bmatrix} \quad (12.19)$$

where $\mathbf{u}_s$ is the vector of the displacements of the DoFs of the structure when the ground displacements $\mathbf{u}_g(t)$ are applied statically, and $\mathbf{u}$ is the vector of the relative displacements of the DoFs of the structure.

The relationship between $\mathbf{u}_s$ and $\mathbf{u}_g(t)$ is given by the following system of equation:

$$\begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix} \begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_g \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{p}_{g,\,s} \end{bmatrix} \quad (12.20)$$

where $\mathbf{p}_{g,\,s}$ is the vector of the support forces needed to impose the displacements $\mathbf{u}_g$ statically. If the system is statically determinated, $\mathbf{p}_{g,\,s}$ is equal to zero (See example of page 12-12).

By introducing Equation (12.19) into Equation (12.18) we obtain the new system of equations:

$$\begin{bmatrix} \mathbf{m} & \mathbf{m}_g \\ \mathbf{m}_g^T & \mathbf{m}_{gg} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_s + \ddot{\mathbf{u}} \\ \ddot{\mathbf{u}}_g \end{bmatrix} + \begin{bmatrix} \mathbf{c} & \mathbf{c}_g \\ \mathbf{c}_g^T & \mathbf{c}_{gg} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_s + \dot{\mathbf{u}} \\ \dot{\mathbf{u}}_g \end{bmatrix} + \begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix} \begin{bmatrix} \mathbf{u}_s + \mathbf{u} \\ \mathbf{u}_g \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{p}_g(t) \end{bmatrix}$$

(12.21)

The first line of the system can be rearranged to:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -(\mathbf{m}\ddot{\mathbf{u}}_s + \mathbf{m}_g\ddot{\mathbf{u}}_g) - (\mathbf{c}\dot{\mathbf{u}}_s + \mathbf{c}_g\dot{\mathbf{u}}_g) - (\mathbf{k}\mathbf{u}_s + \mathbf{k}_g\mathbf{u}_g) \quad (12.22)$$

According to the first line of Equation (12.20) $\mathbf{k}\mathbf{u}_s + \mathbf{k}_g\mathbf{u}_g = \mathbf{0}$ and hence Equation (12.22) becomes:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -(\mathbf{m}\ddot{\mathbf{u}}_s + \mathbf{m}_g\ddot{\mathbf{u}}_g) - (\mathbf{c}\dot{\mathbf{u}}_s + \mathbf{c}_g\dot{\mathbf{u}}_g) \quad (12.23)$$

If we now express the vector $\mathbf{u}_s$ in function of the vector $\mathbf{u}_g$ as

$$\mathbf{u}_s = \underline{\iota}\mathbf{u}_g \quad (12.24)$$

the so-called **influence matrix** $\underline{\iota}$ can be computed, again making use of the first line of Equation (12.20), i.e.:

$$-\mathbf{k}_g\mathbf{u}_g = \mathbf{k}\mathbf{u}_s = \mathbf{k}\underline{\iota}\mathbf{u}_g \quad (12.25)$$

and after rearranging we obtain:

$$\underline{\iota} = -\mathbf{k}^{-1}\mathbf{k}_g \quad (12.26)$$

The influence matrix $\underline{\iota}$ is a $N \times N_g$ matrix where N is the number of DoFs of the structure and N_g is the number of DoFs of the supports.

By introducing Equation (12.26) into Equation (12.23) the final equation of motion of the system is obtained:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -(\mathbf{m}\underline{\iota} + \mathbf{m}_g)\ddot{\mathbf{u}}_g - (\mathbf{c}\underline{\iota} + \mathbf{c}_g)\dot{\mathbf{u}}_g \tag{12.27}$$

Analogously, with the second line of Equation (12.21), an equation for the computation of the forces at the supports $\mathbf{p}_g(t)$ can be setup and solved.

Remarks

In Equation (12.27), the masses associated with the support are often equal to zero, i.e. $\mathbf{m}_g = \mathbf{0}$. If this is the case, Equation (12.27) simplifies to:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\iota\ddot{\mathbf{u}}_g - (\mathbf{c}\underline{\iota} + \mathbf{c}_g)\dot{\mathbf{u}}_g \tag{12.28}$$

And considering that in most cases the damping forces on the LHS of the equation are small (and they are zero if no damping is present) compared to the inertia forces (see [Cho11]), Equation (12.28) can be further simplified to:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\underline{\iota}\ddot{\mathbf{u}}_g \tag{12.29}$$

In the case that the movement of the supports is the same at all supports, $\mathbf{u}_g$ becomes:

$$\mathbf{u}_g = \mathbf{1}u_g \tag{12.30}$$

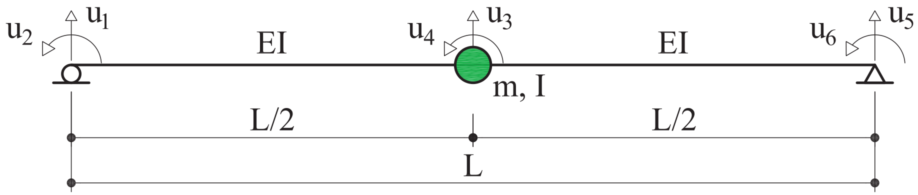
and with $\iota = \underline{\iota}\mathbf{1}$ Equation (12.29) becomes

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\iota\ddot{u}_g \tag{12.31}$$

which is the same as Equation (12.10).

Example: 2-DoF system

The following 2-DoF system is subjected to multiple support ground motion. Two different ground motions are applied to the degrees of freedom u_1 and u_5. Sought is the equation of motion of the system:



The stiffness matrix of the system is assembled by means of the Direct Stiffness Method and the following degrees of freedom are considered:

$$\mathbf{u} = \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} \text{: displacements of the structure} \tag{12.32}$$

$$\mathbf{u_g} = \begin{bmatrix} u_1 \\ u_5 \end{bmatrix} \text{: displacements of the supports (excited, massless)} \tag{12.33}$$

$$\mathbf{u_0} = \begin{bmatrix} u_2 \\ u_6 \end{bmatrix} \text{: displacements of the supports (not excited, massless)} \tag{12.34}$$

The stiffness matrix $\mathbf{K}$ of the system is:

$$\mathbf{K} = \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} 12 & 6\bar{L} & -12 & 6\bar{L} & 0 & 0 \\ 6\bar{L} & 4\bar{L}^2 & -6\bar{L} & 2\bar{L}^2 & 0 & 0 \\ -12 & -6\bar{L} & 12+12 & -6\bar{L}+6\bar{L} & -12 & 6\bar{L} \\ 6\bar{L} & 2L^2 & -6\bar{L}+6\bar{L} & 4\bar{L}^2+4\bar{L}^2 & -6\bar{L} & 2\bar{L}^2 \\ 0 & 0 & -12 & -6\bar{L} & 12 & -6\bar{L} \\ 0 & 0 & 6L & 2\bar{L}^2 & -6L & 4\bar{L}^2 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \tag{12.35}$$

$$\mathbf{K} = \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} 12 & 6\bar{L} & -12 & 6\bar{L} & 0 & 0 \\ 6\bar{L} & 4\bar{L}^2 & -6\bar{L} & 2\bar{L}^2 & 0 & 0 \\ -12 & -6\bar{L} & 24 & 0 & -12 & 6\bar{L} \\ 6\bar{L} & 2\bar{L}^2 & 0 & 8\bar{L}^2 & -6\bar{L} & 2\bar{L}^2 \\ 0 & 0 & -12 & -6\bar{L} & 12 & -6\bar{L} \\ 0 & 0 & 6L & 2\bar{L}^2 & -6\bar{L} & 4\bar{L}^2 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \tag{12.36}$$

with $\bar{L} = L/2$

By swapping DoF 1 and 3 we obtain:

$$\mathbf{K} = \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} 24 & -6\bar{L} & -12 & 0 & -12 & 6\bar{L} \\ -6\bar{L} & 4\bar{L}^2 & 6\bar{L} & 2\bar{L}^2 & 0 & 0 \\ -12 & 6\bar{L} & 12 & 6\bar{L} & 0 & 0 \\ 0 & 2\bar{L}^2 & 6\bar{L} & 8\bar{L}^2 & -6\bar{L} & 2\bar{L}^2 \\ -12 & 0 & 0 & -6\bar{L} & 12 & -6\bar{L} \\ 6\bar{L} & 0 & 0 & 2\bar{L}^2 & -6\bar{L} & 4\bar{L}^2 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \tag{12.37}$$

By swapping DoF 2 and 4 we obtain:

$$\mathbf{K} = \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} 24 & 0 & -12 & -6\bar{L} & -12 & 6\bar{L} \\ 0 & 8\bar{L}^2 & 6\bar{L} & 2\bar{L}^2 & -6\bar{L} & 2\bar{L}^2 \\ -12 & 6\bar{L} & 12 & 6\bar{L} & 0 & 0 \\ -6\bar{L} & 2\bar{L}^2 & 6\bar{L} & 4\bar{L}^2 & 0 & 0 \\ -12 & -6\bar{L} & 0 & 0 & 12 & -6\bar{L} \\ 6\bar{L} & 2\bar{L}^2 & 0 & 0 & -6\bar{L} & 4\bar{L}^2 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \\ u_5 \\ u_6 \end{bmatrix} \tag{12.38}$$

By swapping DoF 2 and 5 we obtain:

$$\mathbf{K} = \frac{EI}{\bar{L}^3} \cdot \left[\begin{array}{cccc|cc} 24 & 0 & -12 & -12 & -6\bar{L} & 6\bar{L} \\ 0 & 8\bar{L}^2 & 6\bar{L} & -6\bar{L} & 2\bar{L}^2 & 2\bar{L}^2 \\ -12 & 6\bar{L} & 12 & 0 & 6\bar{L} & 0 \\ -12 & -6\bar{L} & 0 & 12 & 0 & -6\bar{L} \\ \hline -6\bar{L} & 2\bar{L}^2 & 6\bar{L} & 0 & 4\bar{L}^2 & 0 \\ 6\bar{L} & 2\bar{L}^2 & 0 & -6\bar{L} & 0 & 4\bar{L}^2 \end{array}\right], \quad \mathbf{U} = \begin{bmatrix} u_3 \\ u_4 \\ u_1 \\ u_5 \\ u_2 \\ u_6 \end{bmatrix} \tag{12.39}$$

By means of static condensation we can now eliminate DoF 2 and 6. We racall that:

$$\mathbf{K} = \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} \mathbf{k_{tt}} & \mathbf{k_{t0}} \\ \mathbf{k_{0t}} & \mathbf{k_{00}} \end{bmatrix}, \quad \hat{\mathbf{k}}_{\mathbf{tt}} = \frac{EI}{\bar{L}^3} \cdot (\mathbf{k_{tt}} - \mathbf{k_{0t}^T} \mathbf{k_{00}^{-1}} \mathbf{k_{0t}}) \tag{12.40}$$

By performig the needed calculations we obtain:

$$\mathbf{k_{00}^{-1}} = \frac{1}{|\mathbf{k_{00}}|}\cdot\hat{\mathbf{k}}_{\mathbf{00}} = \frac{1}{16\bar{L}^4}\cdot\begin{bmatrix} 4\bar{L}^2 & 0 \\ 0 & 4\bar{L}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4\bar{L}^2} & 0 \\ 0 & \frac{1}{4\bar{L}^2} \end{bmatrix} \tag{12.41}$$

$$\mathbf{k_{0t}^{T}k_{00}^{-1}k_{0t}} = \begin{bmatrix} -6\bar{L} & 6\bar{L} \\ 2\bar{L}^2 & 2\bar{L}^2 \\ 6\bar{L} & 0 \\ 0 & -6\bar{L} \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{4\bar{L}^2} & 0 \\ 0 & \frac{1}{4\bar{L}^2} \end{bmatrix}\cdot\begin{bmatrix} -6\bar{L} & 2\bar{L}^2 & 6\bar{L} & 0 \\ 6\bar{L} & 2\bar{L}^2 & 0 & -6\bar{L} \end{bmatrix} \tag{12.42}$$

$$\mathbf{k_{0t}^{T}k_{00}^{-1}k_{0t}} = \begin{bmatrix} 18 & 0 & -9 & -9 \\ 0 & 2\bar{L}^2 & 3L & 3L \\ -9 & 3L & 9 & 0 \\ -9 & 3L & 0 & 9 \end{bmatrix} \tag{12.43}$$

$$\hat{\mathbf{k}}_{\mathbf{tt}} = \frac{EI}{\bar{L}^3}\cdot\left(\begin{bmatrix} 24 & 0 & -12 & -12 \\ 0 & 8\bar{L}^2 & 6\bar{L} & -6\bar{L} \\ -12 & 6\bar{L} & 12 & 0 \\ -12 & -6\bar{L} & 0 & 12 \end{bmatrix} - \begin{bmatrix} 18 & 0 & -9 & -9 \\ 0 & 2\bar{L}^2 & 3L & 3L \\ -9 & 3L & 9 & 0 \\ -9 & 3L & 0 & 9 \end{bmatrix}\right) \tag{12.44}$$

$$\hat{\mathbf{k}}_{\mathbf{tt}} = \frac{EI}{\bar{L}^3}\cdot\begin{bmatrix} 6 & 0 & -3 & -3 \\ 0 & 6\bar{L}^2 & 3\bar{L} & -3\bar{L} \\ -3 & 3\bar{L} & 3 & 0 \\ -3 & -3\bar{L} & 0 & 3 \end{bmatrix} \tag{12.45}$$

The equation of motion of the system becomes:

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_1 \\ \ddot{u}_5 \end{bmatrix} + \frac{EI}{\bar{L}^3}\cdot\begin{bmatrix} 6 & 0 & -3 & -3 \\ 0 & 6\bar{L}^2 & 3\bar{L} & -3\bar{L} \\ -3 & 3\bar{L} & 3 & 0 \\ -3 & -3\bar{L} & 0 & 3 \end{bmatrix}\cdot\begin{bmatrix} u_3 \\ u_4 \\ u_1 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p_1(t) \\ p_5(t) \end{bmatrix} \tag{12.46}$$

We racall that:

$$\hat{\mathbf{k}}_{\mathbf{tt}} = \frac{EI}{\bar{L}^3}\cdot\begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix}, \quad \underline{\iota} = -\mathbf{k^{-1}k_g} \tag{12.47}$$

By performig the needed calculations we obtain:

$$\mathbf{k^{-1}} = \frac{1}{|\mathbf{k}|}\cdot\hat{\mathbf{k}} = \frac{\bar{L}^3}{EI}\cdot\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6\bar{L}^2} \end{bmatrix} \tag{12.48}$$

$$\underline{\iota} = -\mathbf{k^{-1}k_g} = -\frac{\bar{L}^3}{EI}\cdot\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6\bar{L}^2} \end{bmatrix}\cdot\frac{EI}{\bar{L}^3}\cdot\begin{bmatrix} -3 & -3 \\ 3\bar{L} & 3\bar{L} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2\bar{L}} & \frac{1}{2\bar{L}} \end{bmatrix} \tag{12.49}$$

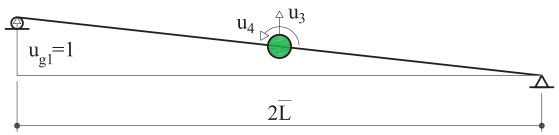
The vector of the effective forces becomes:

$$\mathbf{p}_{\text{eff}}(t) = -(\mathbf{m}\boldsymbol{\iota} + \mathbf{m_g})\ddot{\mathbf{u}}_\mathbf{g}(t) = -\begin{bmatrix} \frac{M}{2} & \frac{M}{2} \\ -\frac{I}{2\bar{L}} & \frac{I}{2\bar{L}} \end{bmatrix} \cdot \begin{bmatrix} \ddot{u}_{g1} \\ \ddot{u}_{g5} \end{bmatrix} \tag{12.50}$$

And the equation of motion of the system finally becomes:

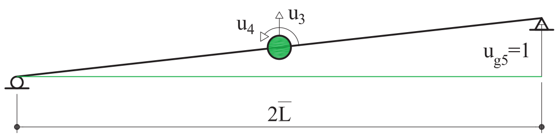
$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{\bar{L}^3} \cdot \begin{bmatrix} 6 & 0 \\ 0 & 6\bar{L}^2 \end{bmatrix} \cdot \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = -\begin{bmatrix} \frac{M}{2} & \frac{M}{2} \\ -\frac{I}{2\bar{L}} & \frac{I}{2\bar{L}} \end{bmatrix} \cdot \begin{bmatrix} \ddot{u}_{g1} \\ \ddot{u}_{g5} \end{bmatrix} \tag{12.51}$$

The following drawings show the interpretation of the elements of the influence matrix $\boldsymbol{\iota}$:



if $u_{g1} = 1$ then:

$u_3 = \frac{1}{2}$ and $u_4 = -\frac{1}{2\bar{L}}$



if $u_{g5} = 1$ then:

$u_3 = \frac{1}{2}$ and $u_4 = \frac{1}{2\bar{L}}$

Remarks:

- See Section 9.7 of [Cho11] for an example with a statically in-determinated system.
- In Finite Element analysis, when applying multiple support ex-citation, support displacements instead of support acceleration are often used. For more details see [Bat96].

12.2 Time-history of the response of elastic systems

As discussed in the previous sections, the equation of motion of a MDoF system under base excitation is:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = -\mathbf{M}\boldsymbol{\iota}\ddot{u}_g(t) \tag{12.52}$$

Where:

- $\mathbf{M}$: Mass matrix (symmetric and positive-definite, or diagonal if only lumped masses are present)
- $\mathbf{K}$: Stiffness matrix (symmetric and positive-definite)
- $\mathbf{C}$: Damping matrix (Classical damping: $\mathbf{C}$ is typically a linear combination of $\mathbf{M}$ and $\mathbf{K}$)
- $\ddot{u}_g(t)$: Ground acceleration
- $\boldsymbol{\iota}$: Influence vector of order N. In the simplest case of a planar system under translational ground motion $\boldsymbol{\iota} = \mathbf{1}$.

If the damping of the MDoF system is classical, Equation (12.52) can be written in the form of N decoupled modal equations, where N is the number of modes of the system. The modal equations are of the following form:

$$m_n^*\ddot{q}_n + c_n^*\dot{q}_n + k_n^*q_n = -\phi_n^T\mathbf{M}\boldsymbol{\iota}\ddot{u}_g \tag{12.53}$$

or:

$$\ddot{q}_n + 2\zeta_n^*\omega_n\dot{q}_n + \omega_n^2q_n = -\frac{\phi_n^T\mathbf{M}\boldsymbol{\iota}}{\phi_n^T\mathbf{M}\phi_n}\ddot{u}_g \tag{12.54}$$

The dynamic response of the MDoF system can be written as:

$$\mathbf{u}(t) = \sum_{n=1}^{N} \boldsymbol{\phi}_{\mathbf{n}} q_n(t) \quad (12.55)$$

$\boldsymbol{\phi}_{\mathbf{n}}$: n^{th} eigenvector of the MDoF system

$q_n(t)$: n^{th} modal coordinate of the MDoF system

Further variables in Equation (12.53) are the **modal mass** m_n^* and the **modal stiffness** k_n^* of the n^{th} mode. These parameters are defined as follows:

$$m_n^* = \phi_n^T \cdot \mathbf{M} \cdot \phi_n \quad (12.56)$$

$$k_n^* = \phi_n^T \cdot \mathbf{K} \cdot \phi_n = \omega_n^2 \cdot m_n^* \quad (12.57)$$

ω_n: n^{th} modal circular frequency of the MDOF system

The modal participation factor Γ_n is a measure for the contribution of the n-th mode to the total response of the system. It is defined as follows:

$$\Gamma_n = \frac{\phi_n^T \mathbf{M} \boldsymbol{\iota}}{\phi_n^T \mathbf{M} \phi_n} \quad (12.58)$$

In addition the so-called **effective modal mass** of the n^{th} mode is defined as:

$$m_{n,\,eff}^* = \Gamma_n^2 \cdot m_n^* \quad (12.59)$$

Unlike the modal mass m_n^* and the modal participation factor Γ_n, the effective modal mass $m_{n,\,eff}^*$ is independent of the normalization of the eigenvectors. The following equation holds:

$$\sum_{n=1}^{N} m_{n,\,eff}^* = \sum_{n=1}^{N} m_n = m_{tot} \quad (12.60)$$

where m_{tot} is the total mass of the dynamic system.

The **effective modal height** h_n^* of the n^{th} mode is:

$$h_n^* = \frac{L_n^\theta}{L_n} \text{ with } L_n^\theta = \sum_{j=1}^{N} h_j \cdot m_j \cdot \phi_{jn} \text{ and } L_n = \phi_n^T \cdot \mathbf{M}\boldsymbol{\iota} \quad (12.61)$$

- Significance of the effective modal mass $m_{n,\,eff}^*$

The effective modal mass $m_{n,\,eff}^*$ is the lumped mass of a single-storey substitute system which is subjected to a base shear force V_{bn} equal to the n^{th} modal base shear force of a multi-storey system.

If in addition the height of the single storey substitute system with the lumped mass $m_{n,\,eff}^*$ equals the modal height h_n^*, the single-storey system is subjected to a base moment M_{bn} which is equal to the n^{th} modal base moment of the multi-storey system.

The following holds:

$$V_{bn} = m_{n,\,eff}^* \cdot S_{pa,\,n} = \sum_{j=1}^{N} f_{jn} \quad (12.62)$$

$$M_{bn} = m^*_{n,eff} \cdot S_{pa,n} \cdot h^*_n = \sum_{j=1}^{N} f_{jn} \cdot h_j \qquad (12.63)$$

where $S_{pa,n}$ is the pseudo-acceleration of the n^{th} mode.

- Distribution of the internal forces

If the internal forces of the entire system are to be determined, the modal equivalent static forces f_{jn} should be computed first:

$$\mathbf{f}_n = \mathbf{s}_n \cdot S_{pa,n} \qquad (12.64)$$

where

$$\mathbf{f}_n = \left[f_{1n} \ f_{2n} \ \dots \ f_{nn} \right] \qquad (12.65)$$

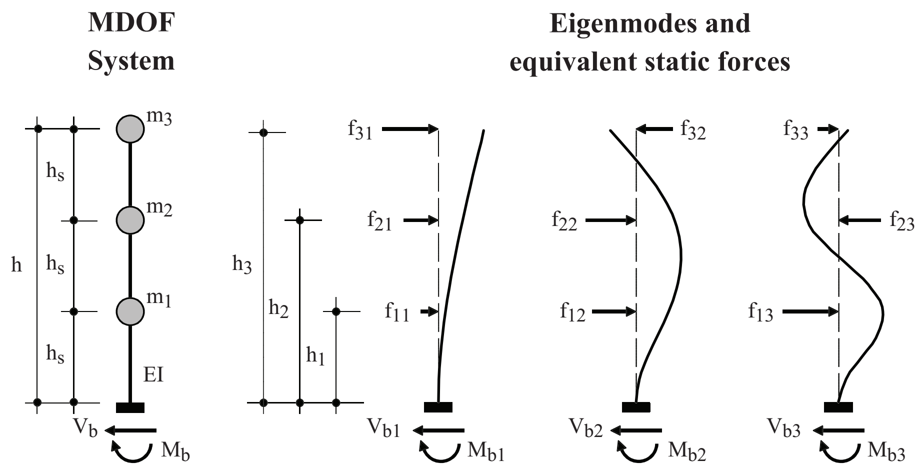
The excitation vector $\mathbf{s}_n$ is defined according to equation (12.66) and specifies the distribution of the inertia forces due to excitation of the n^{th} mode:

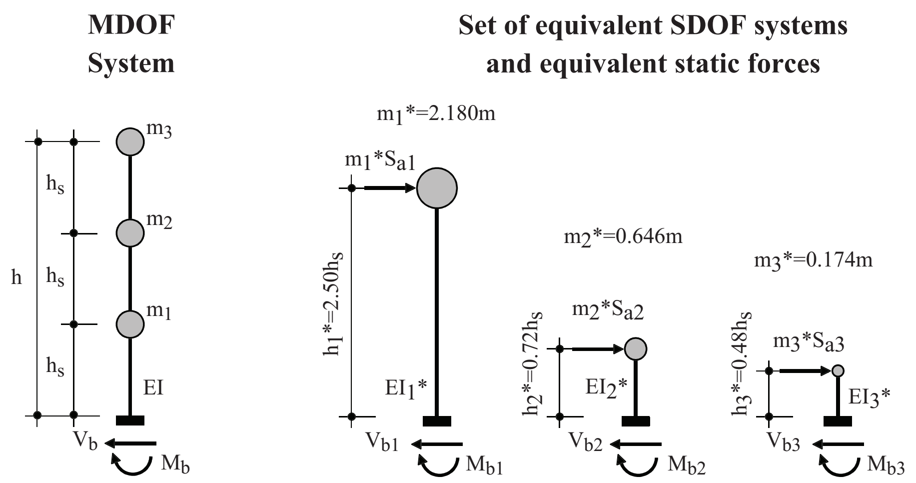
$$\mathbf{s}_n = \Gamma_n \mathbf{M} \boldsymbol{\phi}_n \qquad (12.66)$$

$\mathbf{s}_n$ is independent on the normalization of the eigenvector ϕ_n and we have that:

$$\sum_{n=1}^{N} \mathbf{s}_n = \mathbf{M}\boldsymbol{\iota} \qquad (12.67)$$

- MDoF system with eigenmodes and equivalent SDoF systems





12.3 Response spectrum method

12.3.1 Definition and characteristics

If the maximum response only and not the response to the entire time history according to Equation (12.55) is of interest, the response spectrum method can be applied.

The response spectrum can be computed for the considered seismic excitation and the maximum value of the modal coordinate $q_{n, max}$ can be determined as follows:

$$q_{n, max} = \Gamma_n \cdot S_d(\omega_n, \zeta_n^*) = \Gamma_n \cdot \frac{1}{\omega_n^2} \cdot S_{pa}(\omega_n, \zeta_n^*) \quad (12.68)$$

where:

- Γ_n: modal participation factor of the n-th mode
- $S_d(\omega_n, \zeta_n^*)$: Spectral **displacement** for the circular eigenfrequency ω_n and the modal damping rate ζ_n^*.
- $S_{pa}(\omega_n, \zeta_n^*)$: Spectral **pseudo-acceleration** for the circular eigenfrequency ω_n and the modal damping rate ζ_n^*.

The contribution of the n^{th} mode to the total displacement is:

$$\mathbf{u}_{n, max} = \boldsymbol{\phi}_{\mathbf{n}} \cdot q_{n, max} \quad (12.69)$$

The maxima of different modes do not occur at the same instant. An exact computation of the total maximum response on the basis of the maximum modal responses is hence impossible. Different methods have been developed to estimate the total maximum response from the maximum modal responses.

- "Absolute Sum (ABSSUM)" Combination Rule

$$u_{i, max} \le \sum_{n=1}^{N} |\phi_{in} \cdot q_{n, max}| \quad (12.70)$$

The assumption that all maxima occur at the same instant and in the same direction yields an upper bound value for the response quantity. This assumption is commonly too conservative.

- "Square-Root-of Sum-of-Squares (SRSS)" Combination Rule

$$u_{i, max} \approx \sqrt{\sum_{n=1}^{N} (\phi_{in} \cdot q_{n, max})^2} \quad (12.71)$$

This rule is often used as the standard combination method and yields very good estimates of the total maximum response if the modes of the system are well separated. If the system has several modes with similar frequencies the SRSS rule might yield estimates which are significantly lower than the actual total maximum response.

- "Complete Quadratic Combination (CQC)" Combination Rule

$$u_{i, max} \approx \sqrt{\sum_{j=1}^{N} \sum_{k=1}^{N} u_{i, max}^{(j)} \cdot \rho_{jk} \cdot u_{i, max}^{(k)}} \quad (12.72)$$

where

$u_{i, max}^{(j)}$ and $u_{i, max}^{(k)}$ are the max. modal responses of modes j and k

ρ_{jk} is the correlation coefficient between nodes j and k:

$$\rho_{jk} = \frac{8\sqrt{\zeta_i\zeta_k}(\zeta_i + r\zeta_k)r^{3/2}}{(1-r^2)^2 + 4\zeta_i\zeta_k r(1+r^2) + 4(\zeta_i^2 + \zeta_k^2)r^2} \text{ with } r = \frac{\omega_k}{\omega_j} \quad (12.73)$$

This method based on random vibration theory gives exact results if the excitation is represented by a white noise. If the frequencies of the modes are well spaced apart, the result converge to those of the SRSS rule. More detailed information on this and other combination rules can be found in [Cho11] Chapter 13.7.

- Internal forces

The aforementioned combination rules cannot only be applied on displacements but also on internal forces.

The maximum modal internal forces can be determined from equivalent static forces

$$\mathbf{F}_{n,\,max} = \mathbf{K} \cdot \mathbf{u}_{n,\,max}, \quad (12.74)$$

which, as a first option, are computed from the equivalent static displacements. Alternatively, the equivalent static forces can be determined from the inertia forces:

$$\mathbf{F}_{n,\,max} = \mathbf{s}_n \cdot S_{pa}(\omega_n, \zeta_n^*) = \Gamma_n \mathbf{M} \phi_n \cdot S_{pa}(\omega_n, \zeta_n^*) \quad (12.75)$$

with $\mathbf{s}_n$ being the excitation vector which represents the distribution of the inertia forces of the n^{th} mode (see Equation 12.67).

Attention:

It is wrong to compute the maximum internal forces from the maximum displacement of the total response $\mathbf{u}_{max}$.

- Number of modes to be considered.

All modes which contribute to the dynamic response of the system should be considered. In practical applications, however, only those modes are considered which contribution to the total response is above a certain threshold. It should be noted that in order to achieve the same accuracy for different response measures (e.g. displacements, shear forces, bending moments, etc.) different numbers of modes might need to be considered in the computation.

For a regular building the top displacement can be estimated fairly well on the basis of the fundamental mode only. To estimate the internal forces, however, higher modes need to be considered too.

According to Eurocode 8 "Design of Structures for Earthquake Resistance" [CEN04] all modes should be considered (starting from the lowest) until the sum of the effective modal masses $m_{n,\,eff}$ of all considered modes corresponds to at least 90% of the total mass m_{tot}. As an alternative, Eurocode 8 allows the designer to show that all modes with $m^*_{n,\,eff} > 0.05 m_{tot}$ were considered in the computation.

12.3.2 Step-by-step procedure

The maximum response of a N-storey building can be estimated according to the following procedure:

1) Determine the properties of the MDOF system
   - Choose DOFs
   - Determine mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$.
   - Estimate modal damping ratios ζ_n^*
2) Carry out modal analysis of the MDOF system
   - Determine circular eigenfrequencies ω_n and eigenvectors $\boldsymbol{\phi}_n$

$$(\mathbf{K} - \omega_n^2 \mathbf{M}) \cdot \boldsymbol{\phi}_n = 0$$

   - Compute the modal properties of the MDOF system ($\mathbf{M}^*$, $\mathbf{K}^*$)

$$m_n^* = \boldsymbol{\phi}_n^T \mathbf{M} \boldsymbol{\phi}_n \ , \ k_n^* = \boldsymbol{\phi}_n^T \mathbf{K} \boldsymbol{\phi}_n$$

   - Compute the modal participation factor Γ_n

$$\Gamma_n = \frac{\boldsymbol{\phi}_n^T \mathbf{M} \boldsymbol{\iota}}{\boldsymbol{\phi}_n^T \mathbf{M} \boldsymbol{\phi}_n}$$

3) The maximum response of the n-th mode should be determined as described in the following. This should be done for all modes $n = 1, 2, \ldots, \bar{N}$ which require consideration.
   - For all periods T_n and for the corresponding damping ratios ζ_n^*, the spectral response $S_a(\omega_n, \zeta_n)$ should be determined from the

response spectrum for pseudo-accelerations. (The spectral displacement $S_d(\omega_n, \zeta_n^*)$ should be determined in the same manner)
   - Computation of the maximum displacements

$$\mathbf{u}_{n,\,max} = \boldsymbol{\phi}_n \cdot \Gamma_n \cdot S_d(\omega_n, \zeta_n^*)$$

   - Computation of the maximum equivalent static forces

$$\mathbf{F}_{n,\,max} = \mathbf{s}_n \cdot S_{pa}(\omega_n, \zeta_n) = \Gamma_n \mathbf{M} \boldsymbol{\phi}_n \cdot S_{pa}(\omega_n, \zeta_n)$$

   - Computation of the maximum internal forces on the basis of the forces $\mathbf{F}_{n,\,max}$
4) Estimate the total response in terms of displacements and internal forces by means of suitable combination rules. Different combination rules might be applied (ABSSUM, SRSS, CQC).

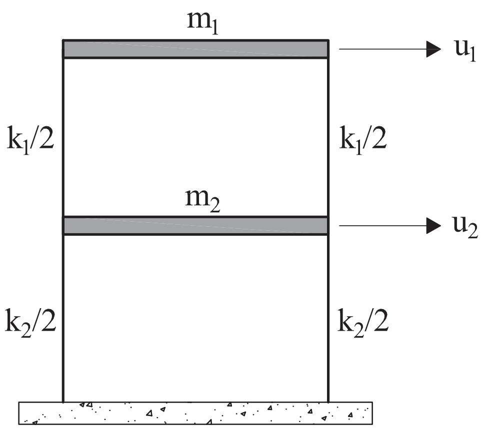
Comment

In order to consider the non-linear behaviour of the structure the equivalent lateral static forces $\mathbf{F}_{n,\,max}$ can be determined from the spectral ordinate $S_{pa}(\omega_n, \zeta_n, q)$ of the **design spectrum** for pseudo-accelerations:

$$\mathbf{F}_{n,\,max} = \mathbf{s}_n \cdot S_{pa}(\omega_n, \zeta_n, q) = \Gamma_n \mathbf{M} \boldsymbol{\phi}_n \cdot S_{pa}(\omega_n, \zeta_n, q) \quad (12.76)$$

12.4 Practical application of the response spectrum method to a 2-DoF system

12.4.1 Dynamic properties



This 2-DoF system corresponds to the system presented in Chapter 9 with the only difference that the 2 DoFs are swapped.

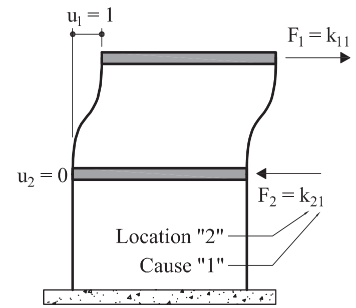
- Degrees of freedom (DoF)
  Horizontal displacements u_1 and u_2 in correspondence of the masses m_1 and m_2
- Masses
  Both story masses have unit value, i.e. $m_1 = m_2 = 1$, hence the mass matrix $\mathbf{M}$ is:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (12.77)$$

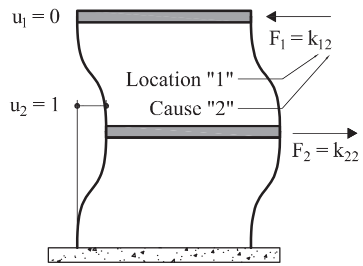
- Stiffness
  The horizontal stiffness of each story is $k = 1$, hence the stiffness matrix $\mathbf{K}$ is:

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad (12.78)$$

1. unit displacement $u_1 = 1$



2. unit displacement $u_2 = 1$



- Damping
  Damping is small and is neglected, hence the damping matrix $\mathbf{C}$ is:

$$\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (12.79)$$

12.4.2 Free vibrations

1) Matrix eigenvalue problem

$$(\mathbf{K}-\omega^2\mathbf{M})\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \mathbf{0} \tag{12.80}$$

The nontrivial solution for the eigenvector $\phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \neq \mathbf{0}$ exists if the determinant is equal to zero:

$$\det(\mathbf{K}-\omega^2\mathbf{M}) = 0 \tag{12.81}$$

$$\det(\mathbf{K}-\omega^2\mathbf{M}) = \det\begin{bmatrix}1-\omega^2 & -1\\ -1 & 2-\omega^2\end{bmatrix} = 0 \tag{12.82}$$

This leads to the quadratic equation in ω^2:

$$(1-\omega^2)\cdot(2-\omega^2)-(-1)\cdot(-1) = 2-3\omega^2+\omega^4-1 = 0 \tag{12.83}$$

or

$$\omega^4-3\omega^2+1 = 0 \tag{12.84}$$

The solution of the quadratic equation yield the eigenvalues:

$$\omega^2 = \frac{3\mp\sqrt{9-4}}{2} = \frac{3\mp\sqrt{5}}{2} \tag{12.85}$$

2) Natural modes of vibration and natural frequencies

For each eigenvalue ω^2 a natural mode of vibration and a natural frequency can be computed.

- **Fundamental mode (first natural mode of vibration)**

The smallest eigenvalue $\omega_1^2 = \frac{3-\sqrt{5}}{2}$ leads to the

1. circular natural frequency $\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}} = 0.62$ (12.86)

When the eigenvalue ω_1^2 is known, the system

$$(\mathbf{K}-\omega_1^2\mathbf{M})\cdot\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = 0 \tag{12.87}$$

can be solved for the corresponding vector $\begin{bmatrix}\phi_{11}\\ \phi_{21}\end{bmatrix}$ (fundamental mode) to within a multiplicative constant:

$$\begin{bmatrix}1-\frac{3-\sqrt{5}}{2} & -1\\ -1 & 2-\frac{3-\sqrt{5}}{2}\end{bmatrix}\cdot\begin{bmatrix}\phi_{11}\\ \phi_{21}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \tag{12.88}$$

The first row yields following equation:

$$\frac{2-(3-\sqrt{5})}{2}\phi_{11}-1\phi_{21} = 0 \tag{12.89}$$

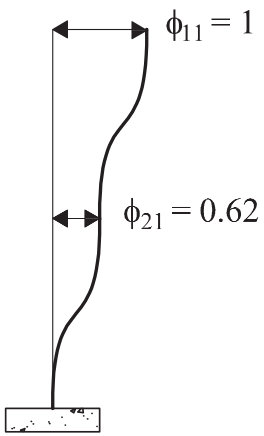
Normalizing the largest coordinate of the eigenvector to unity ($\phi_{11} = 1$), ϕ_{21} becomes:

$$\frac{2-(3-\sqrt{5})}{2} - \phi_{21} = 0 \qquad (12.90)$$

or

$$\phi_{21} = \frac{\sqrt{5}-1}{2} = 0.62 \qquad (12.91)$$

Hence the first natural mode of vibration is:



$$\begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0.62 \end{bmatrix} \qquad (12.92)$$

- **Higher mode of vibration**

The largest eigenvalue $\omega_2^2 = \frac{3+\sqrt{5}}{2}$ leads to the

2. circular natural frequency $\omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}} = 1.62$ (12.93)

In analogy to the fundamental mode, the second mode of vibration can be computed introducing the second eigenvalue ω_2^2 into the system of equations:

$$\begin{bmatrix} 1-\frac{3+\sqrt{5}}{2} & -1 \\ -1 & 2-\frac{3+\sqrt{5}}{2} \end{bmatrix} \cdot \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (12.94)$$

The first row yields following equation:

$$\frac{2-(3+\sqrt{5})}{2}\phi_{12} - 1\phi_{22} = 0 \qquad (12.95)$$

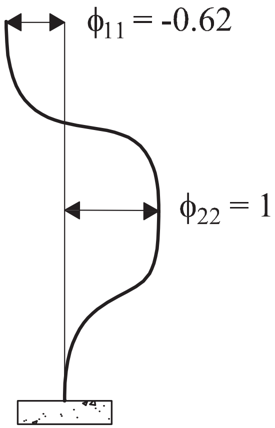
Normalizing the largest coordinate of the eigenvector to unity ($\phi_{22} = 1$), ϕ_{12} becomes:

$$\frac{2-(3+\sqrt{5})}{2}\phi_{12} - 1 = 0 \qquad (12.96)$$

or:

$$\phi_{12} = \frac{-2}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{2} = -0.62 \qquad (12.97)$$

Hence the second natural mode of vibration is:



$$\begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -0.62 \\ 1 \end{bmatrix} \qquad (12.98)$$

3) Orthogonality of modes

In the following the orthogonality of the modes of vibration should be checked.

Hence, following matrix of the eigenvectors is needed:

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & 1 \end{bmatrix} \qquad (12.99)$$

- **Orthogonality with respect to the mass matrix**

The modal mass matrix $\mathbf{M^*}$ is:

$$\mathbf{M^*} = \Phi^T\mathbf{M}\Phi = \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -\frac{\sqrt{5}-1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -\frac{\sqrt{5}-1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\left(\frac{\sqrt{5}-1}{2}\right)^2 & 0 \\ 0 & 1+\left(\frac{\sqrt{5}-1}{2}\right)^2 \end{bmatrix} = \begin{bmatrix} 1.38 & 0 \\ 0 & 1.38 \end{bmatrix} \qquad (12.100)$$

i.e. the matrix $\mathbf{M^*}$ is diagonal.

- **Orthogonality with respect to the stiffness matrix**

The modal stiffness matrix $\mathbf{K}^*$ is:

$$\mathbf{K}^* = \Phi^T \mathbf{K} \Phi = \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -\frac{\sqrt{5}-1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{\sqrt{5}-1}{2} & -1+(\sqrt{5}-1) \\ -\frac{\sqrt{5}-1}{2}-1 & \frac{\sqrt{5}-1}{2}+2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-(\sqrt{5}-1)+2\left(\frac{\sqrt{5}-1}{2}\right)^2 & \frac{\sqrt{5}-1}{2}+\left(\frac{\sqrt{5}-1}{2}\right)^2-1 \\ \frac{\sqrt{5}-1}{2}+\left(\frac{\sqrt{5}-1}{2}\right)^2-1 & 2+(\sqrt{5}-1)+\left(\frac{\sqrt{5}-1}{2}\right)^2 \end{bmatrix} \quad (12.101)$$

Computation of the single elements of $\mathbf{K}^*$:

$$k^*_{11} = 1-(\sqrt{5}-1)+2\left(\frac{\sqrt{5}-1}{2}\right)^2 = -\sqrt{5}+\frac{5-2\sqrt{5}+1}{2} = 0.528 \quad (12.102)$$

$$k^*_{12} = \frac{\sqrt{5}-1}{2}+\left(\frac{\sqrt{5}-1}{2}\right)^2-1 = \frac{2\sqrt{5}-2+5-2\sqrt{5}+1-4}{4} = 0 \quad (12.103)$$

$$k^*_{21} = k^*_{12} = 0 \quad (12.104)$$

$$k^*_{22} = 2+(\sqrt{5}-1)+\left(\frac{\sqrt{5}-1}{2}\right)^2 = 1+\sqrt{5}+\frac{5-2\sqrt{5}+1}{4} = 3.618 \quad (12.105)$$

$$\mathbf{K}^* = \begin{bmatrix} 0.528 & 0 \\ 0 & 3.618 \end{bmatrix} \quad (12.106)$$

i.e. the matrix $\mathbf{K}^*$ is diagonal.

12.4.3 Equation of motion in modal coordinates

The equation of motion in modal coordinates of a system without damping ($\mathbf{C}^* = 0$) is:

$$\mathbf{M}^* \cdot \ddot{\mathbf{q}} + \mathbf{K}^* \cdot \mathbf{q} = -\mathbf{L} \cdot \ddot{u}_g(t) \quad (12.107)$$

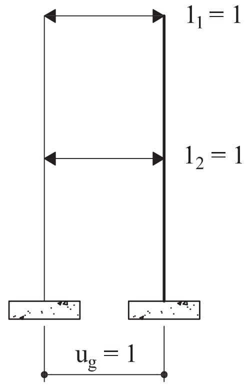
where:

$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, and $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is the vector of the modal coordinates

Computation of the elements of the vector $\mathbf{L}$:

$$\mathbf{L} = \Phi^T \cdot \mathbf{M} \cdot \iota = \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -\frac{\sqrt{5}-1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.62 \\ 0.382 \end{bmatrix} \quad (12.108)$$

The influence vector ι represents the displacement of the masses resulting from the static application of a unit ground displacement $u_g = 1$:



$$\iota = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (12.109)$$

The substitution of $\mathbf{L}$ into the equation of motion in modal coordinates

$$\mathbf{M}^* \cdot \ddot{\mathbf{q}} + \mathbf{K}^* \cdot \mathbf{q} = -\mathbf{L} \cdot \ddot{u}_g(t) \quad (12.110)$$

leads to:

$$\begin{bmatrix} 1.38 & 0 \\ 0 & 1.38 \end{bmatrix} \cdot \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 0.528 & 0 \\ 0 & 3.618 \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = -\begin{bmatrix} 1.62 \\ 0.382 \end{bmatrix} \cdot \ddot{u}_g(t) \quad (12.111)$$

Checking the circular natural frequency computed using modal coordinates against the results of Section 12.4.2:

$$\omega_1 = \sqrt{\frac{k_{11}^*}{m_{11}^*}} = \sqrt{\frac{0.528}{1.38}} = 0.62 \text{ OK!} \quad (12.112)$$

$$\omega_2 = \sqrt{\frac{k_{22}^*}{m_{22}^*}} = \sqrt{\frac{3.618}{1.38}} = 1.62 \text{ OK!} \quad (12.113)$$

- **Additional important modal quantities**

The modal participation factor Γ_n is defined as:

$$\Gamma_n = \frac{L_n}{m_n^*} \quad (12.114)$$

and substituting L_n and m_n^* into this definition gives following values for Γ_1 and Γ_2:

$$\Gamma_1 = \frac{L_1}{m_1^*} = \frac{1.62}{1.38} = 1.17 \quad (12.115)$$

$$\Gamma_2 = \frac{L_2}{m_2^*} = \frac{0.382}{1.38} = 0.28 \quad (12.116)$$

The effective modal mass is defined as:

$$m_{n,\,eff}^* = \Gamma_n^2 \cdot m_n^* \quad (12.117)$$

and substituting Γ_n and m_n^* into this definition gives following values for $m_{1,\,eff}^*$ and $m_{2,\,eff}^*$:

$$m_{1,\,eff}^* = \Gamma_1^2 \cdot m_1^* = 1.17^2 \cdot 1.38 = 1.894 \quad (12.118)$$

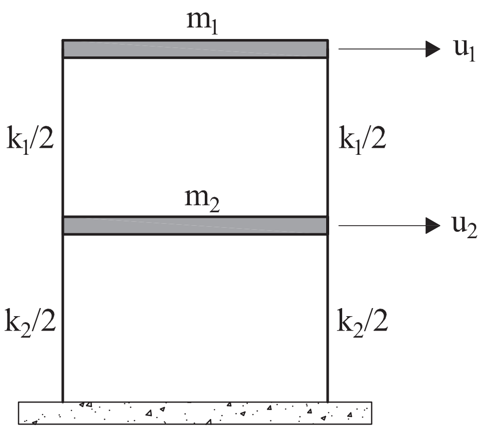
$$m_{2,\,eff}^* = \Gamma_2^2 \cdot m_2^* = 0.28^2 \cdot 1.38 = 0.106 \quad (12.119)$$

$$m_{1,\,eff} + m_{2,\,eff}^* = 1.894 + 0.106 = 2.000 \text{ OK!} \quad (12.120)$$

12.4.4 Response spectrum method

The 2-DoF system analysed in the previous Sections shall be used to illustrate the response spectrum method. For this reason real masses and stiffnesses shall be assumed. The seismic action on the 2-DoF system is represented by the elastic response spectrum of the "El Centro" earthquake.

1) Model



Similar to Section 12.4.1, however with a new definition of masses and stiffnesses:

$$m_1 = m_2 = 1\text{kg}$$

Hence, the mass matrix is:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\text{kg}$$

The stiffness chosen for each story is $k_1 = k_2 = k = 100\ \text{N/m}$ and an appropriate units transformation leads to:

$$k = 100\ \text{N/m} = 100\ \text{kgm/s}^2\text{m}^{-1} = 100\ \text{kg/s}^2 \qquad (12.121)$$

Hence, the stiffness matrix is:

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix}\text{kg/s}^2 \qquad (12.122)$$

2) Natural modes of vibration and natural frequencies

The results of the previous Sections computed using unit masses and unit stiffnesses shall be multiplied by the factor:

$$\sqrt{\frac{k}{m}} = \sqrt{\frac{100\text{kg/s}^2}{1\text{kg}}} = \sqrt{100}\text{s}^{-1} \qquad (12.123)$$

- **Fundamental mode**

Natural frequency: $\omega_1 = 0.62 \cdot \sqrt{100}\text{s}^{-1} = 6.2\ \text{Hz}$

Natural period: $T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{6.2\ \text{Hz}} = 1.02\ \text{s}$

- **Higher vibration mode**

Natural frequency: $\omega_2 = 1.62 \cdot \sqrt{100}\text{s}^{-1} = 16.2\ \text{Hz}$

Natural period: $T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{16.2\ \text{Hz}} = 0.39\ \text{s}$

The eigenvectors are dimensionless quantities and remain unchanged:

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & 1 \end{bmatrix} \qquad (12.124)$$

3) Modal analysis

Equation of motion in u_1-u_2-coordinates (without damping):

$$\mathbf{M} \cdot \ddot{u} + \mathbf{K} \cdot u = -\mathbf{M} \cdot \mathbf{1} \cdot \ddot{u}_g t \tag{12.125}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{kg} \cdot \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix} \mathrm{kg/s^2} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{kg} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \ddot{u}_g(t) \tag{12.126}$$

Variables transformation in modal coordinates q_1 and q_2:

$$\mathbf{u} = \mathbf{\Phi} \cdot \mathbf{q} \tag{12.127}$$

where:

$\mathbf{\Phi}$: Modal Matrix, i.e. the matrix of the eigenvectors

The equation of motion in modal coordinates q_1 and q_2 (without damping) is:

$$\mathbf{M}^* \cdot \ddot{\mathbf{q}} + \mathbf{K}^* \cdot \mathbf{q} = -\mathbf{L} \cdot \ddot{u}_g(t) \tag{12.128}$$

$$\begin{bmatrix} 1.38 & 0 \\ 0 & 1.38 \end{bmatrix} \mathrm{kg} \cdot \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 52.8 & 0 \\ 0 & 362 \end{bmatrix} \mathrm{kg/s^2} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = -\begin{bmatrix} 1.62 \\ 0.382 \end{bmatrix} \mathrm{kg} \cdot \ddot{u}_g(t) \tag{12.129}$$

yielding the equation of motion in modal coordinates of two independent SDoF systems.

4) Peak modal response

The peak modal response of both vibration modes can be computed like in the case of SDoF systems using the spectral value given by the relevant response spectrum.

The peak value of the modal coordinate q_1 is:

$$q_{1,\,max} = \frac{L_1}{m_1^*} \cdot S_d(\omega_1, \zeta_1) = \Gamma_1 \cdot S_d(\omega_1, \zeta_1) \tag{12.130}$$

where:

$S_d(\omega_1, \zeta_1)$: spectral displacement for a natural frequency ω_1 and a damping ζ_1 (here $\zeta_1 = 5\%$)

If an acceleration instead of a displacement response spectrum is used, then the peak value of the modal coordinate q_1 is:

$$q_{1,\,max} = \frac{L_1}{m_1^*} \cdot \frac{1}{\omega_1^2} \cdot S_{pa}(\omega_1, \zeta_1) = \frac{\Gamma_1}{\omega_1^2} \cdot S_{pa}(\omega_1, \zeta_1) \tag{12.131}$$

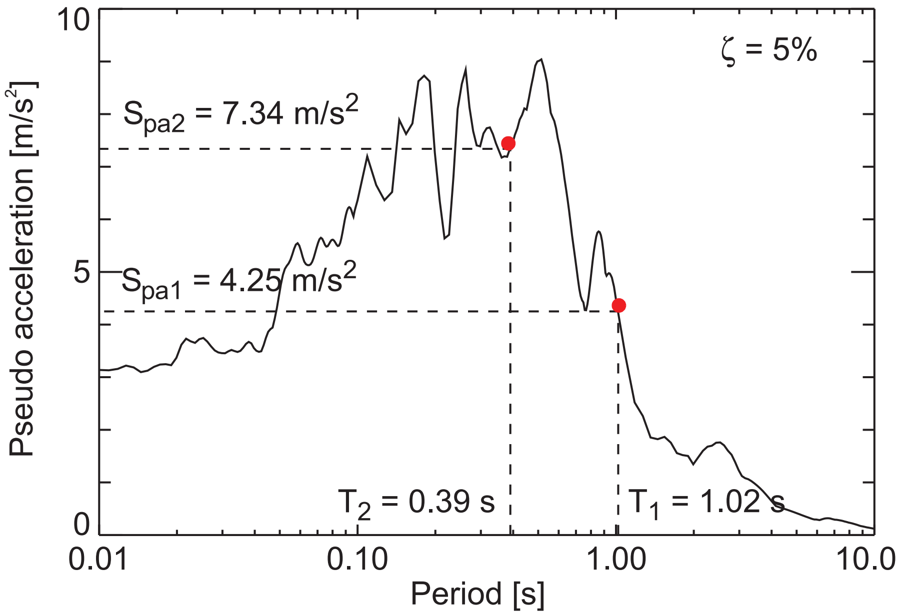
where:

$S_{pa}(\omega_1, \zeta_1)$: spectral pseudo-acceleration for a natural frequency ω_1 and a damping ζ_1 (here $\zeta_1 = 5\%$)

The spectral values given by the elastic acceleration response spectrum of the “El Centro” earthquake for the periods T_1 and T_2 are:

$$S_{pa1} = 4.25\ \text{m/s}^2 \quad \text{and} \tag{12.132}$$

$$S_{pa2} = 7.34\ \text{m/s}^2 \tag{12.133}$$



$$q_{1,\,max} = \frac{1.62\text{kg}}{1.38\text{kg}} \cdot \frac{1}{(6.2\text{Hz})^2} \cdot 4.25\ \text{m/s}^2 = 0.130\text{m} \tag{12.134}$$

$$q_{2,\,max} = \frac{0.38\text{kg}}{1.38\text{kg}} \cdot \frac{1}{(16.2\text{Hz})^2} \cdot 7.34\ \text{m/s}^2 = 0.008\text{m} \tag{12.135}$$

5) Inverse transformation

The peak deformations and internal forces belonging to each mode of vibration in the original reference system are obtained by multiplying the relevant eigenvector with the corresponding peak value of the modal coordinate.

- **Fundamental mode**

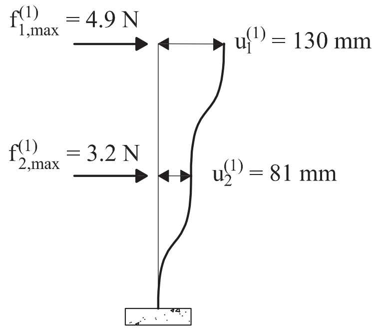
$$\mathbf{u}^{(1)}_{max} = q_{1,\,max} \cdot \boldsymbol{\phi}_1 = 0.130\text{m} \cdot \begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = \begin{bmatrix} 130 \\ 81 \end{bmatrix} \text{mm} \tag{12.136}$$

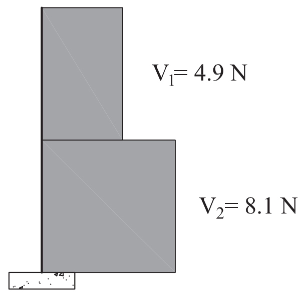
$$\mathbf{f}^{(1)}_{max} = \mathbf{K} \cdot \mathbf{u}^{(1)}_{max} = \begin{bmatrix} 100 & -100 \\ 100 & 200 \end{bmatrix} \cdot \begin{bmatrix} 0.130 \\ 0.081 \end{bmatrix} = \begin{bmatrix} 13.0 - 8.1 \\ -13.0 + 16.2 \end{bmatrix} = \begin{bmatrix} 4.9 \\ 3.2 \end{bmatrix} \text{N} \tag{12.137}$$

Alternatively **(allow an approximate consideration of nonlinearities)**:

$$\mathbf{s}_1 = \Gamma_1 \mathbf{M} \boldsymbol{\phi}_1 = 1.17 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = \begin{bmatrix} 1.17 \\ 0.725 \end{bmatrix} \tag{12.138}$$

$$\mathbf{f}^{(1)}_{max} = \mathbf{s}_1 \cdot S_{pa1} = 4.25 \begin{bmatrix} 1.17 \\ 0.725 \end{bmatrix} = \begin{bmatrix} 4.9 \\ 3.2 \end{bmatrix} \text{N} \tag{12.139}$$





- **Higher vibration mode**

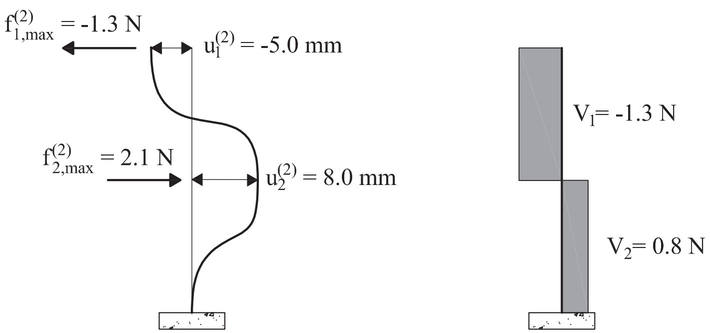
$$\mathbf{u}_{max}^{(2)} = \mathbf{q}_{2,\,max} \cdot \phi_2 = 0.008\mathrm{m} \cdot \begin{bmatrix} -0.62 \\ 1 \end{bmatrix} = \begin{bmatrix} -5.0 \\ 8.0 \end{bmatrix} \mathrm{mm} \tag{12.140}$$

$$\mathbf{f}_{max}^{(2)} = \mathbf{K} \cdot \mathbf{u}_{max}^{(2)} = \begin{bmatrix} 100 & -100 \\ 100 & 200 \end{bmatrix} \cdot \begin{bmatrix} -0.005 \\ 0.008 \end{bmatrix} = \begin{bmatrix} -0.5-0.8 \\ 0.5+1.6 \end{bmatrix} = \begin{bmatrix} -1.3 \\ 2.1 \end{bmatrix} \mathrm{N} \tag{12.141}$$

Alternatively **(allow an approximate consideration of nonlinearities)**:

$$\mathbf{s_2} = \Gamma_2 \mathbf{M} \phi_2 = 0.28 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -0.62 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.173 \\ 0.28 \end{bmatrix} \tag{12.142}$$

$$\mathbf{f}_{max}^{(1)} = \mathbf{s_2} \cdot S_{pa2} = 7.34 \begin{bmatrix} -0.173 \\ 0.28 \end{bmatrix} = \begin{bmatrix} -1.3 \\ 2.1 \end{bmatrix} \mathrm{N} \tag{12.143}$$



6) Combination

The total peak response is obtained from the peak response of the single vibration modes using e.g. the SRSS combination rule (SRSS = Square Root of the Sum of Squares).

- **Peak displacements**

$$u_{1,\,max} = \sqrt{\sum_{k=1}^{2} (u_1^{(n)})^2} = \sqrt{(130\mathrm{mm})^2 + (-5\mathrm{mm})^2} = 130\mathrm{mm} \tag{12.144}$$

$$u_{2,\,max} = \sqrt{\sum_{k=1}^{2} (u_2^{(n)})^2} = \sqrt{(81\mathrm{mm})^2 + (8\mathrm{mm})^2} = 81\mathrm{mm} \tag{12.145}$$

In this case the total peak displacements are almost identical to the peak displacements of the fundamental mode. The relatively small contributions due to the second vibration mode basically disappear because of the SRSS combination rule.

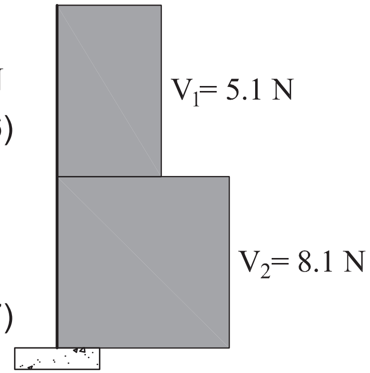
- **Peak sectional forces (Shear force V)**

Upper shear force:

$$V_{1,\,max} = \sqrt{(4.9\text{N})^2 + (-1.3\text{N})^2} = 5.1\text{N} \qquad (12.146)$$

Lower shear force:

$$V_{2,\,max} = \sqrt{(8.1\text{N})^2 + (0.8\text{N})^2} = 8.1\text{N} \qquad (12.147)$$



Compared to the peak sectional forces due to the fundamental mode, the total peak sectional forces show a slight increase in the upper story of the 2-DoF system.

Pay attention to following pitfall!

It is wrong to compute the total peak sectional forces using the total peak displacements:

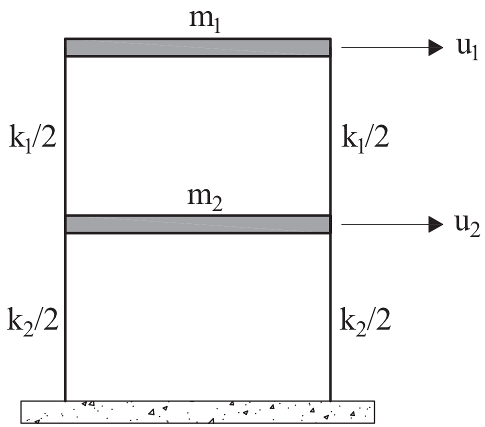
$$V_{1,\,max} \neq 100\text{N/m} \cdot (0.130\text{m} - 0.081\text{m}) = 4.9\text{N} \qquad (12.148)$$

$$V_{2,\,max} \neq 100\text{N/m} \cdot 0.081\text{m} = 8.1\text{N} \qquad (12.149)$$

The sectional forces would be underestimated.

12.4.5 Response spectrum method vs. time-history analysis

1) Model



Case study 1		Case study 2	
Masses:	$m_1 = 1.0\text{kg}$	Masses:	$m_1 = 0.1\text{kg}$
	$m_2 = 1.0\text{kg}$		$m_2 = 1.0\text{kg}$
Stiffnesses:	$k_1 = 100\text{N/m}$	Stiffnesses:	$k_1 = 10\text{N/m}$
	$k_2 = 100\text{N/m}$		$k_2 = 100\text{N/m}$

Case study 1 corresponds to the model analysed in Section 12.4.4. Case study 2 represents a dynamic system where the second vibration mode is important.

2) Results

- **Dynamic properties**

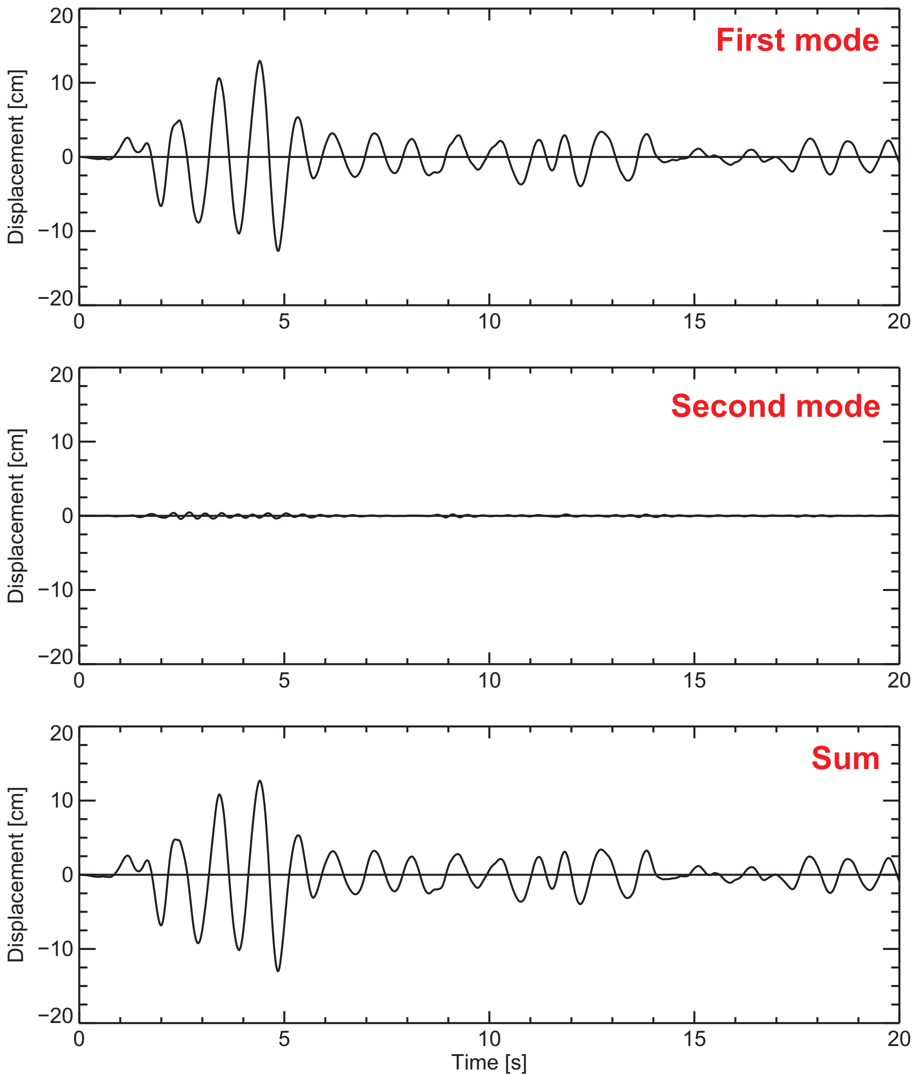
Case study 1		Case study 2	
Periods:	$T_1 = 1.02s$	Periods:	$T_1 = 0.74s$
	$T_2 = 0.39s$		$T_2 = 0.54s$
Eigenvectors:		Eigenvectors:	
1: $\phi_{11} = 1$,	$\phi_{21} = 0.62$	1: $\phi_{11} = 1$,	$\phi_{21} = 0.27$
2: $\phi_{12} = 1$,	$\phi_{22} = -1.62$	2: $\phi_{12} = 1$,	$\phi_{22} = -0.37$
Part. factors:	$\Gamma_1 = 1.17$	Part. factors:	$\Gamma_1 = 2.14$
	$\Gamma_2 = -0.17$		$\Gamma_2 = -1.14$

- Note that in this case the eigenvectors are normalized to yield unit displacement at the top of the second story. Therefore, the eigenvectors and the participation factors of case study 1 differ from the values obtained in previous sections.

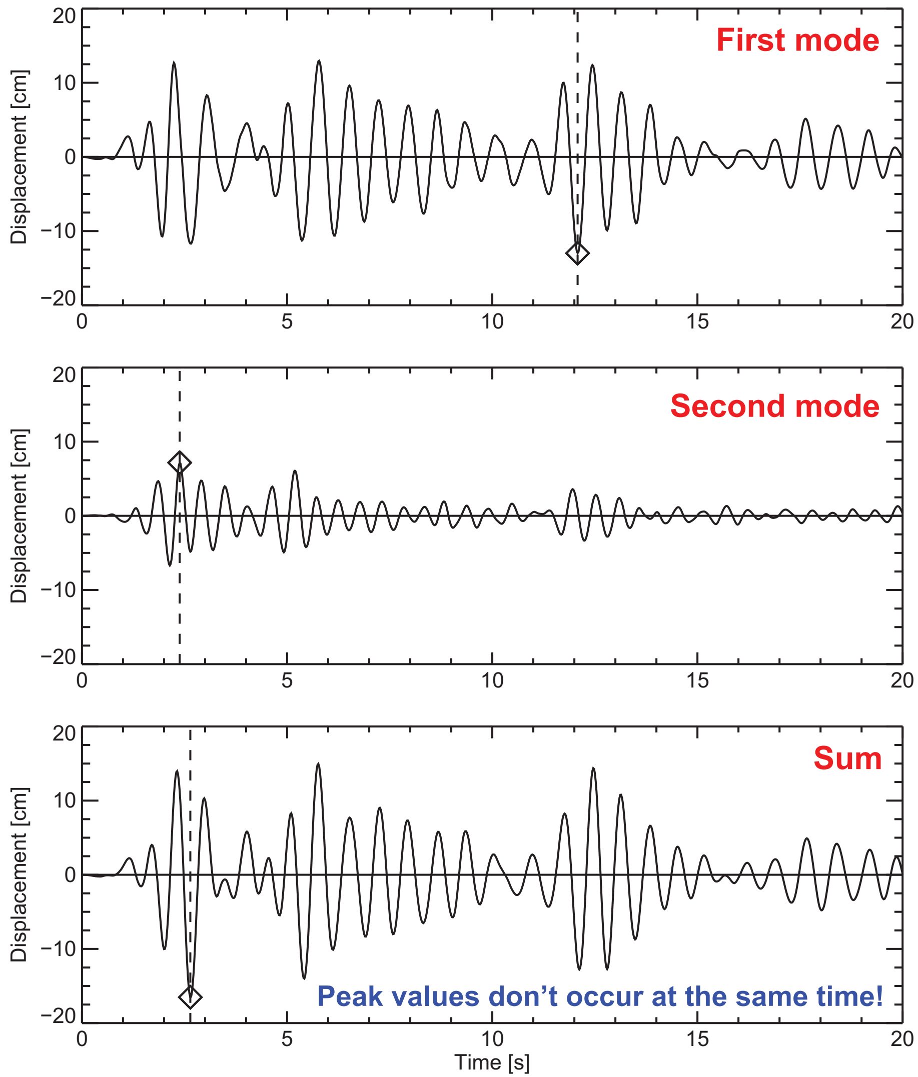
- **Demand**

Case study 1		Case study 2	
Displacements:		Displacements:	
1:	$\Delta = 0.129m$	1:	$\Delta = 0.130m$
2:	$\Delta = 0.005m$	2:	$\Delta = 0.072m$
Sum:	$\Delta = 0.134m$	Sum:	$\Delta = 0.202m$
SRSS:	$\Delta = 0.130m$	SRSS:	$\Delta = 0.148m$
Time-history:	$\Delta = 0.130m$	Time-history:	$\Delta = 0.165m$
Upper shear force:		Upper shear force:	
SRSS:	$V = 5.10N$	SRSS:	$V = 1.36N$
Time-history:	$V = 5.69N$	Time-history:	$V = 1.51N$
Lower shear force:		Lower shear force:	
SRSS:	$V = 8.05N$	SRSS:	$V = 4.40N$
Time-history:	$V = 8.44N$	Time-history:	$V = 4.92N$

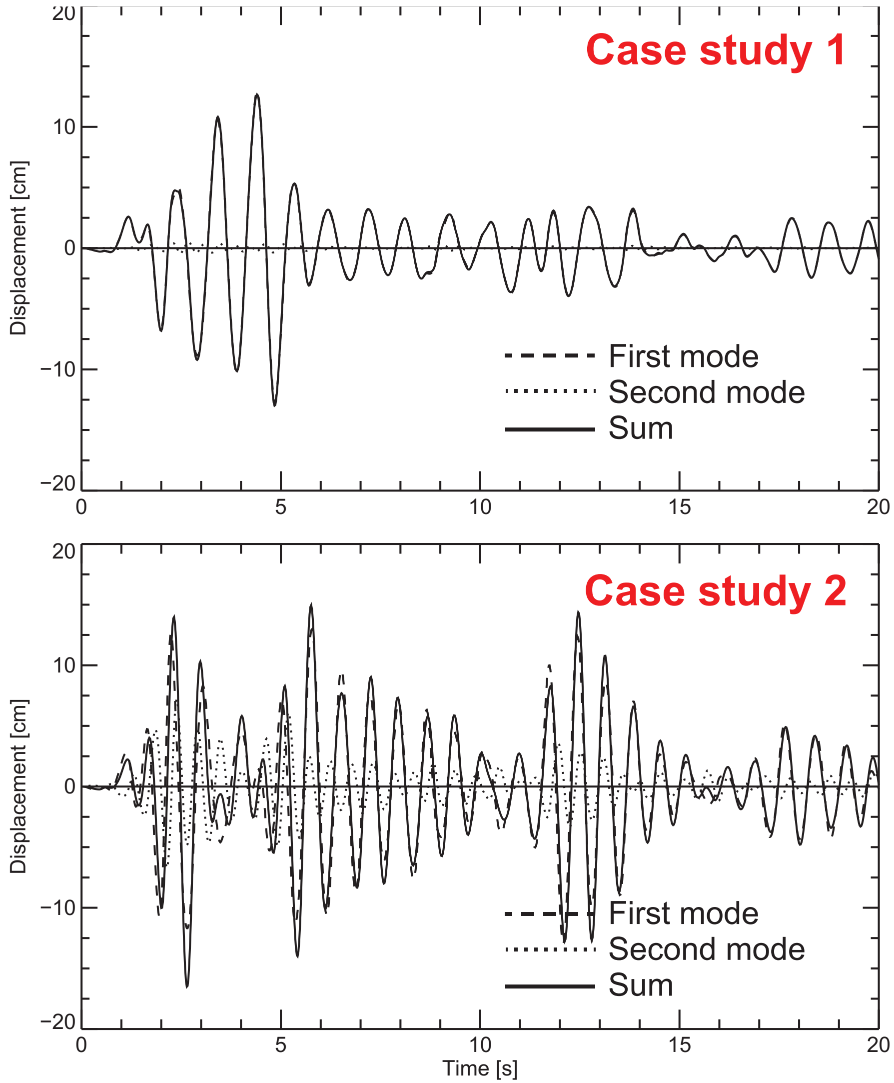
- **Time-histories: Case study 1**



- **Time-histories: Case study 2**



- **Time-histories: Summary**



Blank page

13 Vibration Problems in Structures

13.1 Introduction

There are more and more **vibration problems in structures** because:

- Higher quality materials with higher exploitation
  - slender constructions
  - smaller stiffnesses and masses
- More intensive dynamic excitations
- Increased sensitivity of people

Nevertheless vibration sensitive structures are often designed for static loads only

Goal of this chapter

- Give an overview of possible causes of vibration problems in buildings and of potential countermeasures
- Description of practical cases with vibration rehabilitation

13.1.1 Dynamic action

a) People-induced vibrations

- Pedestrian bridges
- Floors with walking people
- Floors for sport or dance activities
- Floor with fixed seating and spectator galleries
- High-diving platforms

b) Machinery-induced vibrations

- Machine foundations and supports
- Bell towers
- Structure-borne sound
- Ground-transmitted vibrations

c) Wind-induced vibrations

- Buildings
- Towers, chimneys and masts
- Bridges
- Cantilevered roofs

d) Vibrations induced by traffic and construction activity

- Roads and bridges
- Railways
- Construction works

13.1.2 References

[Bac+97] **Bachmann et al.:"Vibration Problems in Structures". ISBN 3-7643-5148-9. Birkhäuser Verlag, Basel 1997.**

[BB88] Baumann K., Bachmann H.: "Durch Menschen verursachte dynamische Lasten und deren Auswirkungen auf Balkentragwerke". IBK Bericht Nr. 7501-3, 1988.

[BW95] Bachmann H., Weber B.: "Tuned Vibration Absorbers for Damping of Lively Structures". Structural Engineering International, No. 1, 1995.

[SIA06] SIA: "Wind – Kommentar zum Kapitel 6 der Normen SIA 261 und 261/1 (2003) Einwirkungen auf Tragwerke". SIA Dokumentation D0188. Zürich 2006.

[SIA03] SIA: "Aktuelle Probleme der Brückendynamik". SIA Dokumentation D0198. Zürich 2003.

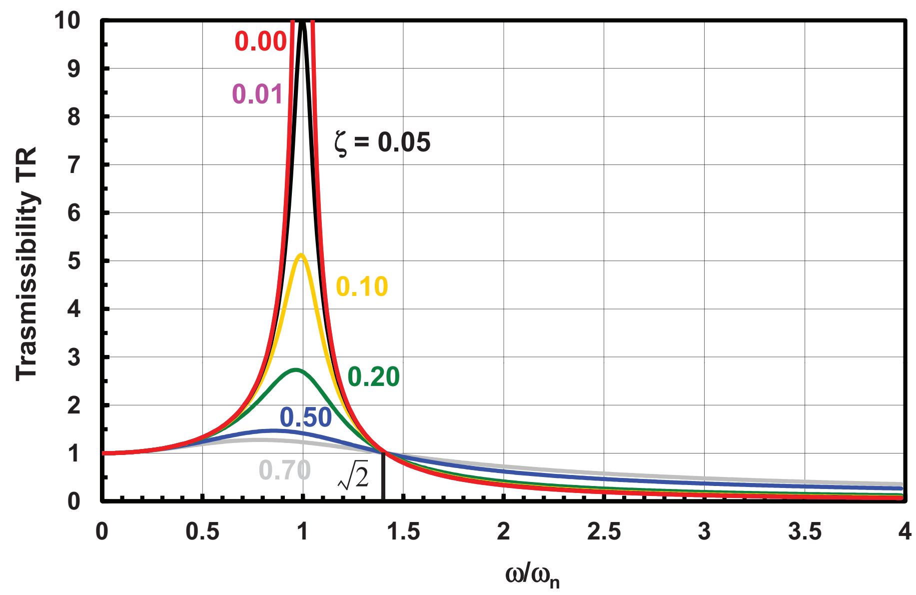
[SIA96] SIA: "Dynamische Probleme bei Brücken- und Hochbauten". SIA Dokumentation D0138. Zürich 1996.

[VB87] Vogt R., Bachmann H.: "Dynamische Kräfte beim Klatschen, Fussstampfen und Wippen". IBK Bericht Nr. 7501-4, 1987.

13.2 Vibration limitation

13.2.1 Verification strategies

- Frequency tuning

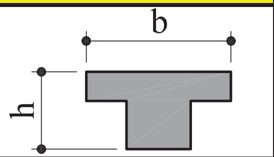
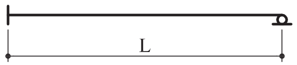
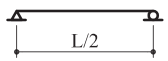


  - High tuning (subcritical excitation)
  - Low tuning (supercritical excitation)

- Amplitude limitation

13.2.2 Countermeasures

- Change of the natural frequency

Strategy	Effects
	Stiffness: $I \sim bh^3$; Mass: $A \sim bh$ $\Rightarrow f \sim \sqrt{\frac{I}{A}} \sim h$
	$f_{a,1} = \frac{\pi^2}{2\pi L^2} \cdot \sqrt{\frac{EI}{\mu}}$ (μ = distributed mass)
	$f_{b,1} = \frac{3.93^2}{2\pi L^2} \cdot \sqrt{\frac{EI}{\mu}} = 1.56 f_{a,1}$ $(2.45 \cdot EI)$
	$f_{c,1} = \frac{4.73^2}{2\pi L^2} \cdot \sqrt{\frac{EI}{\mu}} = 2.27 f_{a,1}$ $(5.14 \cdot EI)$
	$f_{d,1} = \frac{\pi^2}{2\pi (L/2)^2} \cdot \sqrt{\frac{EI}{\mu}} = 4 f_{a,1}$ $(16 \cdot EI)$

- Increase of the damping
  - Installation of dampers or absorbers
  - Plastic energy dissipation
- Tuned Mass Dampers

13.2.3 Calculation methods

- Computation of the natural frequencies

The natural frequencies of structures have to be determined by means of realistic models. Approximate formulas that are often found in design codes and literature shall be checked carefully.

- Computation of the Amplitude

If the frequency of a harmonic of the excitation coincides with a natural frequency of the structure (resonance), the maximum deflection of the structure can be estimated as follows (See Chapter 5):

$$u_p = \frac{F_o}{k} \cdot V(\omega) \cdot \cos(\omega t - \phi) \tag{13.1}$$

for $\omega = \omega_n$ we have $V(\omega) = 1/(2\zeta)$ and:

$$u_{max} = \frac{F_o}{k} \cdot \frac{1}{2\zeta} \tag{13.2}$$

The maximum velocity and the maximum acceleration can be determined from Equation (13.2) as follows:

$$\dot{u}_{max} = \omega \cdot u_{max} = \omega \cdot \frac{F_o}{k} \cdot \frac{1}{2\zeta} \tag{13.3}$$

$$\ddot{u}_{max} = \omega^2 \cdot u_{max} = \omega^2 \cdot \frac{F_o}{k} \cdot \frac{1}{2\zeta} \tag{13.4}$$

The amplitude of the n^{th} harmonic component of a force generated by people excitation is proportional to the mass of the person ($F_o = G \cdot \alpha_n = g \cdot M \cdot \alpha_n$, see Equation (13.8)).

$$\ddot{u}_{max} = \omega^2 \cdot \frac{g \cdot M \cdot \alpha_n}{k} \cdot \frac{1}{2\zeta} = \frac{k}{m} \cdot \frac{g \cdot M \cdot \alpha_n}{k} \cdot \frac{1}{2\zeta} = \frac{M}{m} \cdot \frac{g \cdot \alpha_n}{2\zeta} \quad (13.5)$$

$$\ddot{u}_{max} = \frac{M}{m} \cdot \frac{g \cdot \alpha_n}{2\zeta} \quad (13.6)$$

- Remarks
  - A soft structure is more prone to vibration than a rigid one. See Equations (13.2) to (13.4).
  - The acceleration amplitude is directly proportional on the ratio of the mass of the people to the building mass.

13.3 People induced vibrations

13.3.1 Excitation forces

In Chapter 6 “Forced Vibrations” it has been already mentioned that excitation due to people, like e.g. walking, running, jumping, and so on, can be represented as Fourier-series:

$$F(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \quad (13.7)$$

Equation (13.7) can also be represented in a form according to Equation (13.8):

$$F(t) = G + \sum_{n=1}^{N} G \cdot \alpha_n \cdot \sin(n \cdot 2\pi f_0 \cdot t - \phi_n) \quad (13.8)$$

Where:

- G = Weight of the person
- α_n = Fourier coefficient for the n^{th} harmonic
- $G \cdot \alpha_n$ = Amplitude of the n^{th} harmonic of the excitation force
- f_0 = Step frequency of the excitation force
- ϕ_n = Phase shift of the n^{th} harmonic ($\phi_1 = 0$)
- n = Number of the n^{th} harmonic
- N = Number of considered harmonics

The steady-state response of a SDoF system under periodic excitation can be computed in analogy to Chapter 6 as:

$$u(t) = u_0(t) + \sum_{n=1}^{N} u_n(t) \tag{13.9}$$

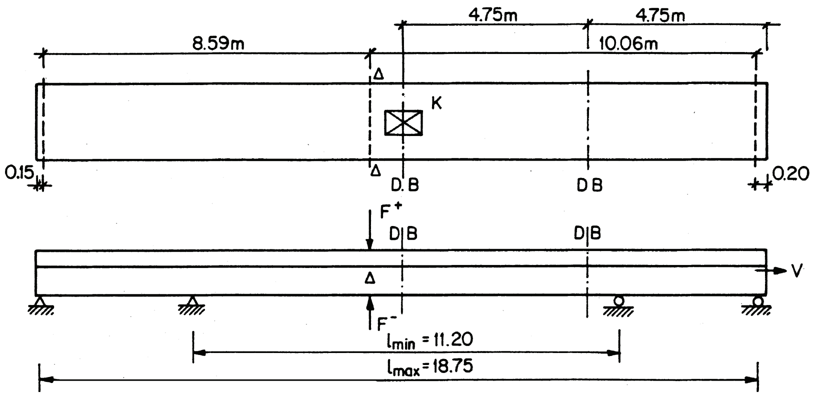
Where

$$u_0(t) = \frac{G}{k} \text{ (Static displacement)} \tag{13.10}$$

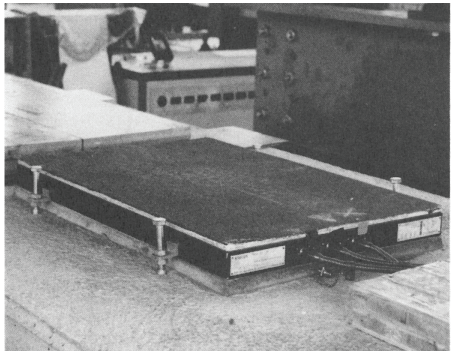
$$u_n(t) = \frac{G \cdot \alpha_n}{k} \cdot \frac{(1-\beta_n^2)\sin(n\omega_0 t - \phi_n) - 2\zeta\beta_n\cos(n\omega_0 t - \phi_n)}{(1-\beta_n^2)^2 + (2\zeta\beta_n)^2} \tag{13.11}$$

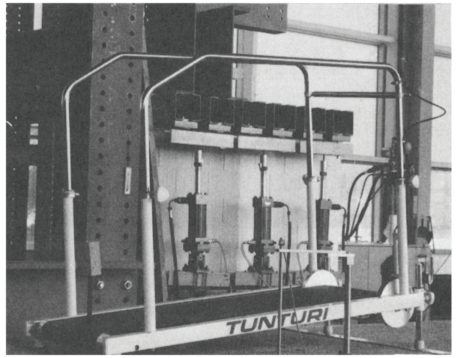
$$\omega_0 = 2\pi f_0 \qquad , \qquad \beta_n = \frac{n\omega_0}{\omega_n} \tag{13.12}$$

- **Measurement of forces (Example)**

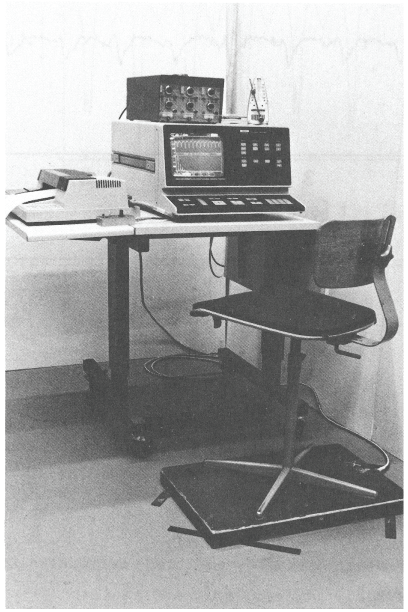


- Jumping (left) and walking (right), see [BB88]

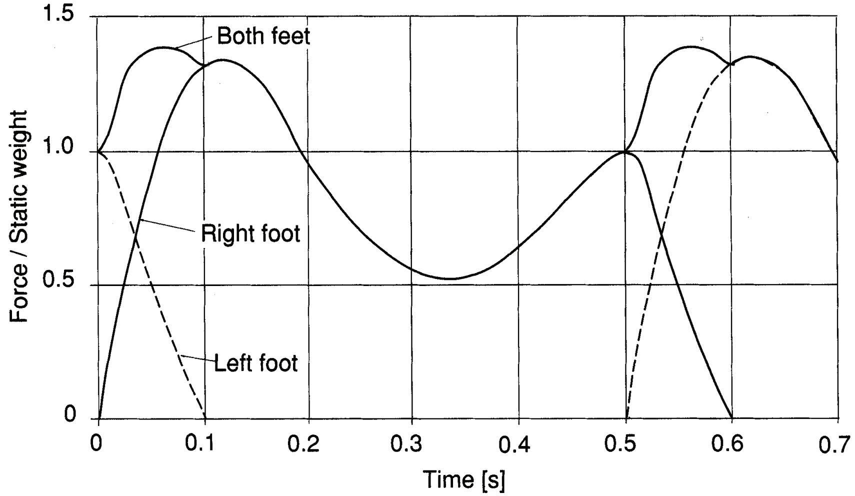




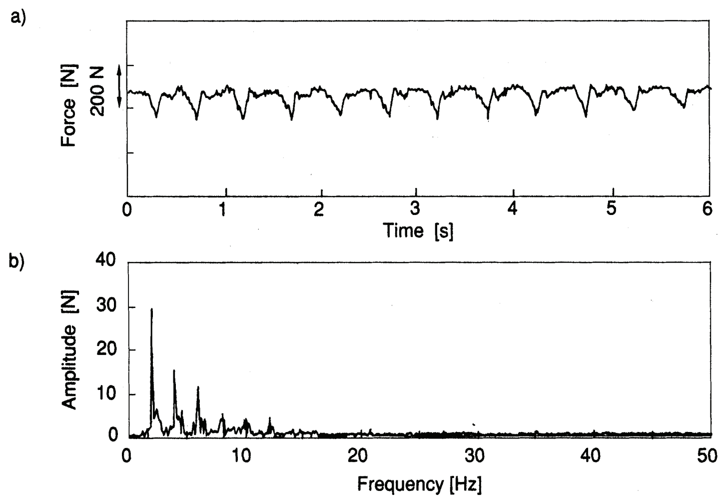
- Clapping, foot stomping and rocking, see [VB87]



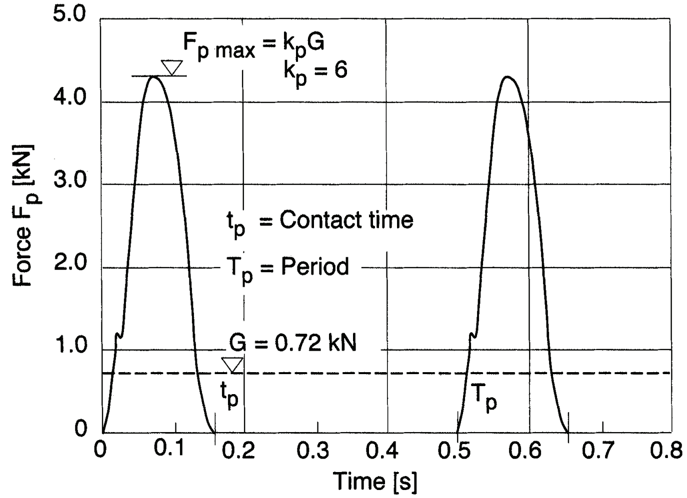
- Walking (see [Bac+97] Figure G.1)



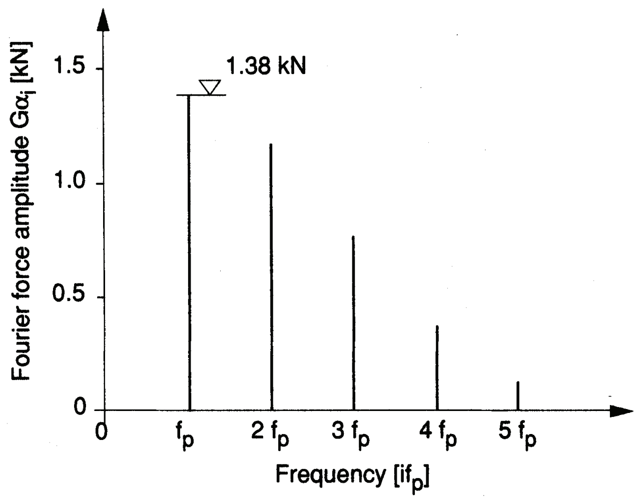
- Clapping (see [Bac+97] Figure G.3)



- Jumping (see [Bac+97] Figure G.2)



- Jumping: Fourier amplitude spectrum (see [Bac+97] Figure G.2)



Representative types of activity			Range of applicability		
Designation	Definition	Design activity rate [Hz]	Actual activities	Activity rate [Hz]	Structure type
"walking"	walking with continuous ground contact	1.6 to 2.4	• slow walking (ambling) • normal walking • fast, brisk walking	~ 1.7 ~ 2.0 ~ 2.3	• pedestrian structures (pedestrian bridges, stairs, piers, etc.) • office buildings, etc.
"running"	running with discontinuous ground contact	2.0 to 3.5	• slow running (jog) • normal running • fast running (sprint)	~ 2.1 ~ 2.5 > 3.0	• pedestrian bridges on jogging tracks, etc.
"jumping"	normal to high rhythmical jumpingon the spot with simultaneous ground contact of both feet	1.8 to 3.4	• fitness training with jumping, skipping and running to rhythmical music • jazz dance training	~ 1.5 to 3.4 ~ 1.8 to 3.5	• gymnasia, sport halls • gymnastic training rooms
"dancing"	approximatly equivalent to "brisk walking"	1.5 to 3.0	• social events with classical and modern dancing (e.g. English Waltz, Rumba etc.)	~ 1.5 to 3.0	• dance halls • concert halls and other community halls without fixed seating
"hand clapping with body bouncing while standing"	rhythmical hand clapping in front of one's chest or above the head while bouncing vertically by forward and backward knee movement of about 50 mm	1.5 to 3.0	• pop concerts with enthusiastic audience	~ 1.5 to 3.0	• concert halls and spectator galleries with and without fixed seating and "hard" pop concerts
"hand clapping"	rhythmical hand clapping in front of one's chest	1.5 to 3.0	• classical concerts, "soft" pop concerts	~ 1.5 to 3.0	• concert halls with fixed seating (no "hard" pop concerts)
"lateral body swaying"	rhythmical lateral body swaying while being seated or standing	0.4 to 0.7	• concerts, social events		• spectator galleries

Characterisation of human activities according to [Bac+97] Table G.1

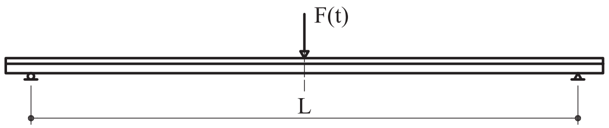
Representative type of activity	Activity rate [Hz]		Fourier coefficient and phase lag					Design density [persons/m²]
			α_1	α_2	ϕ_2	α_3	ϕ_3	
"walking"	vertical	2.0 2.4	0.4 0.5	0.1	$\pi/2$	0.1	$\pi/2$	~ 1
	forward	2.0	0.2 $\alpha_{1/2} = 0.1$	0.1				
	lateral	2.0	$\alpha_{1/2} = 0.1$	$\alpha_{3/2} = 0.1$				
"running"		2.0 to 3.0	1.6	0.7		0.2		-
"jumping"	normal	2.0 3.0	1.8 1.7	1.3 1.1	*) *)	0.7 0.5	*) *)	in fitness training ~ 0.25 (in extreme cases up to 0.5)
	high	2.0 3.0	1.9 1.8	1.6 1.3	*) *)	1.1 0.8	*) *)	*) $\phi_2 = \phi_3 = \pi(1 - f_p t_p)$
"dancing"		2.0 to 3.0	0.5	0.15		0.1		~ 4 (in extreme cases up to 6)
"hand clapping with body bouncing while standing"		1.6 2.4	0.17 0.38	0.10 0.12		0.04 0.02		no fixed seating ~ 4 (in extreme cases up to ~ 6) with fixed seating ~ 2 to 3
"hand clapping"	normal	1.6 2.4	0.024 0.047	0.010 0.024		0.009 0.015		~ 2 to 3
	intensive	2.0	0.170	0.047		0.037		
"lateral body swaying"	seated	0.6	$\alpha_{1/2} = 0.4$	-		-		
	standing	0.6	$\alpha_{1/2} = 0.5$	-		-		~ 3 to 4

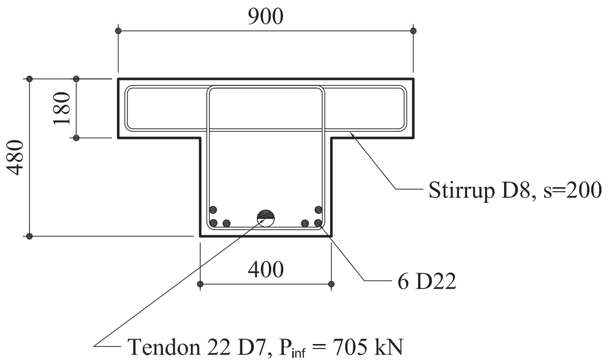
Fourier-coefficients for human activities according to [Bac+97] Table G.2

- Remarks regarding Table G.2
  - Coefficients and phase angles represent averages.
  - Phase angles have strong scattering and therefore, in many cases, it is difficult to provide reasonable values. In such cases (e.g,. running and dancing) in Table G.2 no values are specified.
  - Decisive are cases in which resonance occurs. In such cases the phase angle no longer plays a role.
  - Coefficients and phase were checked and discussed internationally.

13.3.2 Example: Jumping on an RC beam

Here the same example as in Section 6.1.3 is considered again:





- RC Beam

The RC beam has a length of 19 meters. The natural frequency is thus:

$$f_n = 2\text{Hz}. \qquad (13.13)$$

- Excitation

Here "jumping" is described by means of the Fourier-series given in Table G.2. In Section 6.1 "periodic excitation", "jumping" was described by means of a half-sine function.

Jumping frequency: $f_0 = 2\text{Hz}$ (13.14)

Contact time: $t_p = 0.16\text{s}$ (phase angle computation) (13.15)

Weight of the person: $G = 0.70\text{kN}$ (13.16)

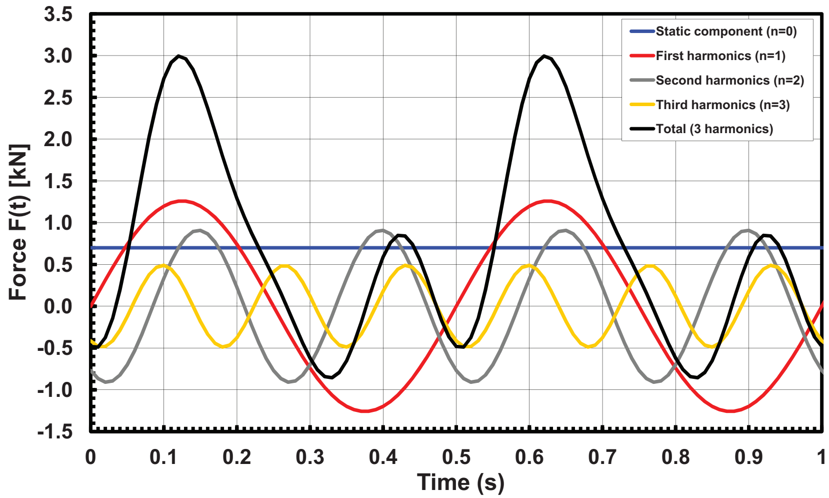
- Results

Excel Table: $u_{max} = 0.043\text{m}$ (13.17)

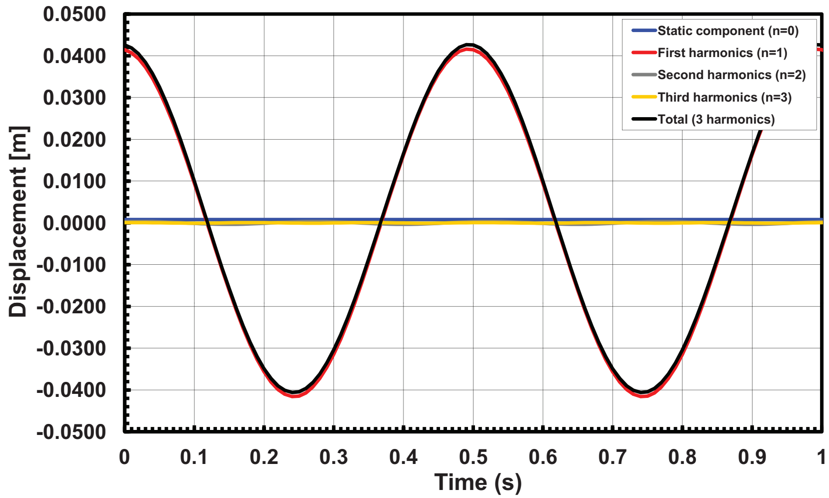
Equation (13.2): $u_{max} = \frac{F}{k} \cdot \frac{1}{2\zeta} = \frac{1.8 \cdot 0.70}{886} \cdot \frac{1}{2 \cdot 0.017} = 0.042\text{m}$ (13.18)

- Remarks
  - Shape of the excitation "similar" as half-sine
  - Maximum deflection very close to the solution obtained by means of the half-sine function

- Excitation



- Response



13.3.3 Footbridges

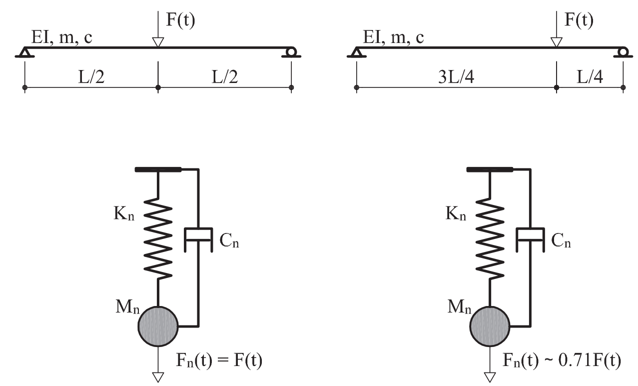
- Frequency tuning
  - **Vertical**: Avoid natural frequencies between 1.6 and 2.4Hz. In the case of structures with low damping (Steel), avoid also natural frequencies from 3.5 to 4.5Hz.
  - **Horizontal transverse**: Avoid natural frequencies between 0.7 and 1.3Hz (absolutely safe: $f_{ht,1} > 3.4$Hz).
  - **Horizontal longitudinal**: Avoid natural frequencies between 1.6 and 2.4Hz.
- Amplitude limitation
  - Calculation of the acceleration maximum amplitude.

$$a_{max} < \text{ca. } 0.5\text{m/s}^2 = 5\%\ g \qquad (13.19)$$

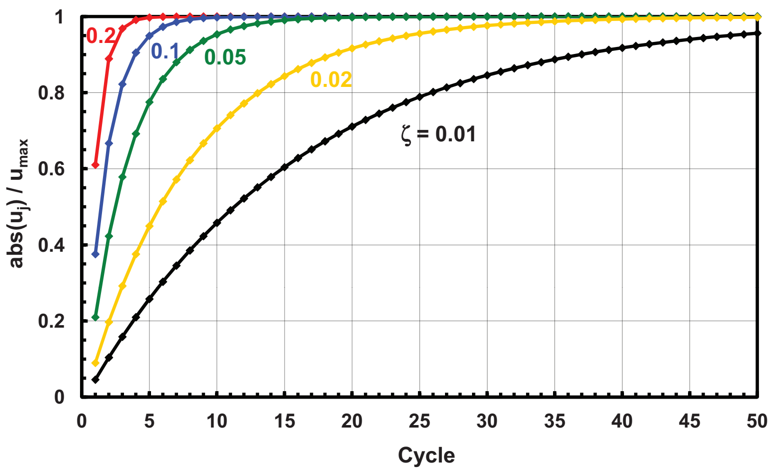
- Special features of the amplitude limitation
  - When walking or running, the effectiveness of people is limited, because the forces are not always applied at midspan;

System

Equivalent SDoF system



- People need a finite number of steps in order to cross the bridge (This limited excitation time may be too short to reach the maximum amplitude)



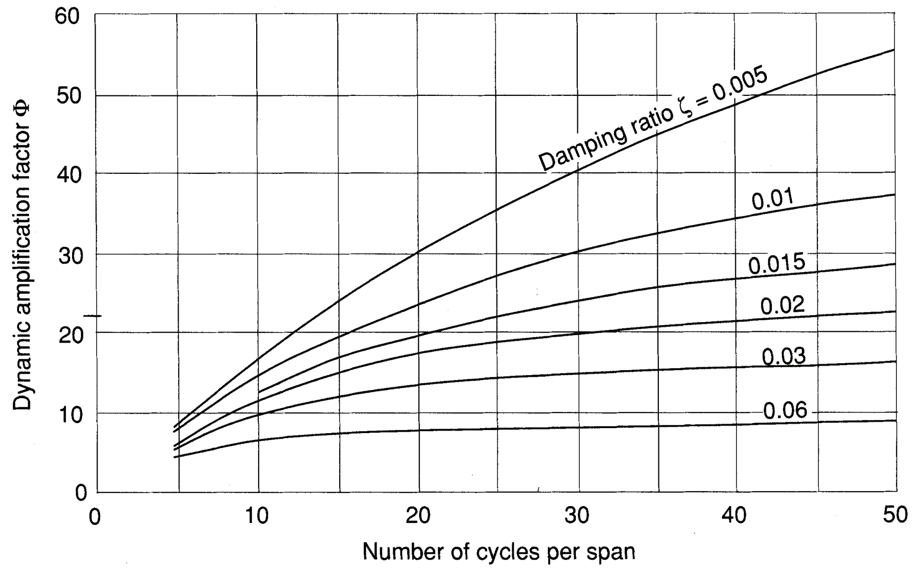
- Not all people walk in the step (Exception: Lateral vibrations $\rightarrow$ Synchronisation effect)

To take into account the specificities of the amplitude limitation, sophisticated methods are available. From [Bac+97] the following one is adopted:

$$a_{max} = 4\pi^2 \cdot f^2 \cdot y \cdot \alpha \cdot \Phi \ [m/s^2] \qquad (13.20)$$

Where:

- y: Static deflection at half the span
- α: Fourier coefficient
- Φ: dynamic amplification factor



The acceleration a_{max} given in Equation (13.20) is the acceleration generated by one person crossing the footbridge. If n people are on the bridge at the same time, the maximum acceleration is typically less than $n \cdot a_{max}$ because not all people walk in step across the bridge.

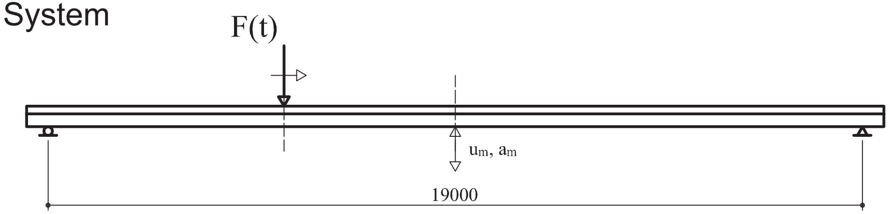
.

The square root of the number of people is often chosen as the multiplication factor, i.e. $\sqrt{n} \cdot a_{max}$

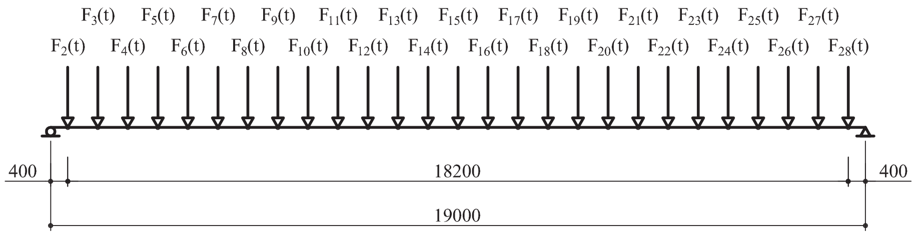
Example: “Walking on an RC beam”

- Situation

System



Discretisation for FE Analysis



  - Stiffness at mid span: $K_n = 886\text{kN/m}$
  - Natural frequency: $f_n = 2\text{Hz}$
  - Damping: $\zeta = 0.017$
- Excitation
  - Walking with $f_0 = 2\text{Hz}$ according to Table G.2.
  - Step length: $S = 0.70\text{m}$
  - Weight of the Person: $G = 1\text{kN}$

- Rough estimate of maximum displacement and acceleration

The maximum displacement, and the maximum acceleration can be estimated by means of Equations (13.2) and (13.4):

$$u_{max,st} = \frac{G}{K_n} = \frac{1}{886} = 0.001128\text{m} = 0.11\text{cm} \qquad (13.21)$$

$$u_{max,1} = \frac{G \cdot \alpha_1}{K_n} \cdot \frac{1}{2\zeta} = \frac{1 \cdot 0.4}{886} \cdot \frac{1}{2 \cdot 0.017} = 0.0133\text{m} = 1.33\text{cm} \qquad (13.22)$$

$$u_{max} = 0.11 + 1.33 = 1.44\text{cm} \qquad (13.23)$$

$$a_{max} = \omega^2 u_{max,1} = (2\pi \cdot 2)^2 \cdot 0.0133 = 2.10\text{m/s}^2 \qquad (13.24)$$

- Estimate of the maximum displacement and acceleration using the improved method

The maximum acceleration is computed by means of Equation (13.20) as follows:

Walking-velocity: $v = S \cdot f_0 = 0.7 \cdot 2 = 1.4\text{m/s}$ (13.25)

Crossing time: $\Delta t = L/v = 19/1.4 = 13.57\text{s}$ (13.26)

Number of cycles: $N = \Delta t \cdot f_n = 13.57 \cdot 2 = 27$ (13.27)

Amplification factor: $\Phi = 23$ (13.28)

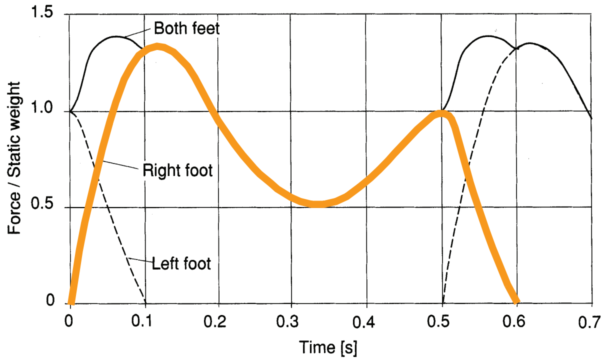
From Equation (13.20):

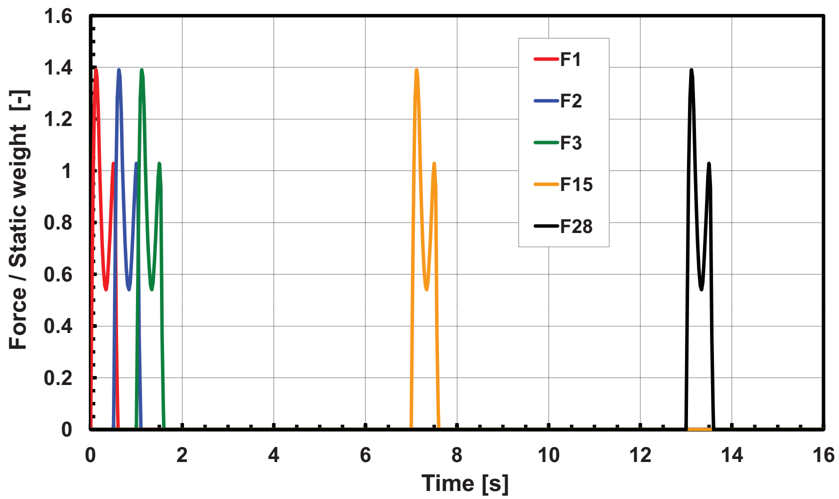
$$a_{max} = 4\pi^2 \cdot 2^2 \cdot \frac{1.00}{886} \cdot 0.40 \cdot 23 = 1.64\text{m/s}^2 \qquad (13.29)$$

- Computation of displacements and accelerations by means of the FE Programme ABAQUS

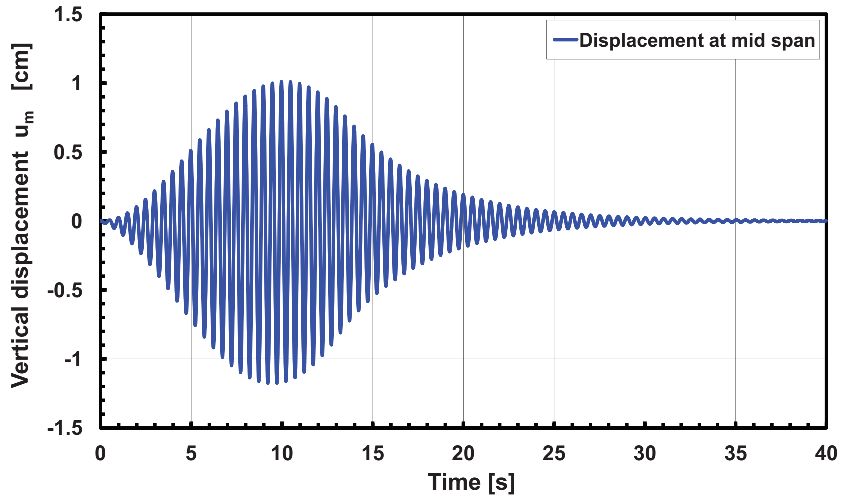
Displacements and accelerations are computed by means of time-history analysis:

- Excitation

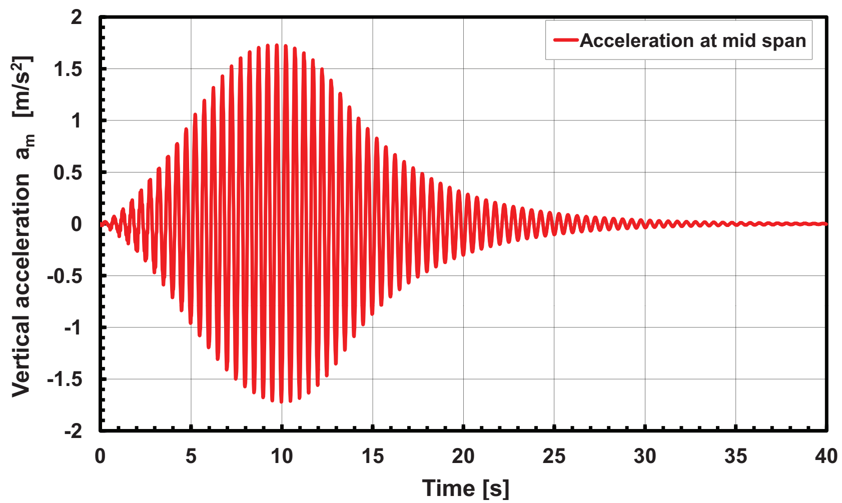




- Time history of the displacement



- Time history of the acceleration



- Remarks
  - The refined method and the time history calculations show lower values compared to the rough method;
  - The refined method and the time history calculations are in good agreement;
  - The time history of the **displacement** is **not** symmetric compared to the time axis, because of the static component of the displacement caused by the weigh of the crossing person;
  - The time history of the **acceleration** is symmetric compared to the time axis, because there is no static component of the acceleration.

- **Swinging footbridge on the Internet**

http://www.londonmillenniumbridge.com/

http://www.youtube.com

13.3.4 Floors in residential and office buildings

- Frequency tuning
  - If the excitation is generated by walking ($f_{max} \cong 2.4\text{Hz}$), the following natural frequencies shall be exceeded:

Damping	Natural frequency [Hz]	Remark
> 5%	> 5	Avoid resonance due to the **second** harmonic
< 5%	> 7.5	Avoid resonance due to the **third** harmonic

- Amplitude limitation
  - Calculation of the acceleration maximum amplitude

$$a_{max} < \text{ca. } 0.05\text{m/s}^2 = 0.5\%\text{ g} \tag{13.30}$$

  - Because of the many non-structural components (wallpaper, furniture, suspended ceilings, technical floors, partitions, ...) it is difficult to estimate the dynamic properties of the floors.
  - Where possible measure the dynamic properties.

Response of people to vibrations

The sensitivity of people to vibration depends on many parameters:

- Position (standing, sitting, lying)
- Direction of the action compared to the spinal column
- Activity (resting, walking, running, ...)
- Type of vibration
- ....

Description	Frequency 1 to 10 Hz a_{max} [m/s²]	Frequency 10 to 100 Hz v_{max} [m/s]
Barely noticeable	0.034	0.0005
Clearly noticeable	0.1	0.0013
Disturbing	0.55	0.0068
Not tolerable	1.8	0.0138

Vertical harmonic vibration action on a standing person. Accepted averages; scatters up to a factor of 2 is possible (from [Bac+97])

- ISO 2631 standard

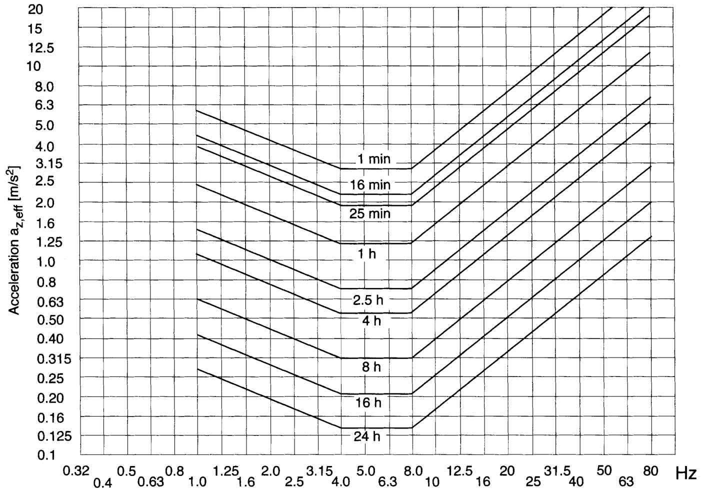
$$a_{eff} = \sqrt{\frac{1}{T}\int_0^T a^2(t)dt} \qquad (13.31)$$

Where T is the period of time over which the effective acceleration was measured.

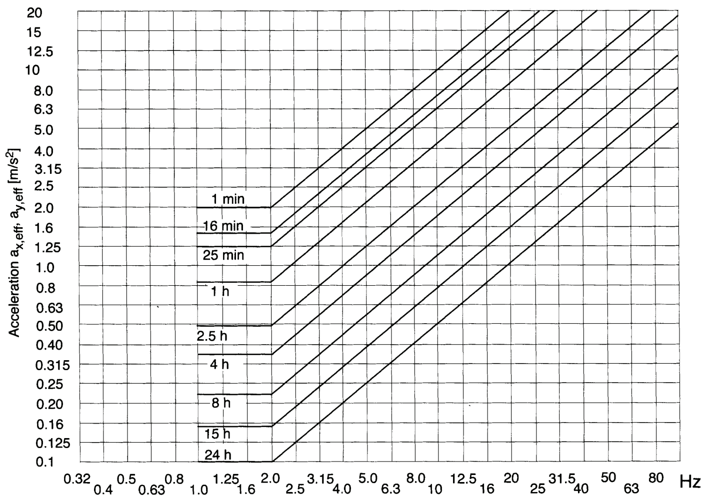
3 limits are defined:

- G1: Reduced comfort boundary
- G2: Fatigue-decreased proficiency boundary ~ 3 x G1
- G3: Exposure limit ~ 6 x G1

- G2 limit for vibrations parallel to the spinal column



- G2 limit for vibrations transverse to the spinal column



13.3.5 Gyms and dance halls

Due to gymnastics or dancing, very large dynamic forces are generated. This is readily understandable when the Fourier coefficients in Table G2 are considered:

- Walking: $\alpha_1 = 0.4$, $\alpha_2 = 0.1$, $\alpha_3 = 0.1$
- Running: $\alpha_1 = 1.6$, $\alpha_2 = 0.7$, $\alpha_3 = 0.2$
- Jumping: $\alpha_1 = 1.9$, $\alpha_2 = 1.6$, $\alpha_3 = 1.1$
- Dancing: $\alpha_1 = 0.5$, $\alpha_2 = 0.15$, $\alpha_3 = 0.1$
  (however: a - many people moving rhythmically. b - certain dances are very similar to jumping)

• Frequency tuning

- If the excitation is generated through jumping ($f_{max} \cong 3.4\text{Hz}$) or dancing ($f_{max} \cong 3.0\text{Hz}$), then the following natural frequencies shall be exceeded:

Construction	Gyms Natural frequency [Hz]	Dance halls Natural frequency [Hz]
Reinforced concrete	> 7.5	> 6.5
Prestressed concrete	> 8.0	> 7.0
Composite structures	> 8.5	> 7.5
Steel	> 9.0	> 8.0

• Amplitude limitation

- Calculation of the acceleration maximum amplitude

$$a_{max} < \text{ca. } 0.5\text{m/s}^2 = 5\%\text{ g} \tag{13.32}$$

- Limits depend on the activity, if e.g. people are sitting in the dance hall, as well, this limit shall be reduced.
- Because of the large forces that can be generated through these activities, the dynamic characteristics of the structure shall be estimated as precisely as possible.

13.3.6 Concert halls, stands and diving platforms

See [Bac+97].

13.4 Machinery induced vibrations

It is not possible to carry out here a detailed treatment of machinery induced vibrations. Therefore, reference to [Bac+97] is made.

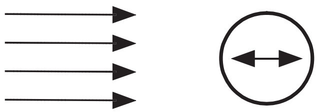
13.5 Wind induced vibrations

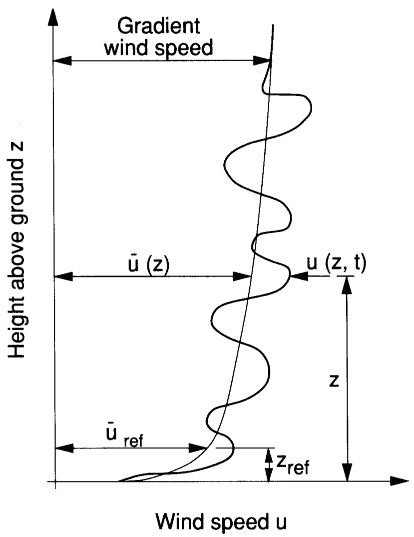
Wind-induced vibrations cover a challenging and wide area. It is not possible to carry out here their detailed treatment. Therefore, reference is made to the relevant literature:

- [Bac+97]
- Simiu E., Scanlan R.H.: “Wind Effects on Structures”. Third Edition. John Wiley & Sons, 1996.

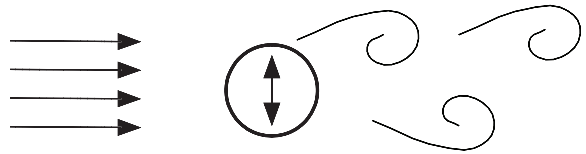
13.5.1 Possible effects

- **Gusts**: Stochastic effects in wind direction
  - Turbulent wind with spatially and temporally variable wind speed.





- **Vortex shedding**: Periodic action transversely to the wind direction
  - Vortex are not shedded left and right at the same time. If the time-interval of the vortex shedding is equal to the oscillation period of the structure, resonance excitation occurs.



$$u_{crit} = \frac{f_e \cdot d}{S} \qquad (13.33)$$

Where

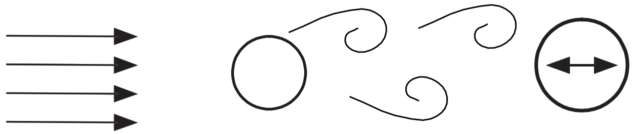
u_{crit}: Critical wind velocity

f_e: Natural frequency of the structure transverse to the wind direction

d: Diameter of the structure

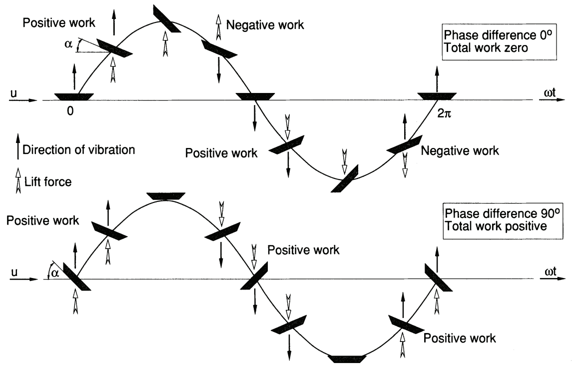
S: Strouhal number (about 0.2 for circular cross sections)

- **Buffeting:** Periodic action in wind direction
  - Vortex detached from an obstacle hit the structure

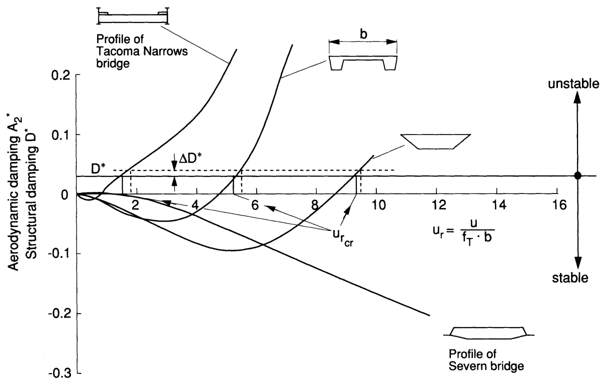


- **“Gallopping” and “Flutter”:**
  Unstable interaction between wind flow and structural motion
  - **Gallopping**: Motion of the structure transversely to the flow direction.
  - **Flutter**: Combined flexural-torsional motion of the structure.

Work done by wind forces during flutter



Stability curves for bridge cross-sections



13.6 Tuned Mass Dampers (TMD)

13.6.1 Introduction

When discussing MDoF systems, in Section 11.1.3 a Tuned Mass Damper (TMD) has already been discussed. However, in that case zero damping was assumed for both the structure and the TMD.

There it was possible to solve the equation of motion simply by means of modal analysis.

Here the theory of the TMD with damping is treated. As we shall see, the damping of the two degrees of freedom is a design parameter, and it shall be possible to chose it freely, therefore:

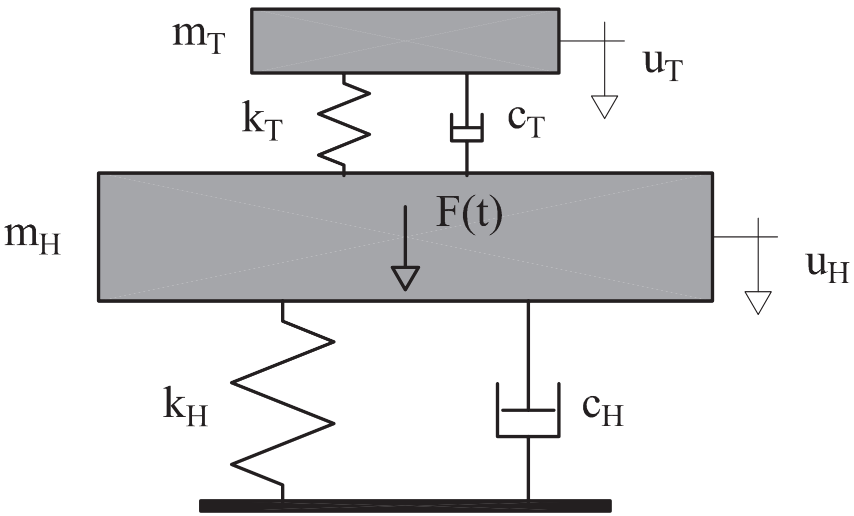
In the case of TMD with damping modal analysis can **not** be used

- References

[BW95] Bachmann H., Weber B.: “Tuned Vibration Absorbers for Damping of Lively Structures”. Structural Engineering International, No. 1, 1995.

[Den85] Den Hartog J.P.: “Mechanical Vibrations”. ISBN 0-486-64785-4. Dover Publications,1985. (Reprint of the original fourth edition of 1956)

13.6.2 2-DoF system



The equations of motion of the 2-DoF system shown above are:

$$\begin{cases} m_H\ddot{u}_H + c_H\dot{u}_H + c_T(\dot{u}_H - \dot{u}_T) + k_H u_H + k_T(u_H - u_T) = F(t) \\ m_T\ddot{u}_T + c_T(\dot{u}_T - \dot{u}_H) + k_T(u_T - u_H) = 0 \end{cases} \quad (13.34)$$

For an harmonic excitation of the type $F(t) = F_H\cos(\omega t)$, a possible ansatz for the steady-state part of the solution is:

$$u_H = U_H e^{i\omega t}, \quad u_T = U_T e^{i\omega t}, \quad F(t) = F_H e^{i\omega t}, \quad (13.35)$$

Using the complex numbers formulation allows a particularly elegant solution to the problem. The equations of motion become:

$$\begin{cases} [-\omega^2 m_H + i\omega(c_H + c_T) + (k_H + k_T)]U_H + [-i\omega c_T - k_T]U_T = F_H \\ [-i\omega c_T - k_T]U_H + [-\omega^2 m_T + i\omega c_T + k_T]U_T = 0 \end{cases}$$

(13.36)

To facilitate the solution of the system, some dimensionless parameters are now introduced:

$\gamma = m_T/m_H$:	Mass ratio (TMD Mass/Mass of the structure)
$\omega_T = \sqrt{k_T/m_T}$:	Natural frequency of the TMD
$\omega_H = \sqrt{k_H/m_H}$:	Natural frequency of the structure without TMD
$\beta = \omega_T/\omega_H$:	Ratio of the natural frequencies
$\Omega = \omega/\omega_H$:	Ratio of the excitation frequency to the natural frequency of the structure
ζ_T:	Damping ratio of the TMD
ζ_H:	Damping ratio of the structure
$U_{H0} = F_H/k_H$:	Static deformation of the structure

Substituting these dimensionless parameters into Equation (13.36) we obtain:

$$\begin{cases} [-\Omega^2 + 2i\Omega(\zeta_H + \beta\gamma\zeta_T) + (1 + \beta^2\gamma)]U_H + [-2i\Omega\beta\gamma\zeta_T - \beta^2\gamma]U_T = U_{H0} \\ [-2i\Omega\beta\gamma\zeta_T - \beta^2\gamma]U_H + [-\Omega^2\gamma + 2i\Omega\beta\gamma\zeta_T + \beta^2\gamma]U_T = 0 \end{cases}$$

(13.37)

The system of equation can be easily solved using “Maple”, and we obtain the following expression for the amplification function U_H/U_{H0}:

$$\frac{U_H}{U_{H0}} = \frac{(\beta^2-\Omega^2)+2i\Omega\beta\zeta_T}{[(\beta^2-\Omega^2)-\Omega^2\beta^2(1-\gamma)+\Omega^2(\Omega^2-4\beta\zeta_H\zeta_T)]+2i[(\beta^2-\Omega^2)\zeta_H+(1-\Omega^2-\Omega^2\gamma)\beta\zeta_T]} \quad (13.38)$$

The complex expression given in Equation (13.38) shall now be converted into the form:

$$z = x+iy \qquad \text{or} \qquad U_H = U_{H0}(x+iy) \quad (13.39)$$

The displacement U_H has therefore two components: 1) One that is in phase with the displacement U_{H0} and 2) one with a phase shift equal to $\pi/4$. From the vectorial sum of these two components the norm of U_H can be computed as:

$$|U_H| = U_{H0}\sqrt{x^2+y^2} \quad (13.40)$$

Equation (13.38) has however the form

$$U_H = U_{H0}\frac{(A+iB)}{(C+iD)} \quad (13.41)$$

and must be first rearranged as follows:

$$U_H = U_{H0}\frac{(A+iB)\cdot(C-iD)}{(C+iD)\cdot(C-iD)} = U_{H0}\frac{(AC+BD)+i(BC-AD)}{C^2+D^2} \quad (13.42)$$

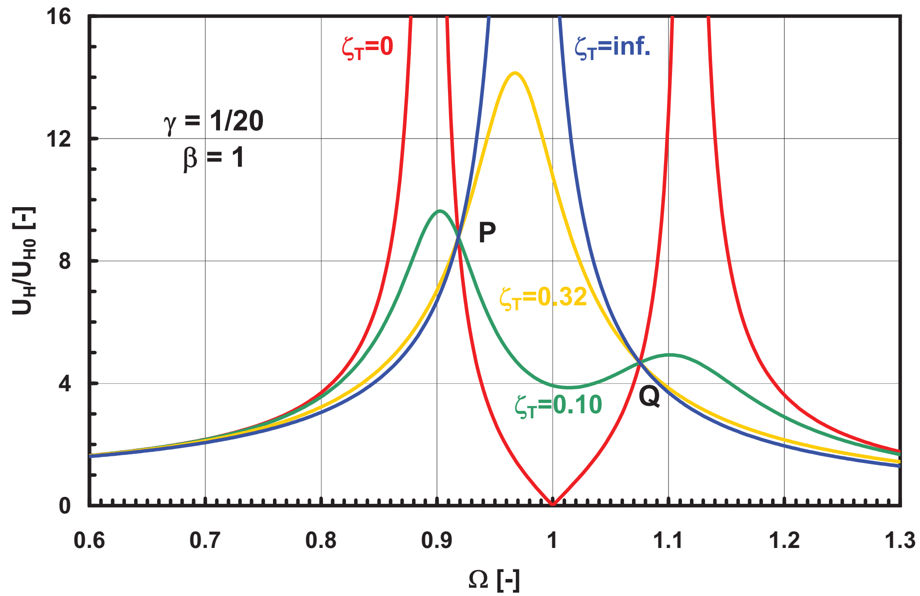
$$|U_H| = U_{H0}\sqrt{\frac{A^2+B^2}{C^2+D^2}} \quad (13.43)$$

Thus, the norm of the dynamic amplification function U_H/U_{H0} can be easily calculated:

$$\left|\frac{U_H}{U_{H0}}\right| = \sqrt{\frac{(\beta^2-\Omega^2)^2+(2\Omega\beta\zeta_T)^2}{[(\beta^2-\Omega^2)-\Omega^2\beta^2(1-\gamma)+\Omega^2(\Omega^2-4\beta\zeta_H\zeta_T)]^2+4[(\beta^2-\Omega^2)\zeta_H+(1-\Omega^2-\Omega^2\gamma)\beta\zeta_T]^2}} \quad (13.44)$$

A similar procedure can be followed to compute the dynamic amplification function U_T/U_{H0}.

Next figure show a representation of Equation (13.44) in function of Ω for an undamped structure $\zeta_H = 0$. Curves for different values of the parameters β, γ and ζ_T are provided.



13.6.3 Optimum TMD parameters

Based on observations and consideration at the previous image Den Hartog found optimum TMD parameters for an undamped structure:

$$f_{T,\,opt} = \frac{f_H}{1 + m_T/m_H} = \frac{f_H}{1+\gamma} \quad \text{or} \quad \beta_{opt} = \frac{1}{1+\gamma} \tag{13.45}$$

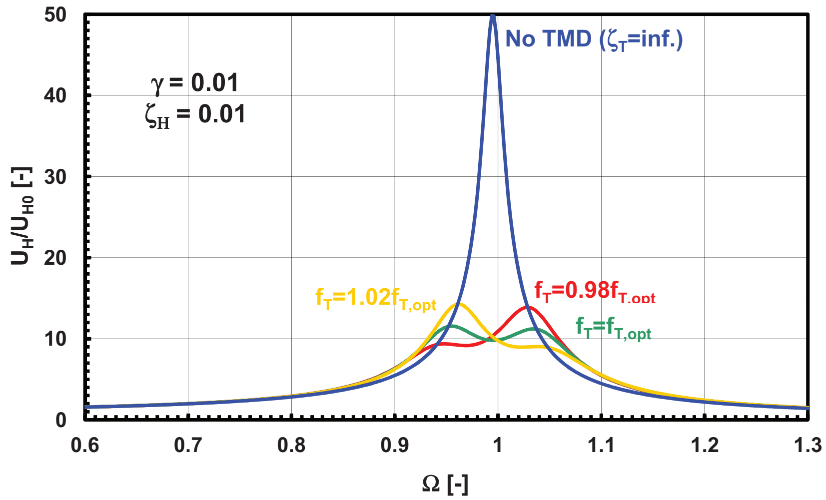
$$\zeta_{T,\,opt} = \sqrt{\frac{3m_T/m_H}{8(1+m_T/m_H)^3}} = \sqrt{\frac{3\gamma}{8(1+\gamma)^3}} \tag{13.46}$$

These optimum TMD parameters can be applied also to lowly damped structures providing good response results.

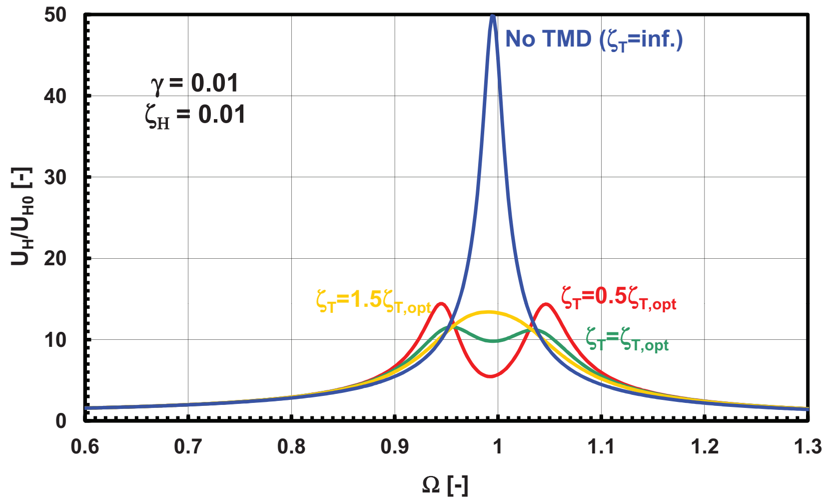
13.6.4 Important remarks on TMD

- The frequency tuning of the TMD shall be quite precise
- The compliance with the optimum damping is less important
- Design charts for TMDs shall be computed numerically
- TMDs are most effective when the damping of the structure is low
- It is not worth increasing the mass ratio too much
- For large mass ratios, the amplitude of the TMD oscillations reduce
- Meaningful mass ratios γ are 3-5%
- The exact tuning of the TMD occurs experimentally, therefore great care should be paid to construction details.

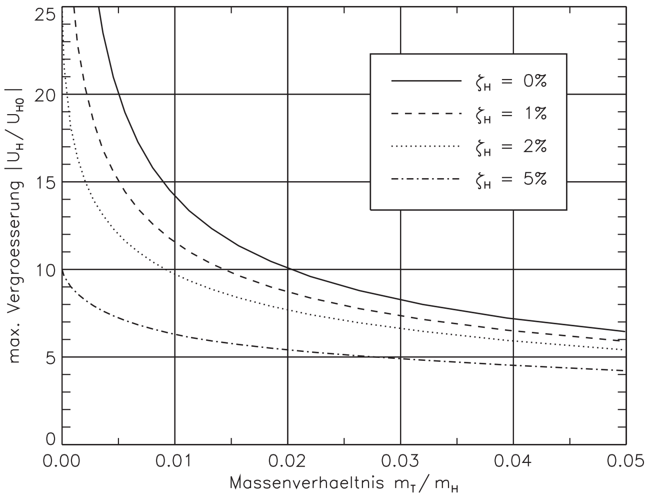
- Amplification function with TMD: Variation of TMD frequency



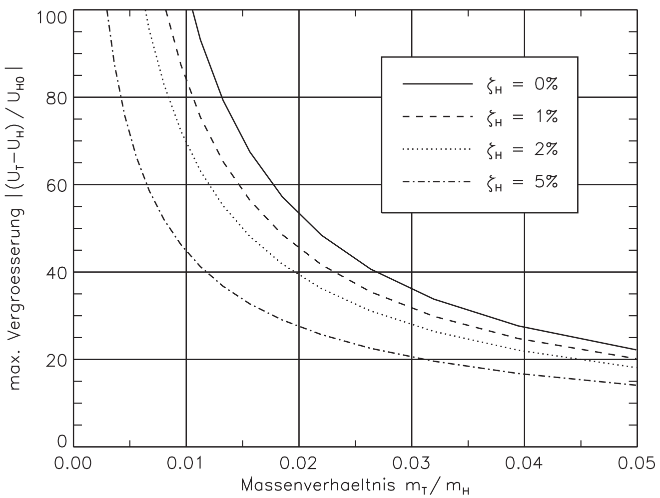
- Amplification function with TMD: Variation of TMD damping



- Design charts: Displacement of the structure (from [BW95])



- Design charts: Relative TMD displacement (from [BW95])

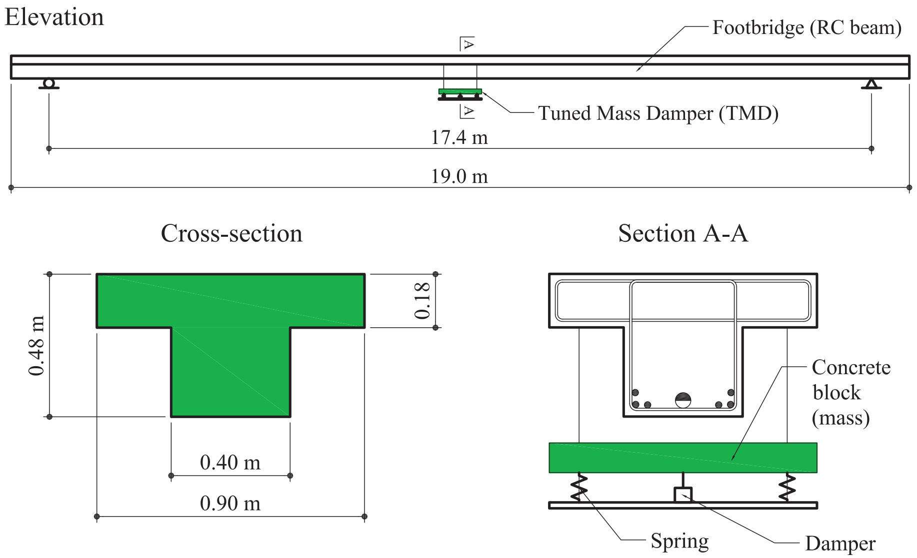


Blank page

14 Pedestrian Footbridge with TMD

14.1 Test unit and instrumentation

The test unit is a post-tensioned RC beam. The beam is made of lightweight concrete and the post-tensioning is without bond. The dimension of the beam were chosen to make it particularly prone to vibrations induced by pedestrians. A Tuned Mass Damper (TMD) is mounted at midspan.



On the test unit the following quantities are measured:

- Displacement at midspan
- Acceleration at midspan
- Acceleration at quarter point of the span

Figure 14.1: View of the test setup.

- Characteristics of the TMD

A close-up of the TMD is shown in Figure 14.2. We can see:

- The 4 springs that define the stiffness K_T of the TMD
- The 4 viscous dampers that define the damping constant c_{opt} of the TMD
- The mass M_T, which is made up by a concrete block and two side container filled with lead spheres. The lead spheres are used for the fine-tuning of the TMD.

The properties of the TMD are given in Section 14.2.

Figure 14.2: Close-up of the TMD.

14.2 Parameters

14.2.1 Footbridge (Computed, without TMD)

Modal mass: $M_H = 5300\text{kg}$

Modal stiffness: $K_H = 861\text{kN/m}$

Natural frequency: $f_H = 2.03\text{Hz}$

(Computed with TMD mass: $f = 1.97\text{Hz}$)

14.2.2 Tuned Mass Damper (Computed)

Mass: $M_T = 310\text{kg}$

Mass ratio: $\mu = \frac{M_T}{M_H} = 0.0585 = 5.85\%$

Natural frequency: $f_{opt} = \frac{f_H}{1+\mu} = 1.92\text{Hz}$

(Measured: $f_T = 1.91\text{Hz}$)

Stiffness: $K_T = M_T \cdot (2\pi f_{opt})^2 = 50.9\text{kN/m}$

Damping rate $\zeta_{opt} = \sqrt{\frac{3\mu}{8(1+\mu)^3}} = 0.14 = 14\%$

(Measured: $\zeta_T = 13\%$)

Damping constant: $c_{opt} = 2\zeta_{opt}\sqrt{K_T M_T} = 1.18\text{kNs/m}$

14.3 Test programme

Following tests are carried out:

No.	Test	Action location	TMD
1	Free decay	Midspan	Locked
2	Sandbag	Midspan	Locked
3	Sandbag	Quarter-point	Locked
4	Sandbag	Midspan	Free
5	Walking 1 Person 3Hz	Along the beam	Locked
6	Walking 1 Person 2Hz	Along the beam	Locked
7	Walking 1 Person 2Hz	Along the beam	Free
8	Walking in group 2Hz	Along the beam	Locked
9	Walking in group 2Hz	Along the beam	Free
10	Jumping 1 Person 2Hz	Midspan	Locked
11	Jumping 1 Person 2Hz	Midspan	Free

Typical results of the experiments are presented and briefly commented in the following sections.

14.4 Free decay test with locked TMD

Time history of the displacement at midspan

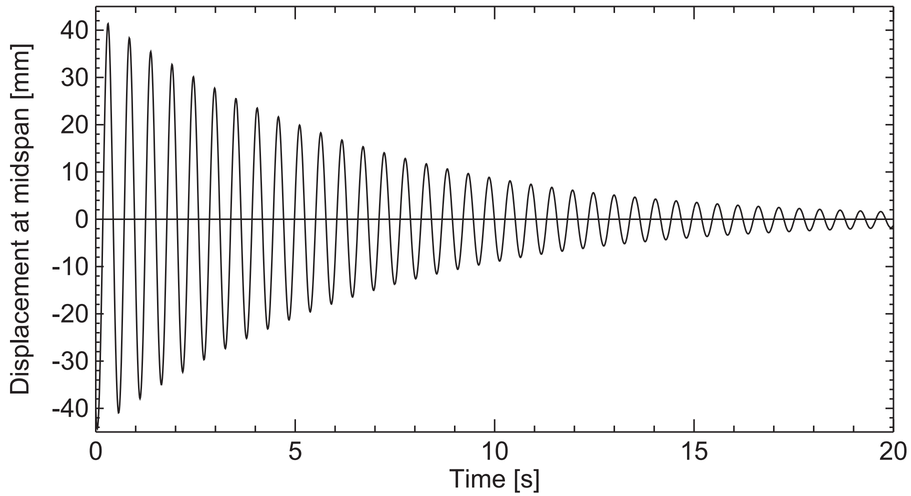


*Figure 14.3: Free decay test with **locked** TMD: Displacement at midspan.*

Evaluation:

	Logarithmic decrement	Damping ratio
Region 1 Average amplitude: ~30mm	$\delta = \frac{1}{8}\ln\frac{41.36}{21.66} = 0.081$	$\zeta_H = \frac{0.081}{2\pi} = 1.29\%$
Region 2 Average amplitude: ~14mm	$\delta = \frac{1}{8}\ln\frac{19.91}{9.68} = 0.090$	$\zeta_H = \frac{0.090}{2\pi} = 1.43\%$
Region 3 Average amplitude: ~6mm	$\delta = \frac{1}{8}\ln\frac{8.13}{3.90} = 0.092$	$\zeta_H = \frac{0.092}{2\pi} = 1.46\%$

Fourier-spectrum of the displacement at midspan

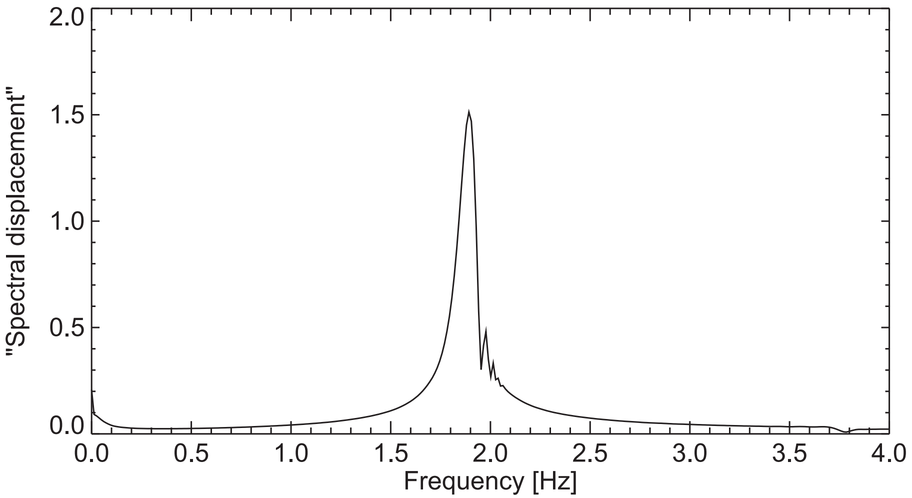


Figure 14.4: Free decay test with ***locked*** *TMD: Fourier-spectrum of the displacement at midspan.*

The measured natural frequency of the footbridge with locked TMD is equal to:

$$f = 1.89\text{Hz} \tag{14.1}$$

This value is less than the value given in Section 14.2.1. This can be explained with the large amplitude of vibration at the start of the test, which causing the opening of cracks in the web of the beam, hence reducing its stiffness.

The second peak in the spectrum corresponds to $f = 1.98\text{Hz}$, wich is in good agreement with Section 14.2.1.

14.5 Sandbag test

The sandbag test consists in hanging a 20 kg sandbag 1 meter above the footbridge, letting it fall down and measuring the response of the system.

In order to excite the different modes of vibration of the footbridge, the test is repeated several times changing the position of the impact of the sandbag on the bridge. The considered locations are:

- at midspan (Section 14.5.1)
- at quarter-point of the span (Section 14.5.2).

These tests are carried out with locked TMD. In order to investigate the effect of the TMD on the vibrations of the system, the test of Section 14.5.1 is repeated with free TMD (see Section 14.5.3).

Remark

- The results presented in Section 14.5.1 and those presented in Section 14.5.2 and 14.5.3 belongs to two different series of tests carried out at different point in time. Between these test series the test setup was completely disassembled and reassembled. Slight differences in the assemblage of the test setup (support!) may have led to slightly different natural frequencies of the system.

14.5.1 Locked TMD, Excitation at midspan

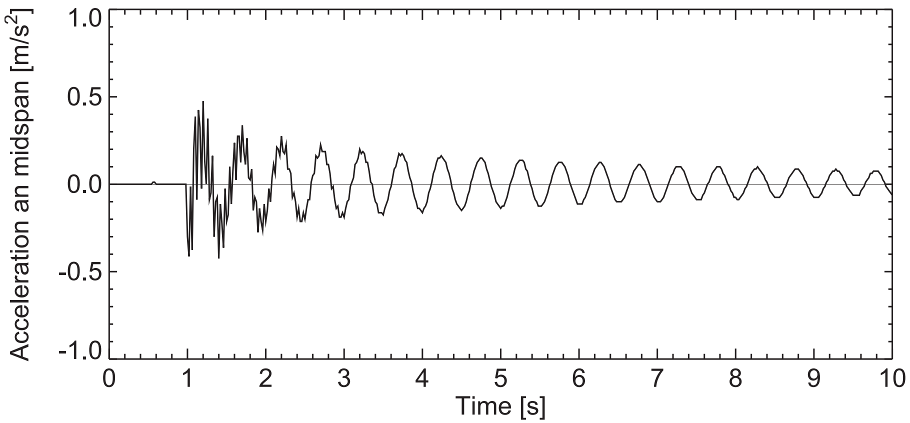


*Figure 14.5: Sandbag test with **locked** TMD: Acceleration at midspan.*

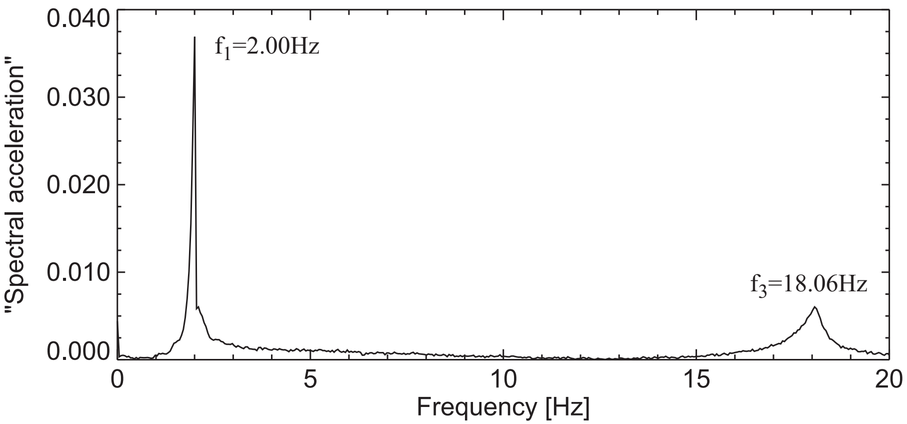


*Figure 14.6: Sandbag test with **locked** TMD: Fourier-spectrum of the acceleration at midspan.*

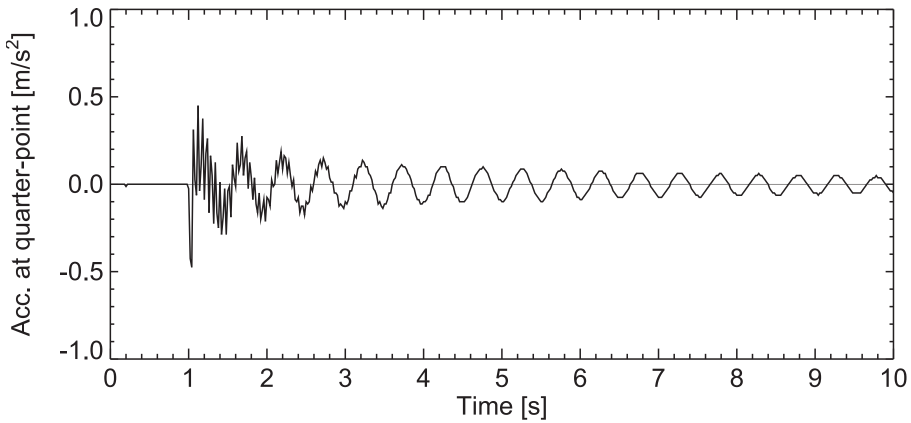


*Figure 14.7: Sandbag test with **locked** TMD: Acceleration at quarter-point of the span.*

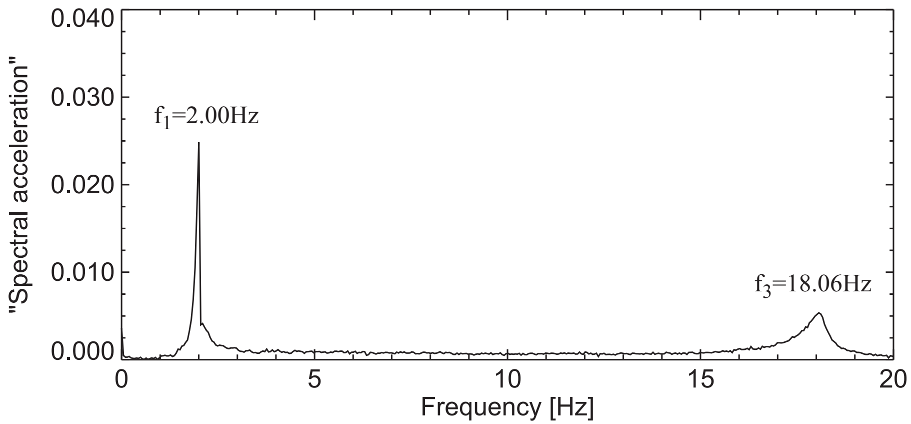


*Figure 14.8: Sandbag test with **locked** TMD: Fourier-spectrum of the acceleration at quarter-point of the span.*

Remarks

- With the sandbag test in principle all frequencies can be excited. Figures 14.5 and 14.7 show a high-frequency vibration, which is superimposed on a fundamental vibration;
- The Fourier amplitude spectrum shows prominent peaks at the first and third natural frequencies of the system (Footbridge with locked TMD);
- The second mode of vibration of the system is not excited, because the sandbag lands in a node of the second eigenvector.
- At midspan, the amplitude of the vibration due to the first mode of vibration is greater than at quarter-point. The amplitude of the vibration due to the third mode of vibration, however, is about the same in both places. This is to be expected, if the shape of the first and third eigenvectors is considered.
- The vibration amplitude is relatively small, therefore, the measured first natural frequency $f_1 = 2.0\mathrm{Hz}$ in good agreement with the computation provided in Section 14.2.1.

14.5.2 Locked TMD, Excitation at quarter-point of the span

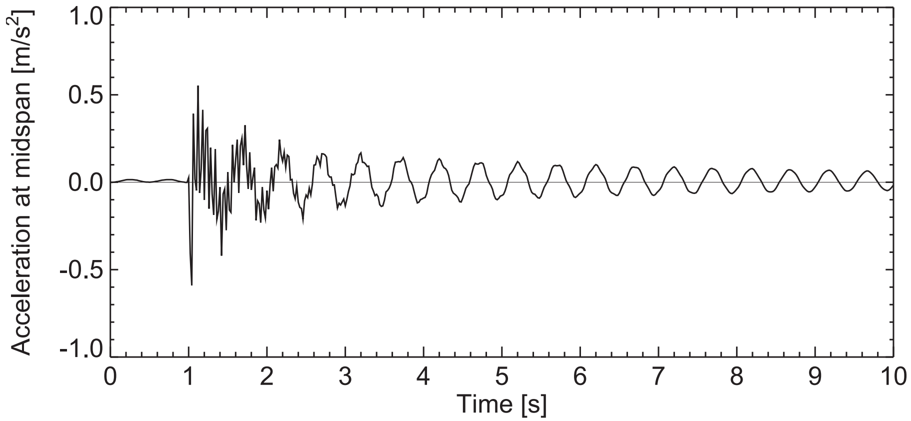


*Figure 14.9: Sandbag test with **locked** TMD: Acceleration at midspan.*

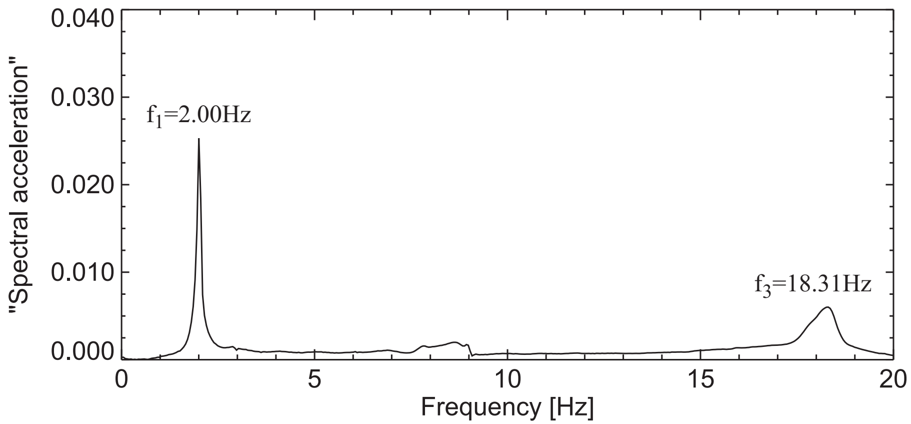


*Figure 14.10: Sandbag test with **locked** TMD: Fourier-spectrum of the acceleration at midspan.*

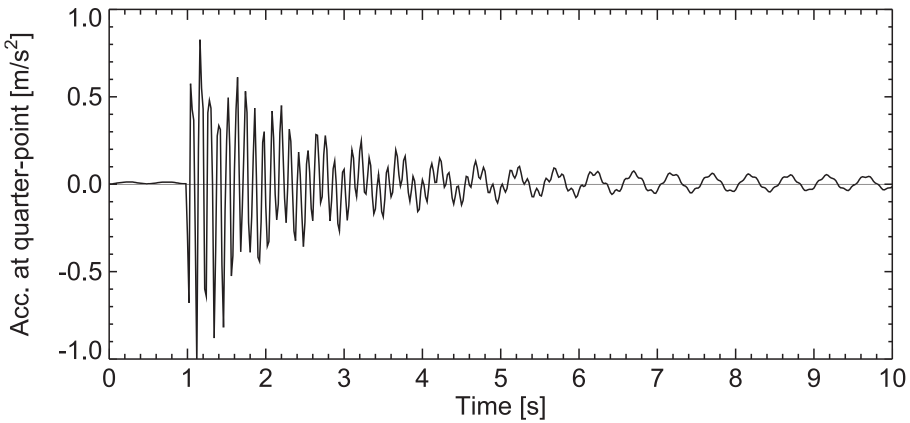


*Figure 14.11: Sandbag test with **locked** TMD: Acceleration at quarter-point of the span.*

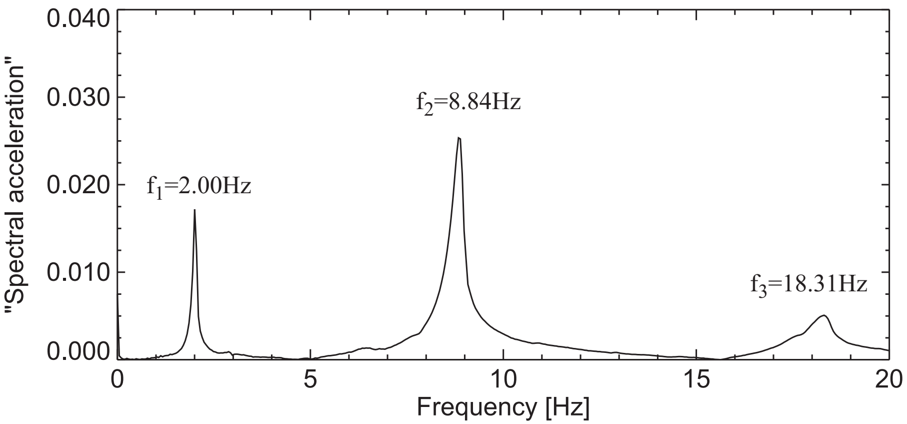


*Figure 14.12: Sandbag test with **locked** TMD: Fourier-spectrum of the acceleration at quarter-point of the span.*

Remarks

- When the sandbag lands at quarter-point of the bridge, the second mode of vibration of the system is strongly excited. Its contribution to the overall vibration at quarter-point of the foot-bridge is clearly shown in Figures 14.11 and 14.12.
- The acceleration sensor located at midspan of the footbridge lays in a node of the second mode of vibration, and as expected in figures 14.9 and 14.10 the contribution of the second mode is vanishingly small.

14.5.3 Free TMD: Excitation at midspan

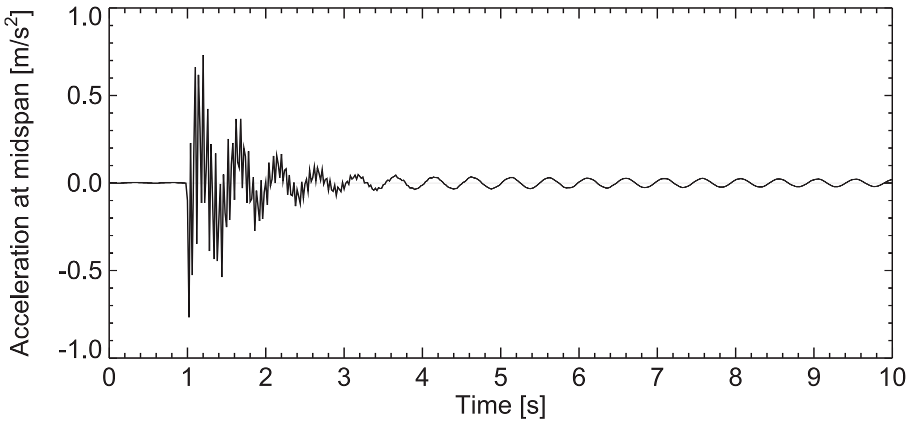


*Figure 14.13: Sandbag test with **free** TMD: Acceleration at midspan.*

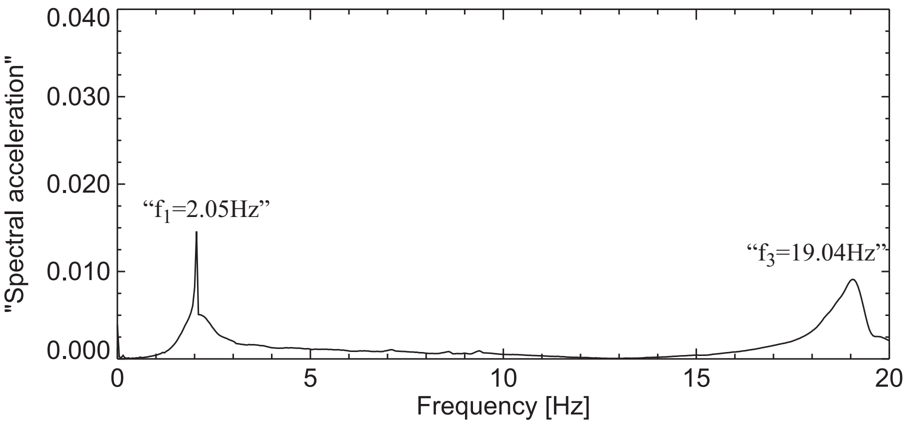


*Figure 14.14: Sandbag test with **free** TMD: Fourier-spectrum of the acceleration at midspan.*

Remarks

- With active (free) TMD the “first” and the “third “natural frequencies of the bridge are excited. As expected, these frequencies are slightly larger than the natural frequencies of the system (bridge with locked TMD), which are given in Figure 14.6. This is because the mass of the TMD is no longer locked and can vibrate freely.
- The effect of the TMD is clearly shown in Figure 14.14. The amplitude of the peak in the “first natural frequency” is much smaller than in Figure 14.6. The amplitude of the peak at the “third natural frequency” is practically the same. The “third mode of vibration” is only marginally damped by the TMD.
- In the two comments above, the natural frequencies are mentioned in quotes, because by releasing the TMD number and properties of the natural vibrations of the system change. A direct comparison with the natural vibrations of system with locked TMD is only qualitatively possible.

14.6 One person walking with 3 Hz

One 65 kg-heavy person (G = 0.64 kN) crosses the footbridge. He walks with a frequency of about 3 Hz, which is significantly larger than the first natural frequency of the bridge.

Remarks

- The static deflection of the bridge when the person stands at midspan is:

$$d_{st} = \frac{G}{K_H} = \frac{0.69}{861} = 0.00080\text{m} = 0.80\text{mm}$$

- The maximum measured displacement at midspan of the bridge is about 2 mm (see Figure 14.15), which corresponds to about 2.5 times d_{st}. As expected, the impact of dynamic effects is rather small.
- In the Fourier spectrum of the acceleration at midspan of the bridge (see Figure 14.17), the frequencies that are represented the most correspond to the first, the second and the third harmonics of the excitation. However, frequencies corresponding to the natural modes of vibrations of the system are also visible.

Test results

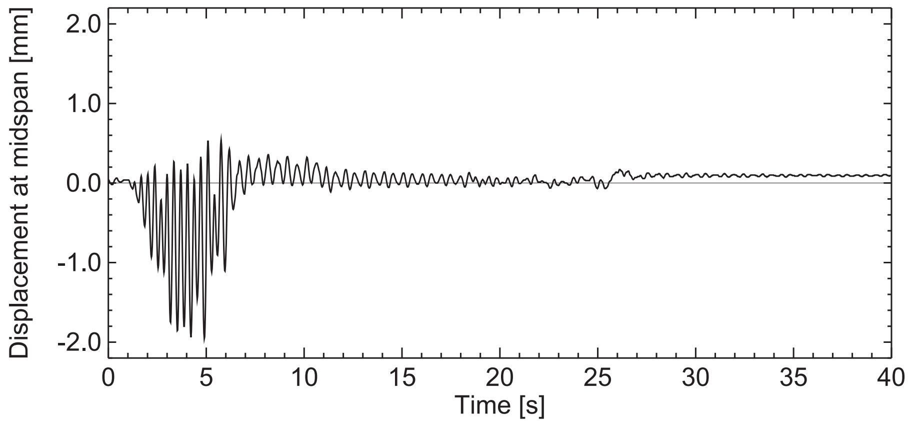


*Figure 14.15: One person walking with 3 Hz: Displacement at midspan with **locked** TMD.*

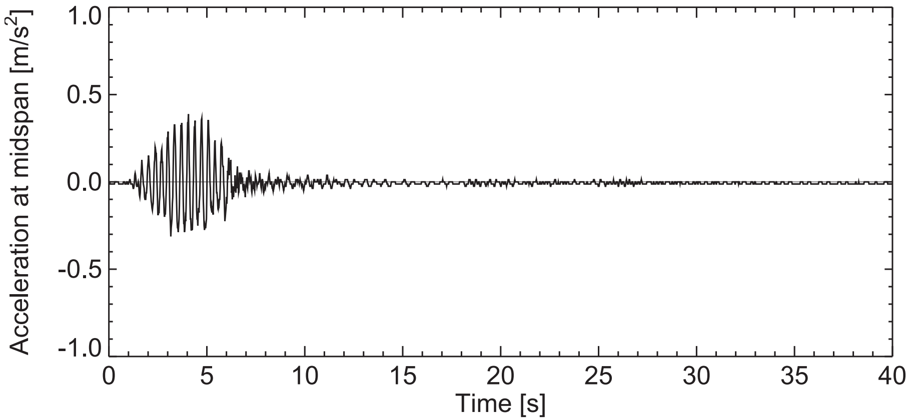


*Figure 14.16: One person walking with 3 Hz: Acceleration at midspan with **locked** TMD.*

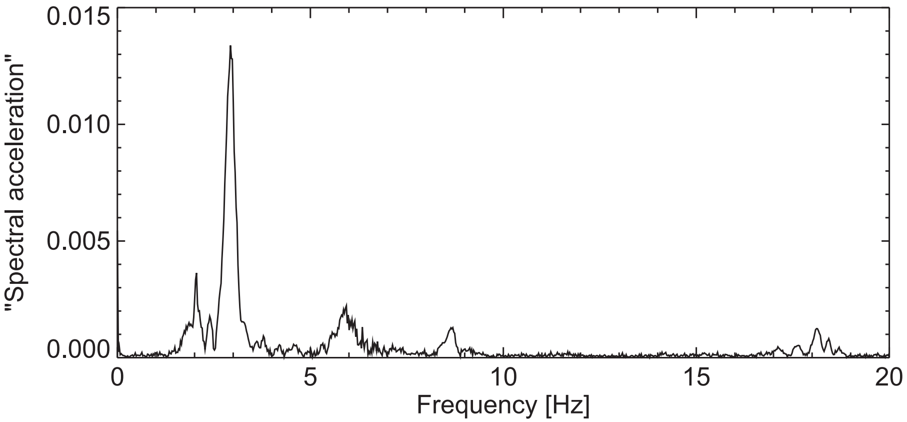


*Figure 14.17: One person walking with 3 Hz: Fourier-Spectrum of the acceleration at midspan with **locked** TMD.*

14.7 One person walking with 2 Hz

One 95 kg-heavy person (G = 0.93 kN) crosses the footbridge. He walks with a frequency of 1.95 Hz, which is approximately equal to the first natural frequency of the bridge. The length of the step is 0.70 m.

Sought is the response of the bridge under this excitation. A similar problem was solved theoretically in Section 13.3.3.

14.7.1 Locked TMD (Measured)

First the maximum amplitudes are calculated by hand:

Static displacement: $d_{st} = \frac{G}{K_H} = \frac{0.93}{861} = 0.00108\text{m} = 1.08\text{mm}$

(Measured: $d_{st} = 1.22\text{mm}$)

Walking velocity: $v = S \cdot f_0 = 0.7 \cdot 1.95 = 1.365\text{m/s}$

Crossing time: $\Delta t = L/v = 17.40/1.365 = 12.74\text{s}$

Number of cycles: $N = \Delta t \cdot f_n = 12.74 \cdot 1.95 = 25$

Amplification factor: $\Phi = 22$ (From page 13-20 with $\zeta_H = 1.6\%$)

Max. acceleration: $a_{max} = 4\pi^2 \cdot 1.95^2 \cdot 0.00108 \cdot 0.4 \cdot 22$
$= 1.43\text{m/s}^2$

(Measured: $a_{max} = 1.63\text{m/s}^2$)

Max. dyn. displ.: $d_{dyn,max} = 1.08 \cdot 0.4 \cdot 22 = 9.50\text{mm}$

Max. displacement: $d_{max} = 9.50 + 1.08 = 10.58\text{mm}$

(Measured: $d_{max} = 12.04\text{mm}$)

Test results

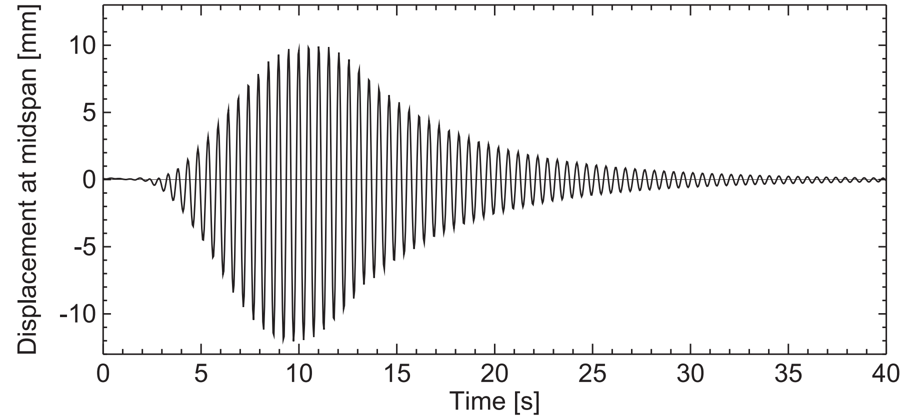


Figure 14.18: One person walking with 2 Hz: Displacement at midspan with ***locked*** *TMD.*

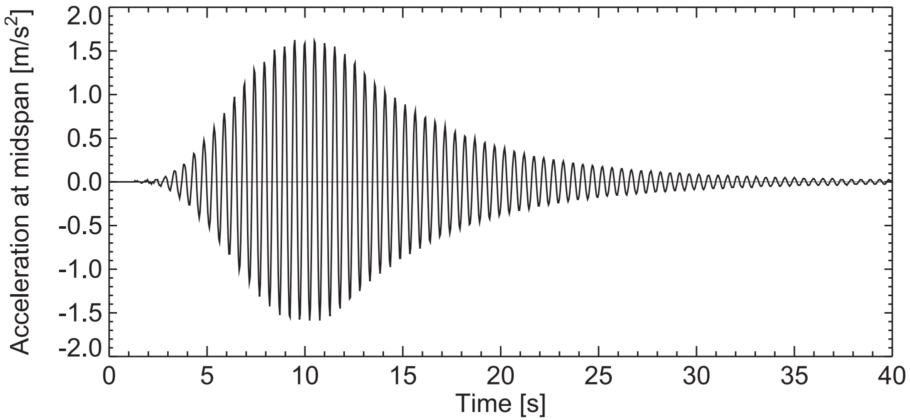


Figure 14.19: One person walking with 2 Hz: Acceleration at midspan with ***locked*** *TMD.*

14.7.2 Locked TMD (ABAQUS-Simulation)

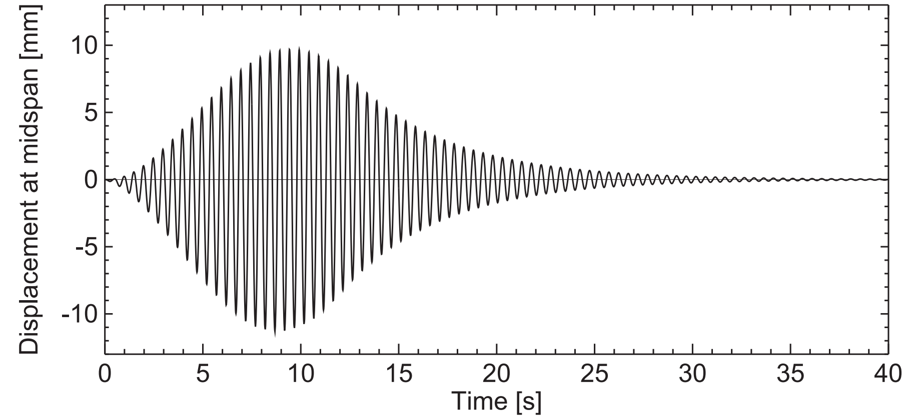


Figure 14.20: One person walking with 2 Hz: Displacement at midspan with ***locked*** *TMD. (ABAQUS-Simulation).*

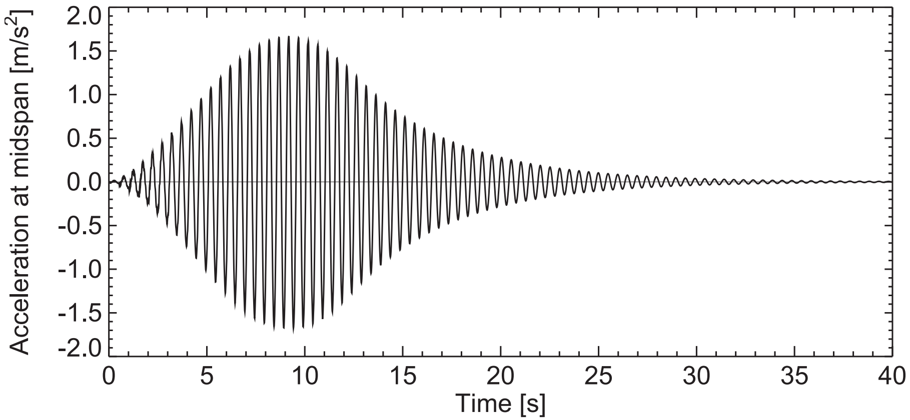


Figure 14.21: One person walking with 2 Hz: Acceleration at midspan with ***locked*** *TMD. (ABAQUS-Simulation).*

The curves in Figures 14.20 and 14.21 were computed using the FE program ABAQUS. A similar calculation is described in detail in Section 13.3.3. The input data used in that section were only slightly adjusted here in order to better describe the properties of the test.

Maximum vibration amplitude

Static displacement: $d_{st} = 1.08\text{mm}$
(Measured: $d_{st} = 1.22\text{mm}$)

Maximum displacement: $d_{max} = 11.30\text{mm}$
(Measured: $d_{max} = 12.04\text{mm}$)

Amplification factor: $V = \frac{d_{max}}{d_{st}} = \frac{11.30}{1.08} = 10.5$

Maximum acceleration: $a_{max} = 1.68\text{m/s}^2$
(Measured: $a_{max} = 1.63\text{m/s}^2$)

The maximum amplitudes of the numerical simulation and of the experiment agree quite well and also the time-histories shown in Figures 14.18 and 14.21 look quite similar.

Please note that during the first 2 seconds of the experiment, displacements and accelerations are zero, because the person started to walk with a slight delay.

14.7.3 Free TMD

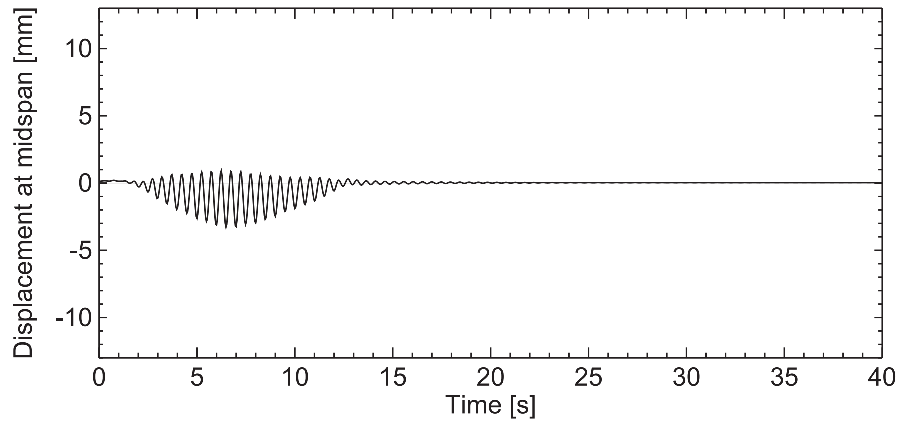


*Figure 14.22: One person walking with 2 Hz: Displacement at midspan with **free** TMD.*

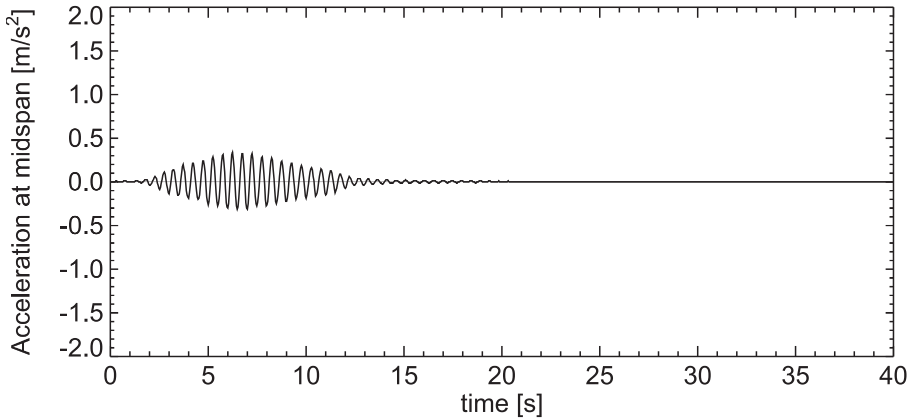


*Figure 14.23: One person walking with 2 Hz: Acceleration at midspan with **free** TMD.*

Estimate of the maximum vibration amplitude

Amplification factor: about 5.5 (from page 13-41)

Maximum dyn. displ.: $d_{dyn,max} = 1.08 \cdot 0.4 \cdot 5.5 = 2.38\text{mm}$

Maximum displacement: $d_{max} = 2.38 + 1.08 = 3.46\text{mm}$
(Measured: $d_{max} = 3.27\text{mm}$)

Maximum acceleration.: $a_{max} = 4\pi^2 \cdot 1.95^2 \cdot 0.00108 \cdot 0.4 \cdot 5.5$
$= 0.36\text{m/s}^2$
(Measured: $a_{max} = 0.34\text{m/s}^2$)

14.7.4 Remarks about "One person walking with 2 Hz"

- The effect of the TMD can be easily seen in Figures 14.22 and 14.23. The maximum acceleration at midspan reduces from 1.63m/s^2 to 0.34m/s^2, which corresponds to a permissible value.

14.8 Group walking with 2 Hz

All student participating to the test (24 people) cross the footbridge in a continuous flow. A metronome is turned on to ensure that all students walk in the same step and with a frequency of about 2 Hz.

The test is carried out both with locked (Section 14.8.1) and free (Section 14.8.2) TMD.

In Figures 14.24 to 14.27 the first 40 seconds of the response of the bridge are shown.

Remarks

The results of the experiments with several people walking on the bridge are commented by using the results of tests with one person walking (see Section 14.7) as comparison. For this reason the maximum vibration amplitudes shown in Figures 14.18, 14.19, 14.22, 14.23 and 14.24 to 14.27 are summarised in Tables 14.1 and 14.2.

Case	Group	1 person	ratio
Maximum acceleration at midspan. **Locked** TMD	2.05 m/s^2	1.63 m/s^2	1.26
Maximum acceleration at midspan. **Free** TMD	0.96 m/s^2	0.34 m/s^2	2.82
Ratio	2.14	4.79	

Table 14.1: Comparison of the accelerations at midspan.

Case	Group	1 person	ratio
Maximum displacement at midspan. **Locked** TMD	20.52 mm	12.04 mm	1.70
Maximum displacement at midspan. **Free** TMD	12.28 mm	3.27 mm	3.76
Ratio	1.67	3.68	

Table 14.2: Comparison of the displacements at midspan.

It is further assumed that only about 16 of the 24 persons are on the footbridge at the same time.

The following remarks can thereby be made:

- The maximum acceleration measured at midspan of the bridge with locked TMD is only about 1.26-times greater than the acceleration which has been generated by the single person. According to section 13.3.3 we could have expected a larger acceleration from the group ($\sqrt{16} = 4$). One reason why the maximum acceleration is still relatively small, is the difficulty to walk in the step when the “ground is unsteady.” With a little more practice, the group could probably have achieved much larger accelerations. It is further to note that the person who walked of the bridge for the test presented in Section 14.7 was with his 95 kg probably much heavier than the average of the group.
- The maximum displacement measured at midspan of the bridge with locked TMD is 1.70 times larger than the displacement generated by the single person. The amplification factor

of the displacement is larger that the amplification factor of the accelerations, because the static deflection caused by the group is significantly larger than that caused by the single person.

- The activation of the TMD results in a reduction of the maximum acceleration caused by the single person by a factor of 4.79. In the case of the group the reduction factor is only 2.14. It should be noted here that when the TMD is active (free), the vibrations are significantly smaller, and therefore it is much easier for the group to walk in step. It is therefore to be assumed that in the case of the free TMD, the action was stronger than in the case of the locked TMD. This could explain the seemingly minor effectiveness of the TMD in the case of the group.

14.8.1 Locked TMD

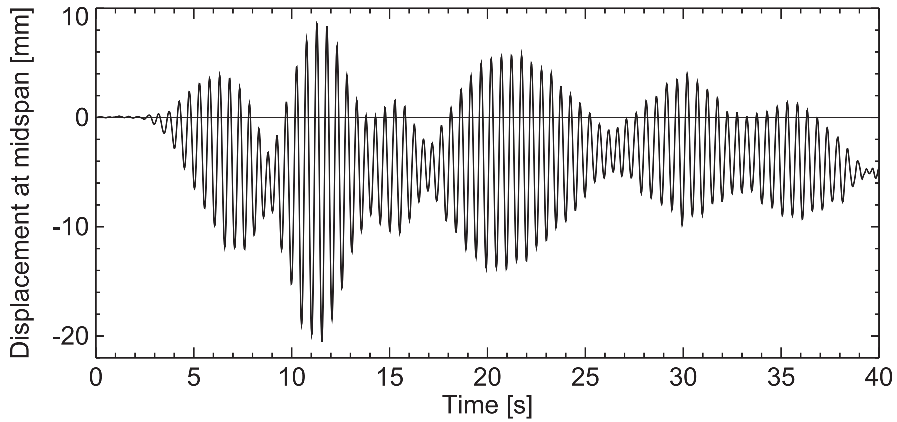


Figure 14.24: Group walking with 2 Hz: Displacement at midspan with ***locked*** *TMD.*

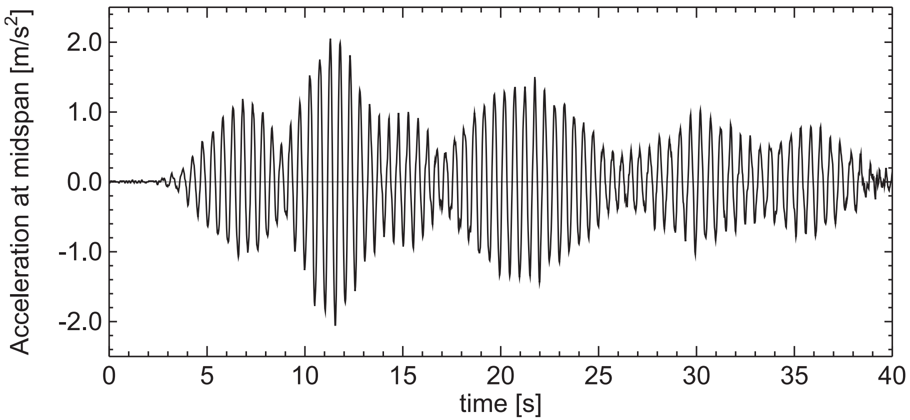


Figure 14.25: Group walking with 2 Hz: Acceleration at midspan with ***locked*** *TMD.*

14.8.2 Free TMD

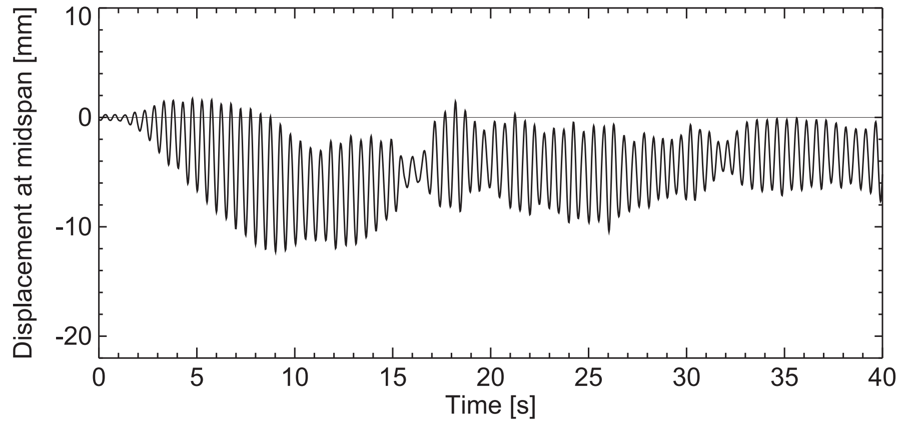


Figure 14.26: Group walking with 2 Hz: Displacement at midspan with ***free*** *TMD.*

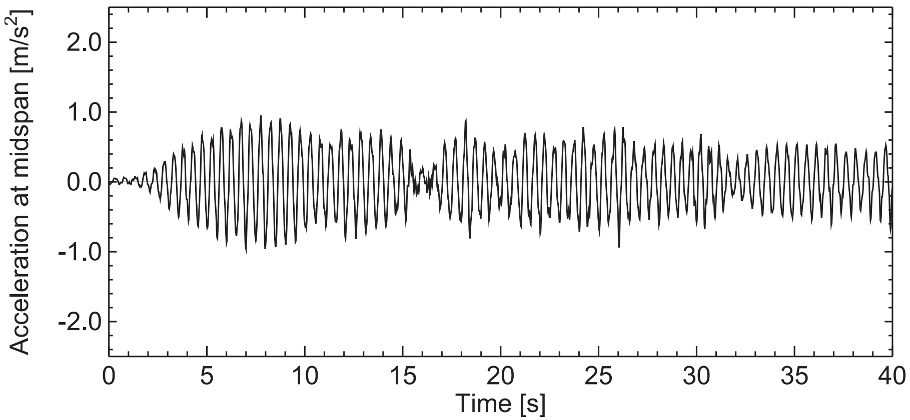


Figure 14.27: Group walking with 2 Hz: Acceleration at midspan with ***free*** *TMD.*

14.9 One person jumping with 2 Hz

One 72 kg-heavy person (G = 0.71 kN) keeps jumping at midspan of the footbridge. He is jumping with a frequency of 1.95 Hz, which is approximately equal to the first natural frequency of the bridge.

Sought is the response of the bridge under this excitation. A similar problem was solved theoretically in Section 6.1.3.

14.9.1 Locked TMD

First the maximum amplitudes are calculated by hand:

Static displacement: $d_{st} = \frac{G}{K_H} = \frac{0.71}{861} = 0.0008\text{m} = 0.82\text{mm}$

(Measured: $d_{st} = 0.93\text{mm}$)

Amplification factor: $V = \frac{1}{2\zeta} = \frac{1}{(2 \cdot 0.016)} = 31.25$

Maximum acceleration: $a_{max} = 4\pi^2 \cdot 1.95^2 \cdot 0.0008 \cdot 1.8 \cdot 31.25 = 6.92\text{m/s}^2$

(Measured: $a_{max} = 7.18\text{m/s}^2$)

Max. dyn. displacement: $d_{dyn,max} = 0.82 \cdot 1.8 \cdot 31.25 = 46.13\text{mm}$

Maximum displacement: $d_{max} = 46.13 + 0.82 = 46.95\text{mm}$

(Measured: $d_{max} = 51.08\text{mm}$)

Course “Fundamentals of Structural Dynamics” An-Najah 2013

Test results

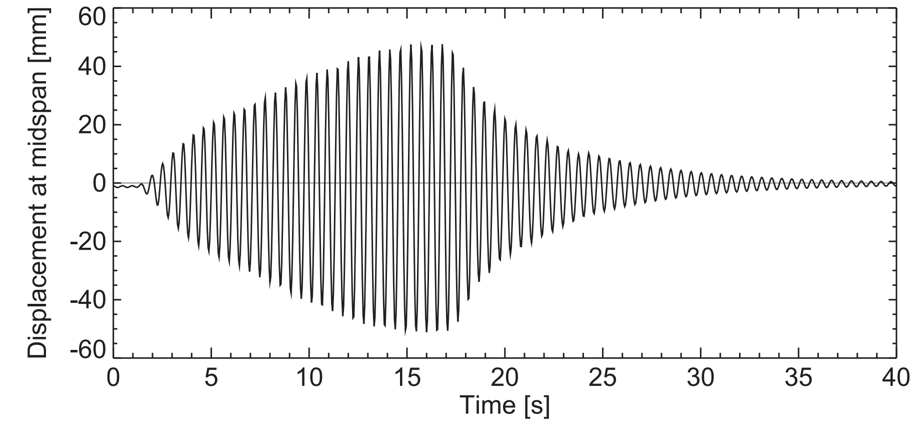


*Figure 14.28: One person jumping with 2 Hz: Displacement at midspan with **locked** TMD.*

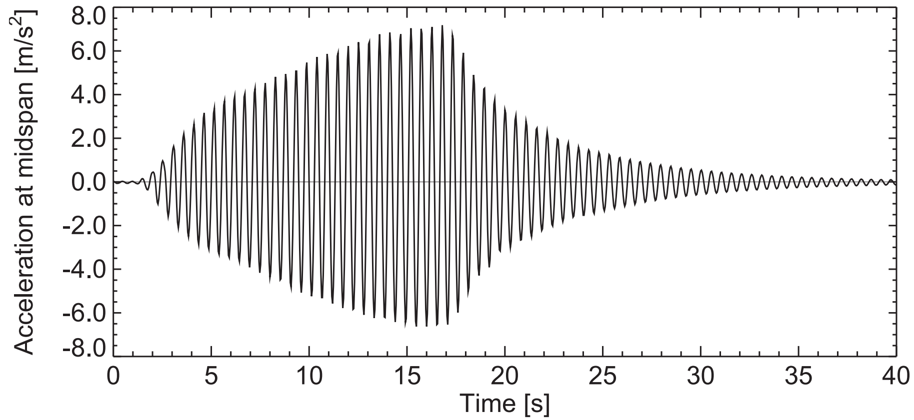


*Figure 14.29: One person jumping with 2 Hz: Acceleration at midspan with **locked** TMD.*

14.9.2 Free TMD

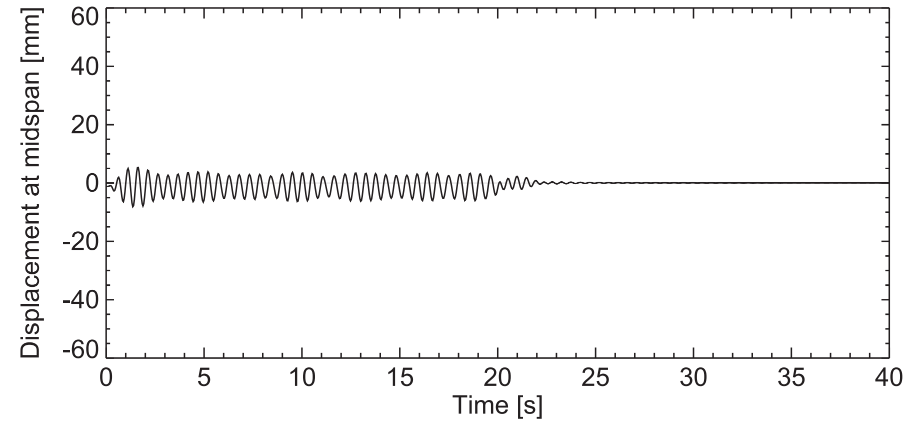


*Figure 14.30: One person jumping with 2 Hz: Displacement at midspan with **free** TMD.*

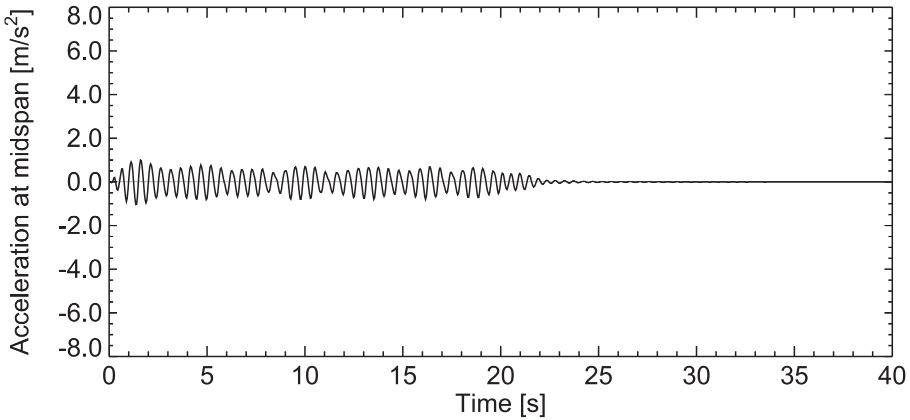


*Figure 14.31: One person jumping with 2 Hz: Acceleration at midspan with **free** TMD.*

Estimate of the maximum vibration amplitude

Amplification factor: about 5.5 (from page 13-41)

Maximum dyn. displ.: $d_{dyn,max} = 0.82 \cdot 1.8 \cdot 5.5 = 8.12\text{mm}$

Maximum displacement: $d_{max} = 8.12 + 0.82 = 8.94\text{mm}$

(Measured: $d_{max} = 8.12\text{mm}$)

Maximum acceleration: $a_{max} = 4\pi^2 \cdot 1.95^2 \cdot 0.0008 \cdot 1.8 \cdot 5.5$

$= 1.22\text{m/s}^2$

(Measured: $a_{max} = 1.04\text{m/s}^2$)

14.9.3 Remarks about “One person jumping with 2 Hz”

- When jumping, the footbridge can be much strongly excited than when walking.
- The achieved acceleration $a_{max} = 7.18\text{m/s}^2 = 73\%\,g$ is very large and two jumping people could easily produce the lift-off of the footbridge.
- The effect of the TMD can be easily seen in Figures 14.30 and 14.31. The maximum acceleration at midspan reduces from 7.18m/s^2 to 1.04m/s^2, what, however, is still perceived as unpleasant.