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SEVENTH FRAMEWORK PROGRAMME

SASPARM

Support Action for Strengthening Palestinian-administrated Areas capabilities for seismic Risk Mitigation

Call ID FP7-INCO.2011-6.2

MODULE 3 : GROUND RESPONSE ANALYSES AND NEAR-SURFACE SITE CHARACTERIZATION

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Review of Fourier Analysis & Discrete Inverse Theory

NNU, May 2 – 4, 2013

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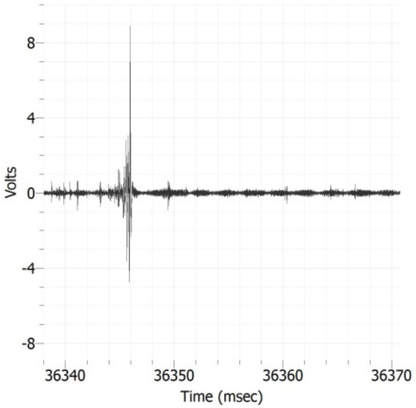
Outline

- Seismograms as signals
- Signal transformations: background mathematics
- Signal transformations
- 2D array transformations
- Forward and inverse modeling
- Two selected algorithms: Occam & neighborhood algorithms

Seismograms as signals

Seismic recordings

Seismogram: A recording of particle displacement/velocity/acceleration or pressure with time
In surface wave methods, the most common seismograms are velocity-meters.



Other distinctive features:

Frequency range characteristics (frequency bandwidth, cut-off frequencies)
Particle motion range characteristics (strong, intermediate, weak motion)
Permanent vs. temporary deployment
1-component vs. 3-component
Surface (1D – 2D) vs. downhole
Single station vs. multi-station

Complex numbers (background)

$$z = x + jy$$ (rectangular coordinates)

↕

$$z = r\cos(\theta) + jr\sin(\theta)z$$
$$z = re^{i\theta}$$
(polar coordinates)

Euler's formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Polar vs. rectangular coordinates:

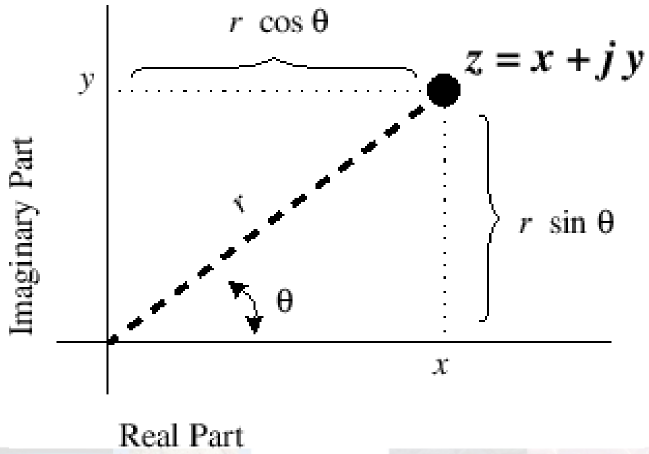
$$x = r\cos(\theta)$$
$$y = r\sin(\theta).$$

↕

-Magnitude: $r = \sqrt{x^2 + y^2}$

-Angle: $\theta = \tan^{-1}(y, x).$

The complex plane:



Source: https://ccrma.stanford.edu/~jos/st/Fourier_Transforms_Continuous_Discrete_Time_Frequency.html

Signal Transformations Background Mathematics

Fourier Transforms

Sampling: Continuous vs. Discrete
Duration: Infinite vs. Finite

Hence, four different definitions for the Fourier Transform:

Time Duration		
Finite	Infinite	
Discrete FT (DFT) $X(k)=\sum_{n=0}^{N-1} x(n)e^{-j\omega_k n}$ $k=0,1,\ldots,N-1$	Discrete Time FT (DTFT) $X(\omega)=\sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$ $\omega\in[-\pi,+\pi)$	discr. time n
Fourier Series (FS) $X(k)=\frac{1}{P}\int_0^P x(t)e^{-j\omega_k t}dt$ $k=-\infty,\ldots,+\infty$	Fourier Transform (FT) $X(\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ $\omega\in(-\infty,+\infty)$	cont. time t
discrete freq. k	continuous freq. ω	

Source: https://ccrma.stanford.edu/~jos/st/Fourier_Transforms_Continuous_Discrete_Time_Frequency.html

Fourier Transforms

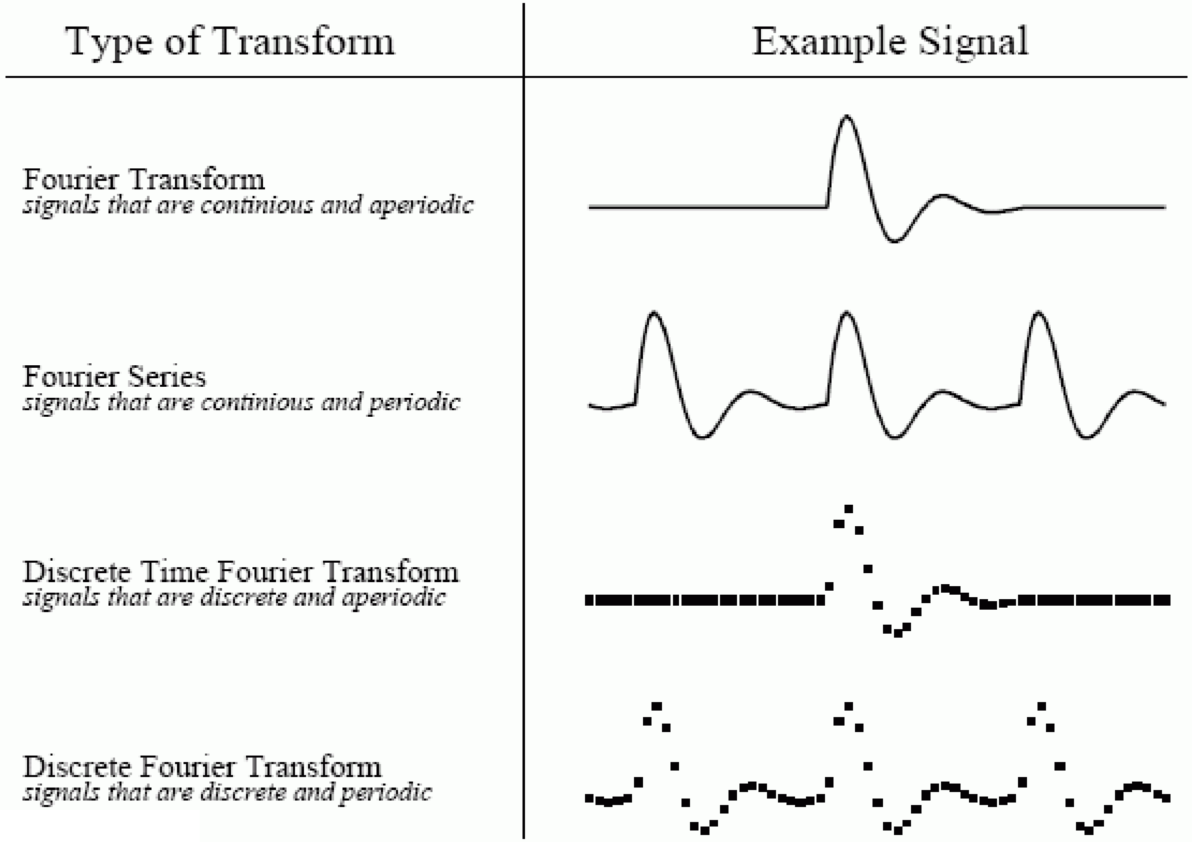


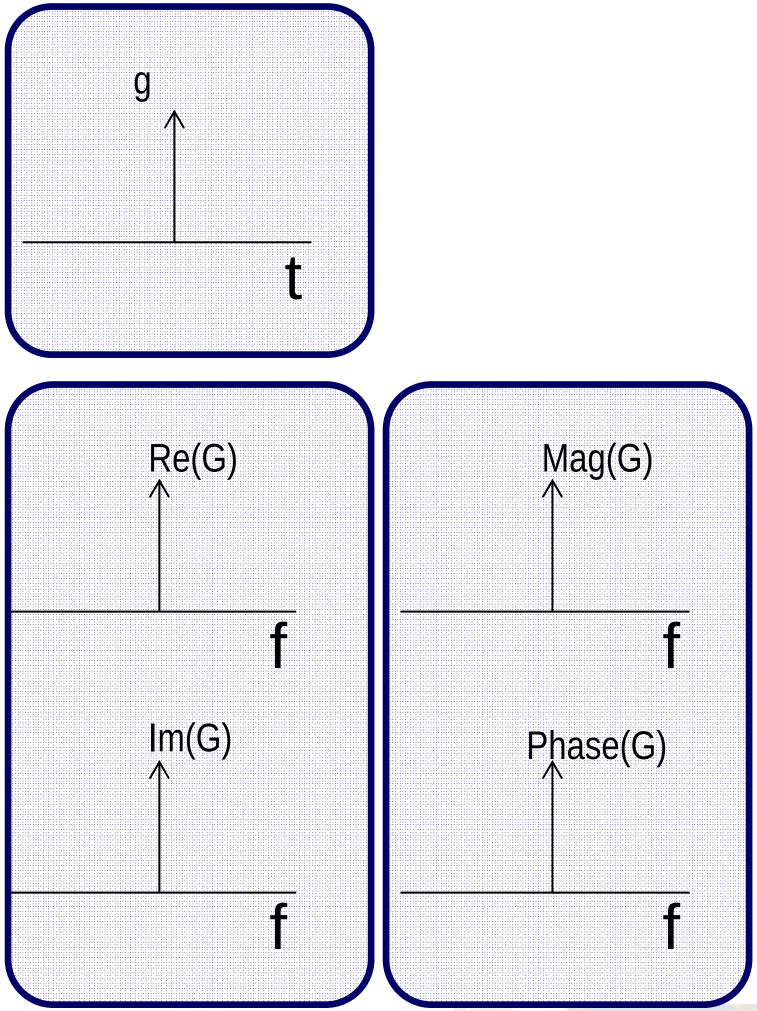
Illustration of the four Fourier transforms. A signal may be continuous or discrete, and it may be periodic or aperiodic. Together these define four possible combinations, each having its own version of the Fourier transform. The names are not well organized; simply memorize them.

Everything we do in a computer is DFT!!

Source: http://www.dspguide.com/ch8/1.htm

Fourier Transforms

Notation for FT: $g(t) \supset G(f)$



$$G(f) = \mathrm{Re}\{G\} + \mathrm{Im}\{G\} = Mag\{G\}e^{iPhase\{G\}}$$

$$\mathrm{Re}\{G\} = Mag\{G\}\cos(Phase\{G\})$$

$$\mathrm{Im}\{G\} = Mag\{G\}\sin(Phase\{G\})$$

$$Mag\{G\} = \sqrt{\mathrm{Re}\{G\}^2 + \mathrm{Im}\{G\}^2}$$

$$Phase\{G\} = \arctan\left(\frac{\mathrm{Im}\{G\}}{\mathrm{Re}\{G\}}\right)$$

Discrete Fourier Transform (DFT)

$$X(\omega_k) = \sum_{n=0}^{N-1} x(t_n) e^{-j\omega_k t_n} \qquad \begin{cases} n = 0,1,2,\ldots,N-1 \\ k = 0,1,2,\ldots,N-1 \end{cases}$$

$[t_o, t_1, t_2, \ldots, t_{N-1}]$

$[x(t_o), x(t_1), x(t_2), \ldots, x(t_{N-1})]$

$[f_o, f_1, f_2, \ldots, f_{N-1}]$

$[X(f_o), X(f_1), X(f_2), \ldots, X(f_{N-1})]$

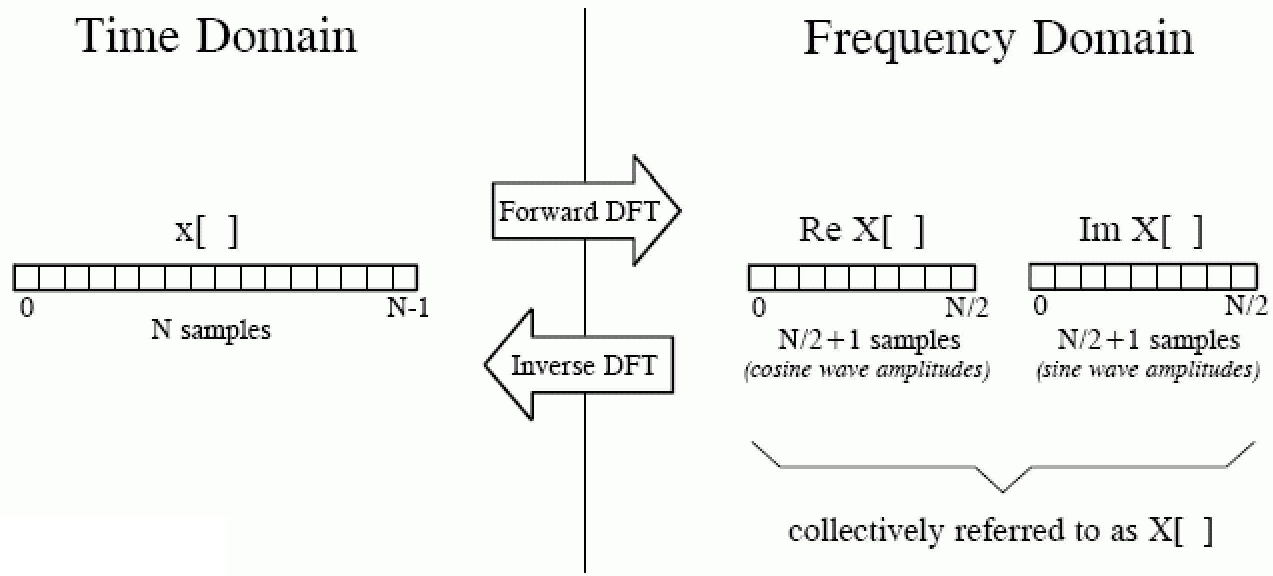
Sampling time interval: dt

Sampling frequency interval: $f_s = \frac{1}{dt}$

Inverse Discrete Fourier Transform (IDFT)

$$x(t_n) = \frac{1}{N} \sum_{k=0}^{N-1} X(\omega_k) e^{j\omega_k t_n} \qquad \begin{cases} n = 0,1,2,\ldots,N-1 \\ k = 0,1,2,\ldots,N-1 \end{cases}$$

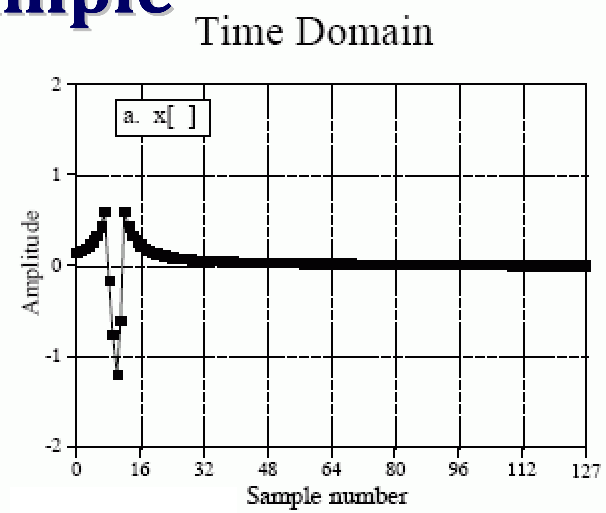
Discrete Fourier Transform (DFT)

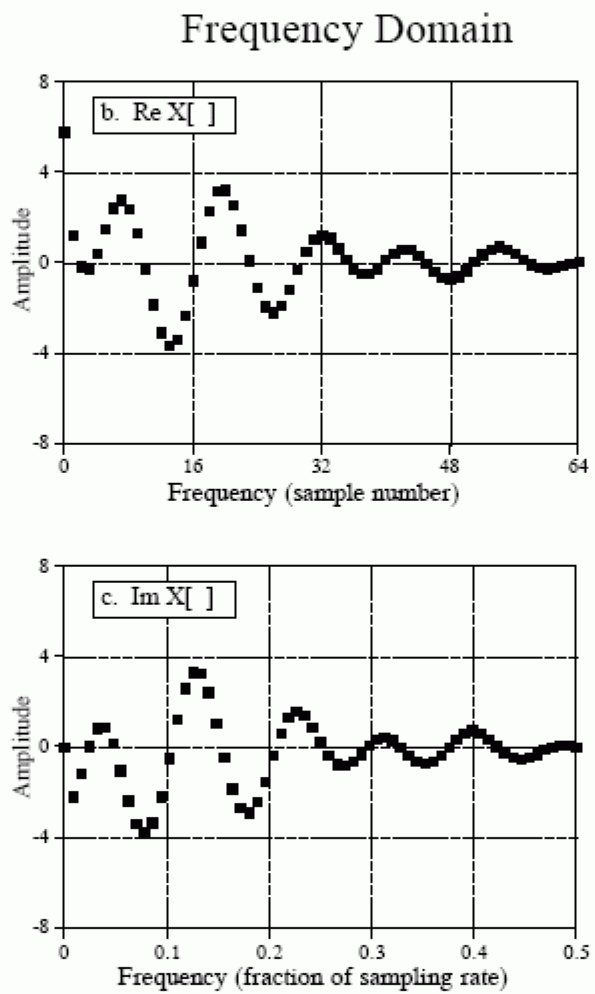


DFT terminology. In the time domain, $x[\ ]$ consists of N points running from 0 to $N-1$. In the frequency domain, the DFT produces two signals, the real part, written: $Re\,X[\ ]$, and the imaginary part, written: $Im\,X[\ ]$. Each of these frequency domain signals are $N/2+1$ points long, and run from 0 to $N/2$. The Forward DFT transforms from the time domain to the frequency domain, while the Inverse DFT transforms from the frequency domain to the time domain. (Take note: this figure describes the **real DFT**. The **complex DFT**, discussed in Chapter 31, changes N complex points into another set of N complex points).

Source: http://www.dspguide.com/ch8/1.htm

DFT example





Example of the DFT. The DFT converts the time domain signal, *x*[ ], into the frequency domain signals, *Re X*[ ] and *Im X*[ ]. The horizontal axis of the frequency domain can be labeled in one of three ways: (1) as an array index that runs between 0 and *N*/2, (2) as a fraction of the sampling frequency, running between 0 and 0.5, (3) as a natural frequency, running between 0 and π. In the example shown here, (b) uses the first method, while (c) use the second method.

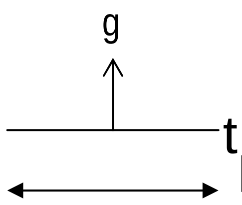
Source: http://www.dspguide.com/ch8/1.htm

DFT basis functions

$$c_k[i] = \cos(2\pi ki / N)$$
$$s_k[i] = \sin(2\pi ki / N)$$

c_k and s_k contain N points: i=0,…,N-1

k determines the frequency of the basis function, which can range from 0 to N/2



g(t) Ↄ *G(f)* = *Re{G(f)}* + *Im{G(f)}*

N/2 numbers representing amplitude of c_k

N/2 numbers representing amplitude of s_k

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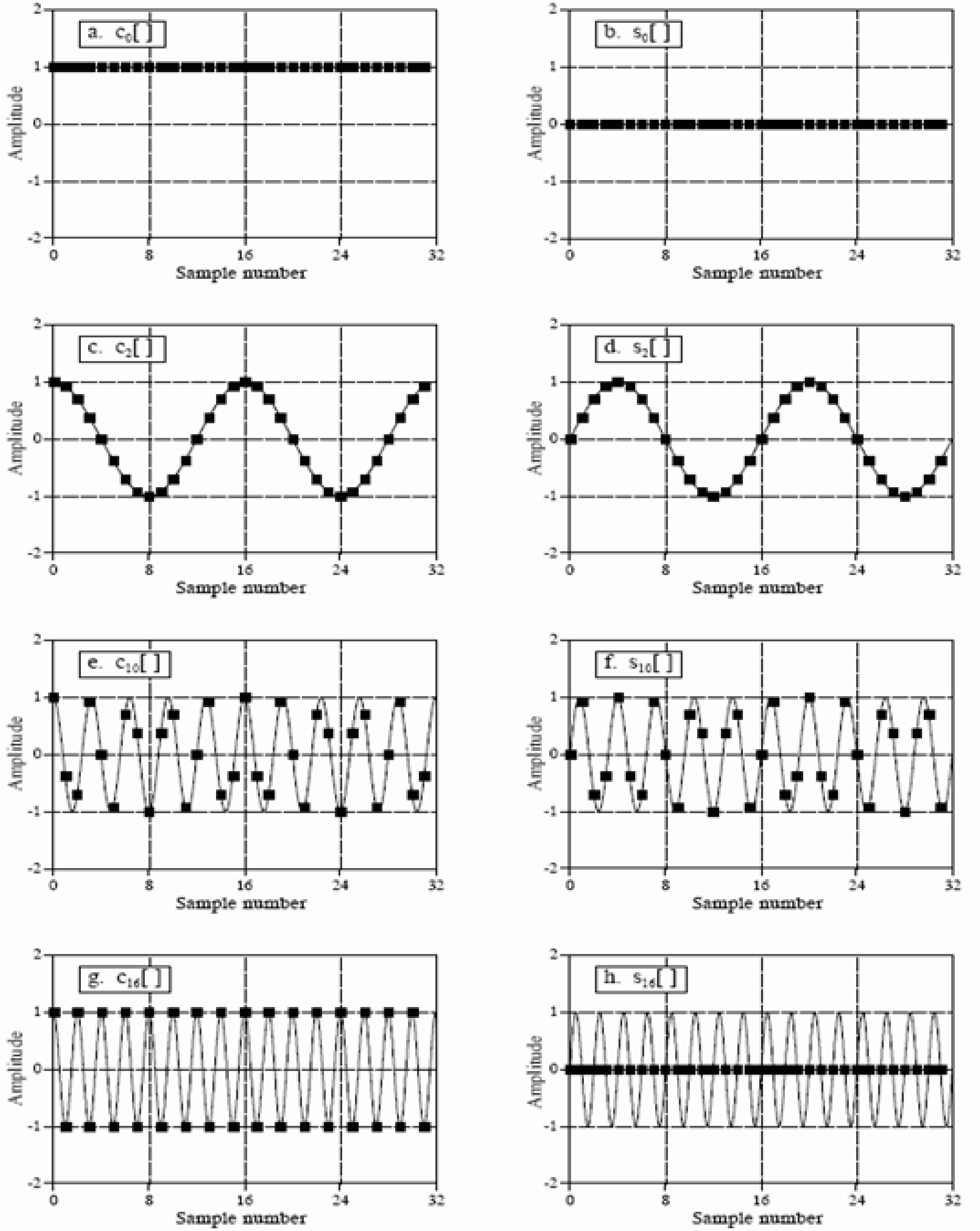
DFT basis functions

Example:
32 point DFT contains
17 discrete cosine waves (real part)
and 17 discrete sine waves (imaginary part)



c_0 means 0 cycles in N points
c_1 means one cycle in N points
c_2 means two cycles in N points
…

Source: http://www.dspguide.com/ch8/1.htm

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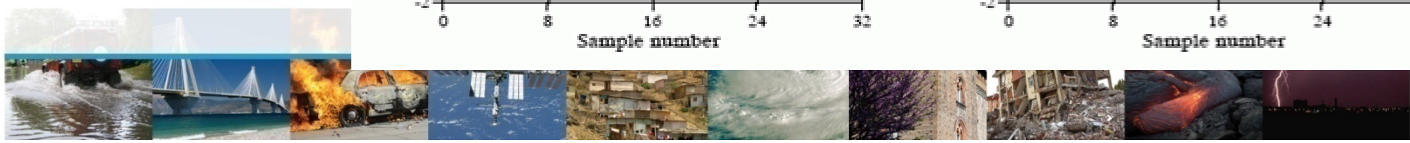


DFT basis functions

$$x[i] = \sum_{k=0}^{N/2} \mathrm{Re}\{\bar{X}\}\cos(2\pi ki / N) + \sum_{k=0}^{N/2} \mathrm{Im}\{\bar{X}\}\sin(2\pi ki / N)$$

Synthesis Equation

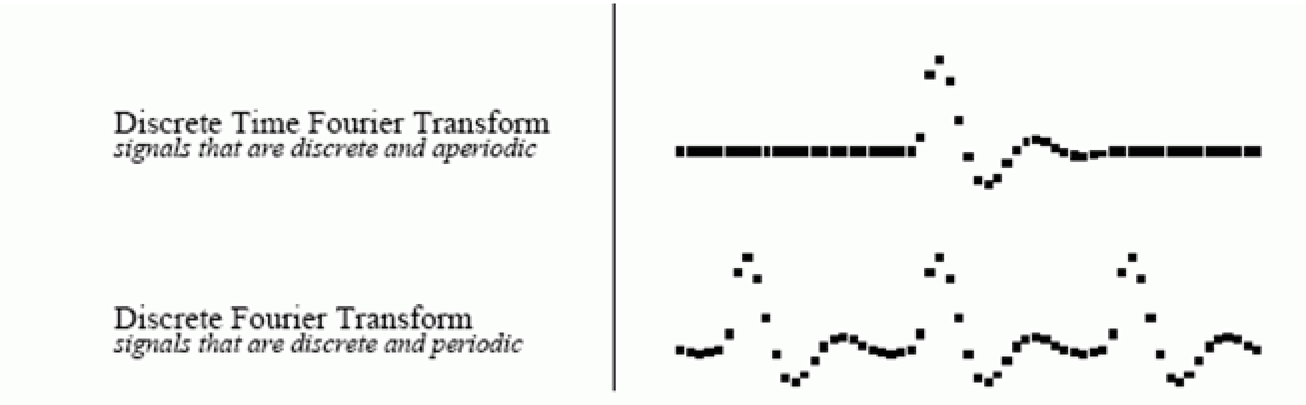
$$\mathrm{Re}\{\bar{X}[k]\} = \frac{\mathrm{Re}\{X[k]\}}{N/2}$$

$$\mathrm{Im}\{\bar{X}[k]\} = -\frac{\mathrm{Im}\{X[k]\}}{N/2}$$

$$\mathrm{Re}\{\bar{X}[0]\} = \frac{\mathrm{Re}\{X[k]\}}{N}$$

$$\mathrm{Re}\{\bar{X}[N/2]\} = \mathrm{Re}\frac{\{X[N/2]\}}{N}$$

DFT vs. DTFT



Discrete FT (DFT)	Discrete Time FT (DTFT)
$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\omega_k n}$	$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$
$k = 0, 1, \ldots, N-1$	$\omega \in [-\pi, +\pi)$

DFT vs. DTFT

The DFT is the transform of a limited number of samples of a periodic signal (whether or not the actual signal is in fact periodic).

The DTFT is a transform of the ENTIRE sampled signal from -∞ to + ∞, and the input isn't necessarily periodic.

One is mathematical and precise (DTFT), the other is physically realizable (DFT).

DFT: time domain is discrete and periodic with period T.
frequency domain is discrete and periodic.
DTFT: time domain is discrete and not necessarily periodic.
frequency domain is continuous and periodic.

Discrete FT (DFT)	Discrete Time FT (DTFT)
$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\omega_k n}$	$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$
$k = 0, 1, \ldots, N-1$	$\omega \in [-\pi, +\pi)$

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DFT in time domain

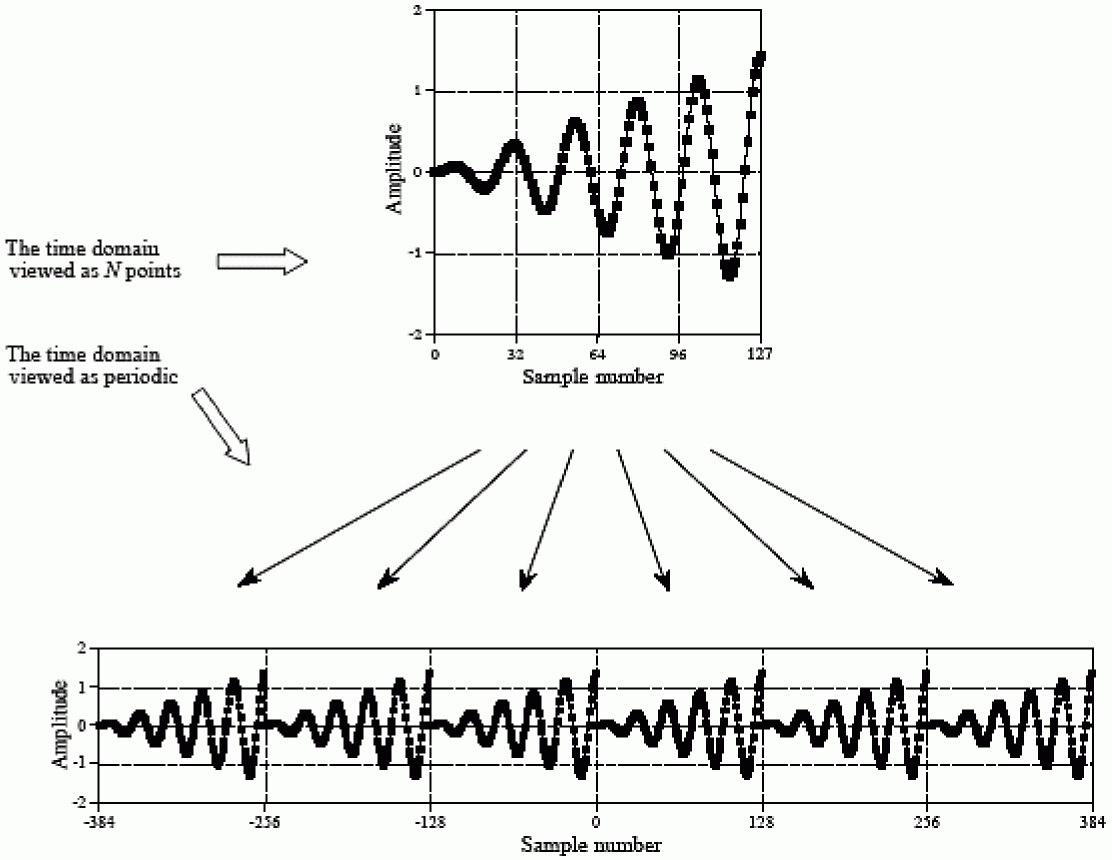

FIGURE 10-8
Periodicity of the DFT's time domain signal. The time domain can be viewed as *N* samples in length, shown in the upper figure, or as an infinitely long periodic signal, shown in the lower figure.

Source: http://www.dspguide.com/ch10/3.htm

DFT in frequency domain

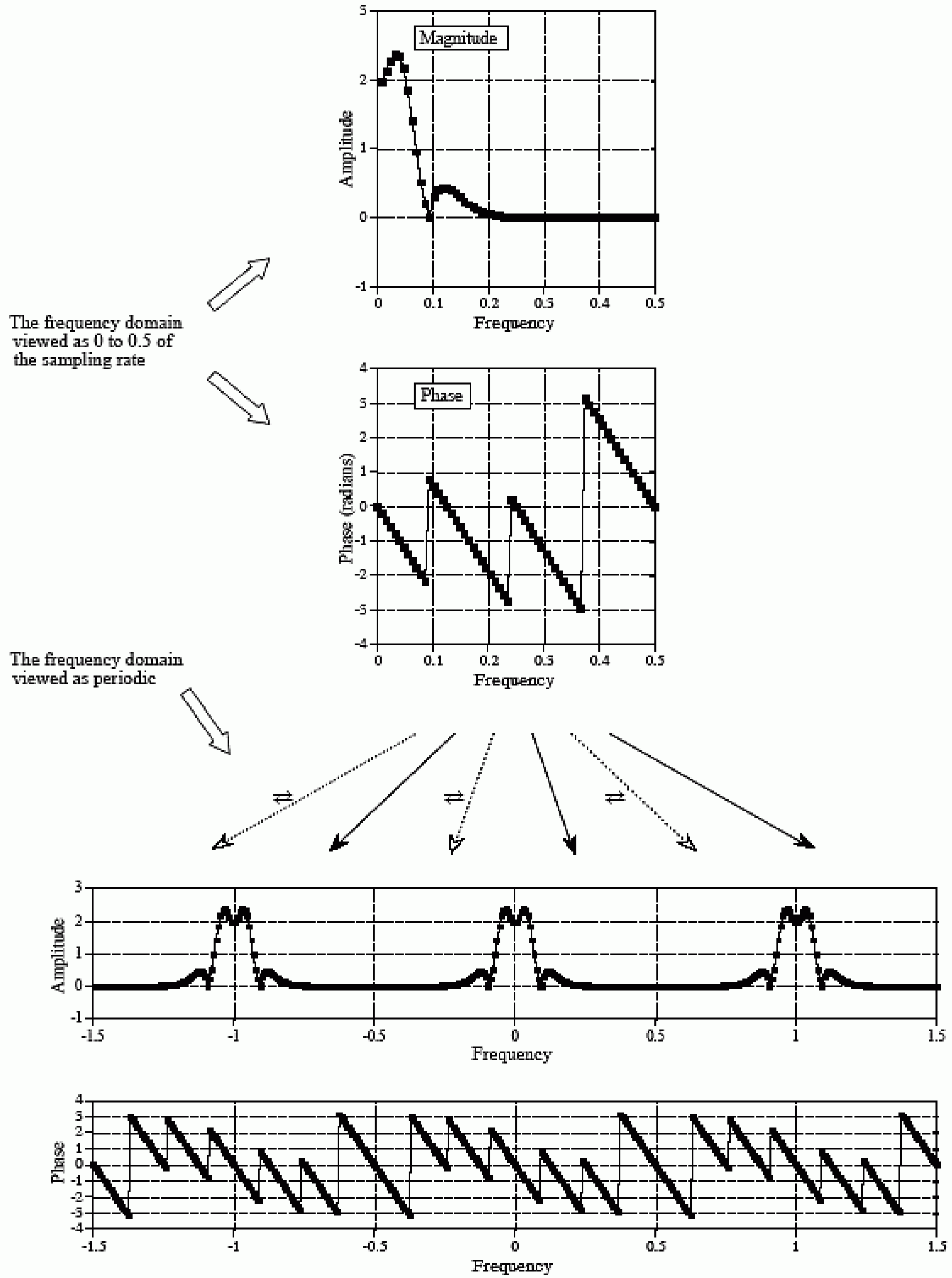

FIGURE 10-9
Periodicity of the DFT's frequency domain. The frequency domain can be viewed as running from 0 to 0.5 of the sampling rate (upper two figures), or an infinity long periodic signal with every other 0 to 0.5 segment flipped left-for-right (lower two figures).

Source: http://www.dspguide.com/ch10/3.htm

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DFT

Problem: match points with cosines, of different frequency and phase?
Answer: family of solutions….

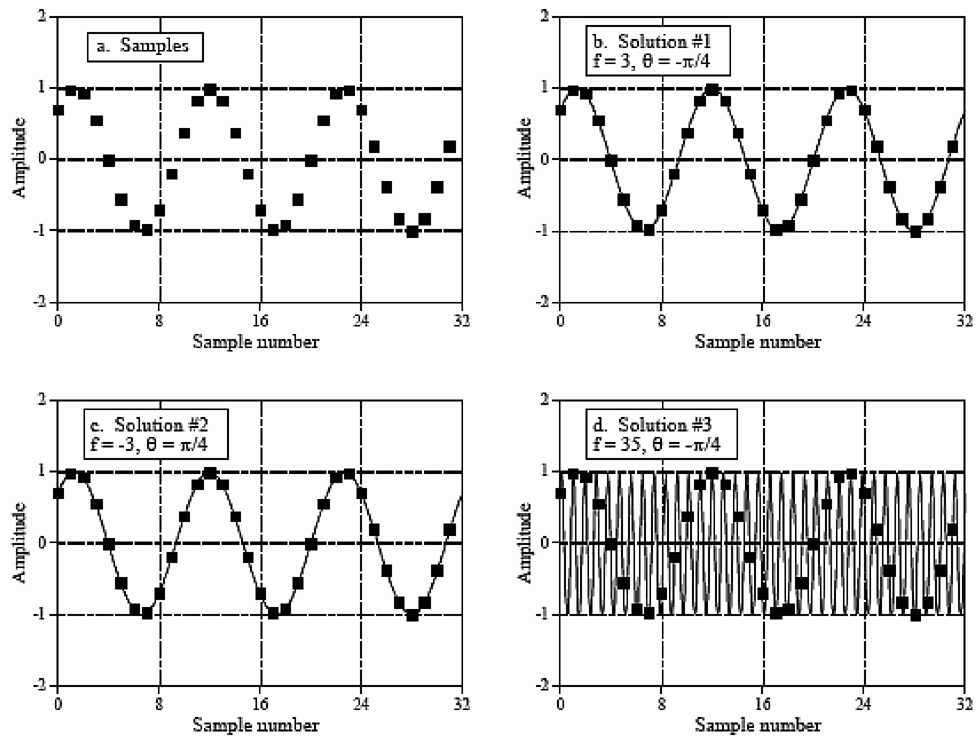

FIGURE 10-11
The meaning of negative frequencies. The problem is to find the frequency spectrum of the discrete signal shown in (a). That is, we want to find the frequency and phase of the sinusoid that passed through all of the samples. Figure (b) is a solution using a *positive* frequency, while (c) is a solution using a *negative* frequency. Figure (d) represents a family of solutions to the problem.

Why 'bother' with negative frequencies?

Source: http://www.dspguide.com/ch10/3.htm

Why ‘bother’ with negative frequencies (and “negative” time)?

Some DSP operations don’t require knowledge of negative frequencies.
Example in spectral analysis sufficient to see frequency domain from 0 to ½ fs.

However, many DSP operations require knowledge of negative frequencies, as signals may overflow between periods causing time and frequency domain aliasing.

DFT and aliasing

Periodicity in time domain + processing

time domain aliasing

Periodicity in frequency domain + processing

frequency domain aliasing

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DFT and aliasing

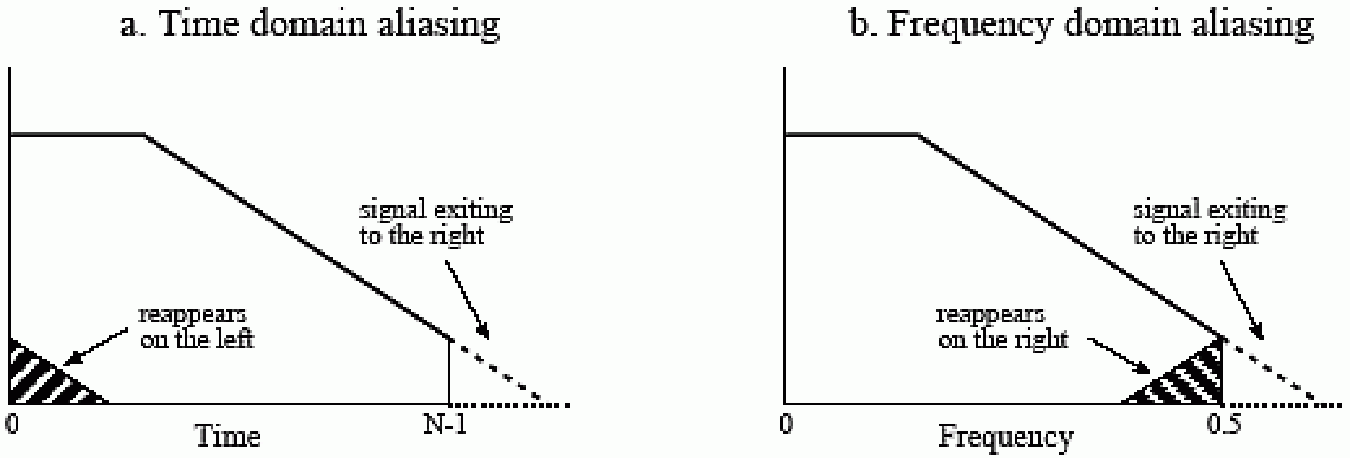


FIGURE 10-10
Examples of aliasing in the time and frequency domains, when only a single period is considered. In the time domain, shown in (a), portions of the signal that exits to the right, reappear on the left. In the frequency domain, (b), portions of the signal that exit to the right, reappear on the right as if they had been folded over.

Source: http://www.dspguide.com/ch10/3.htm

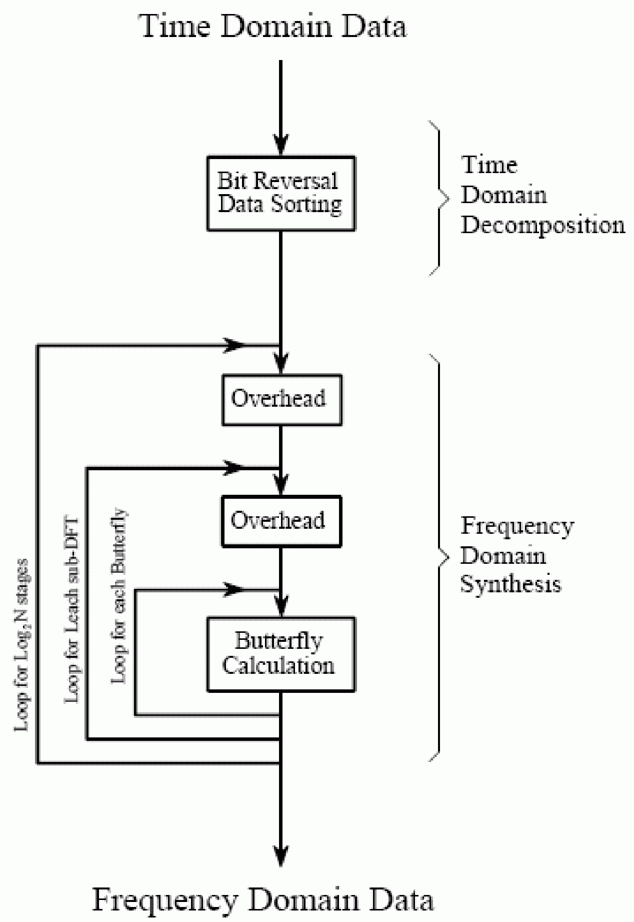
How can we calculate the DFT?

-solve simultaneous linear equations
-correlation method
-Fast Fourier Transform

Why FFT? IT IS FAST!!! From walking… to a jet aircraft!!!

FFT for signal processing engineers

FIGURE 12-7
Flow diagram of the FFT. This is based on three steps: (1) decompose an N point time domain signal into N signals each containing a single point, (2) find the spectrum of each of the N point signals (nothing required), and (3) synthesize the N frequency spectra into a single frequency spectrum.



Source: http://www.dspguide.com/ch10/3.htm

FFT for signal processing engineers

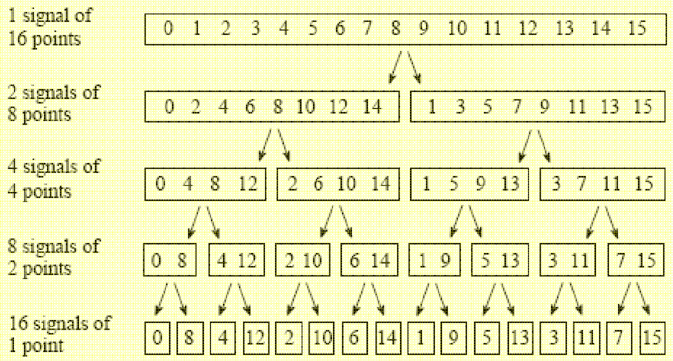


FIGURE 12-2
The FFT decomposition. An N point signal is decomposed into N signals each containing a single point. Each stage uses an *interlace decomposition*, separating the even and odd numbered samples.

Sample numbers in normal order			Sample numbers after bit reversal	
Decimal	*Binary*		*Decimal*	*Binary*
0	0000		0	0000
1	0001		8	1000
2	0010		4	0100
3	0011		12	1100
4	0100		2	0010
5	0101		10	1010
6	0110	⇨	6	0100
7	0111		14	1110
8	1000		1	0001
9	1001		9	1001
10	1010		5	0101
11	1011		13	1101
12	1100		3	0011
13	1101		11	1011
14	1110		7	0111
15	1111		15	1111

FIGURE 12-3
The FFT bit reversal sorting. The FFT time domain decomposition can be implemented by sorting the samples according to bit reversed order.

A lot of complex operations!!!!

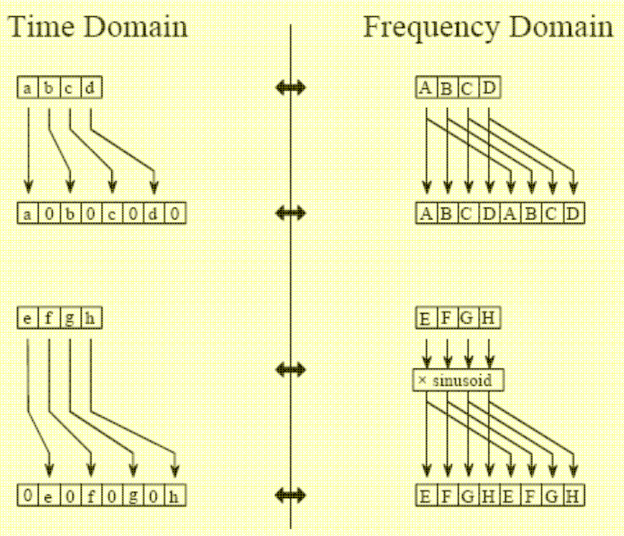
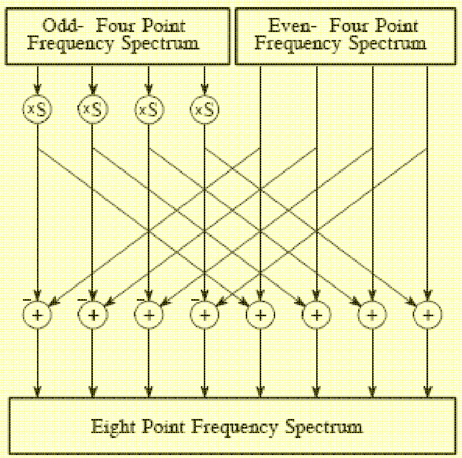
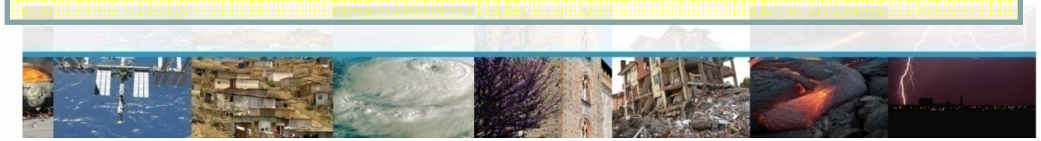


FIGURE 12-4
The FFT synthesis. When a time domain signal is diluted with zeros, the frequency domain is duplicated. If the time domain signal is also shifted by one sample during the dilution, the spectrum will additionally be multiplied by a sinusoid.

FIGURE 12-5
FFT synthesis flow diagram. This shows the method of combining two 4 point frequency spectra into a single 8 point frequency spectrum. The ×S operation means that the signal is multiplied by a sinusoid with an appropriately selected frequency.





Useful operations: Convolution

$$(x*y)_n = \sum_{m=0}^{N-1} x(m)y(n-m)$$

Convolution is cumulative:

$$x*y = y*x$$

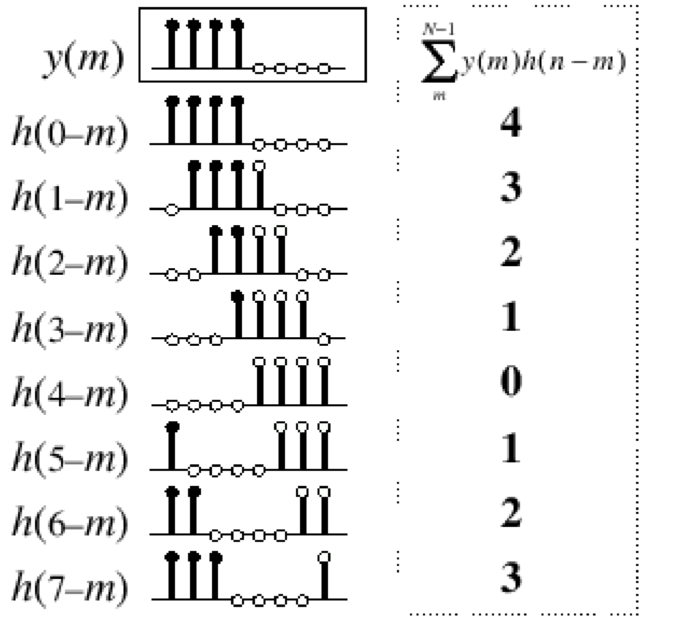
Convolution is associative:

$$x*(y*z) = (y*x)*z$$

Convolution Theorem
(multiplication in one domain is convolution in the other domain):

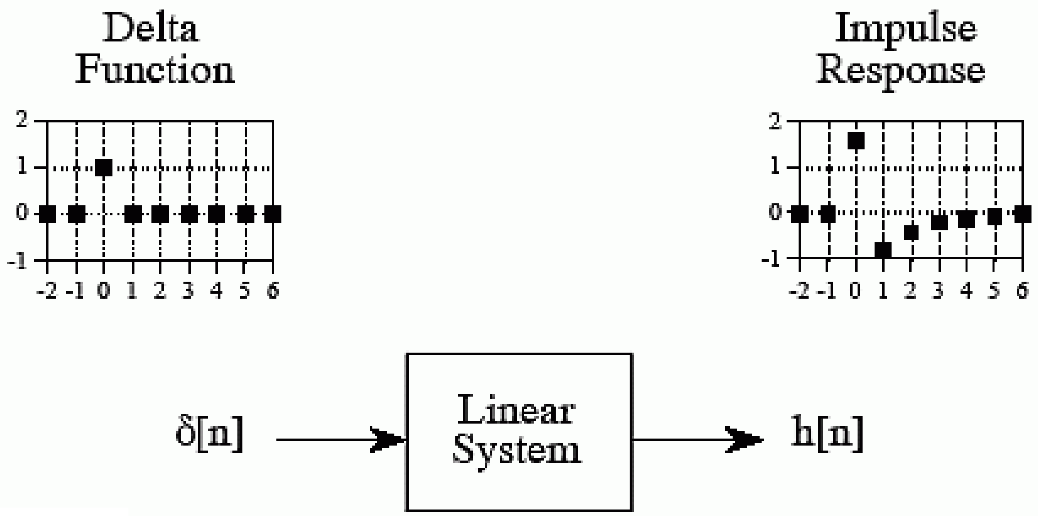
(f *g) Ɔ F x G
(F*G) C f x g

$$y = [1,1,1,1,0,0,0,0]$$
$$h = [1,0,0,0,0,1,1,1]$$
$$(y*h)(n) = [4,3,2,1,0,1,2,3]$$



Source: https://ccrma.stanford.edu/~jos/mdft/Convolution_Example_3_Matched.html

Convolution as Impulse Response



Definition of *delta function* and *impulse response*. The delta function is a normalized impulse. All of its samples have a value of zero, except for sample number zero, which has a value of one. The Greek letter delta, $\delta[n]$, is used to identify the delta function. The *impulse response* of a linear system, usually denoted by $h[n]$, is the output of the system when the input is a delta function.

$$\delta(t) * h(t) = h(t) * \delta(t) = h(t)$$

Source: http://www.dspguide.com/ch6/2.htm

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Useful operations: Correlation

$$(x \circ y)_n = \sum_{m=0}^{N-1} x(m)\, y(m-n)$$

Normalized correlation:

$$(x \circ y)_n = \frac{\sum_{m=0}^{N-1} x(m)\, y(m-n)}{\sqrt{\sum_{m=0}^{N-1} (x(m))^2 \sum_{m=0}^{N-1} (y(m-n))^2}}$$

Special Case: autocorrelation

$$(x \circ x)_n = \sum_{m=0}^{N-1} x(m)\, x(m-n)$$

Correlation is a statistical measure of how similar two waveforms are.

The DFT of the correlation is called cross-spectral density, or cross-spectrum:

$$(x \circ y) \quad ↄ \quad X^{*}\, Y$$

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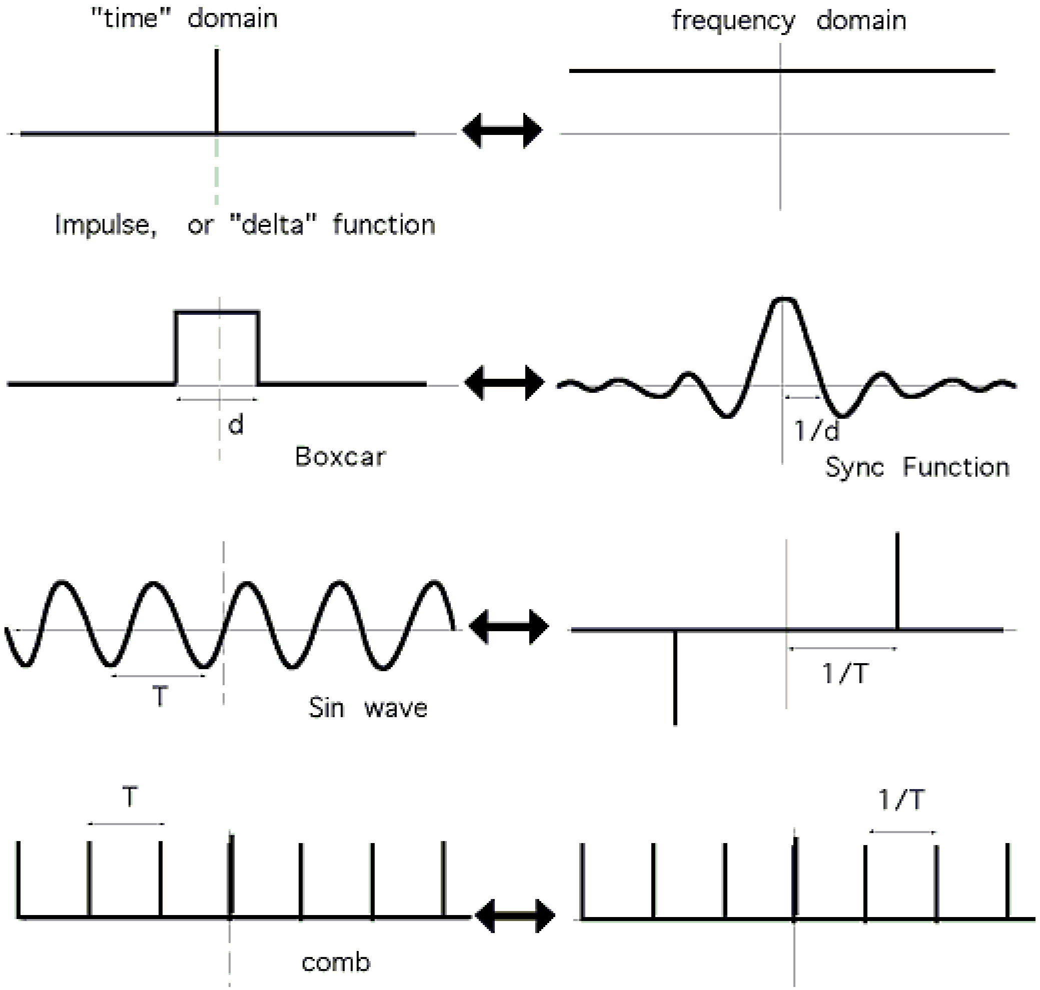
Summary: Useful operations

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\|a\|}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\|a\|}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

Source: http://faculty.kfupm.edu.sa/EE/muqaibel/Courses/EE207%20Signals%20and%20Systems/EE207%20SignalsAndSystems3.htm

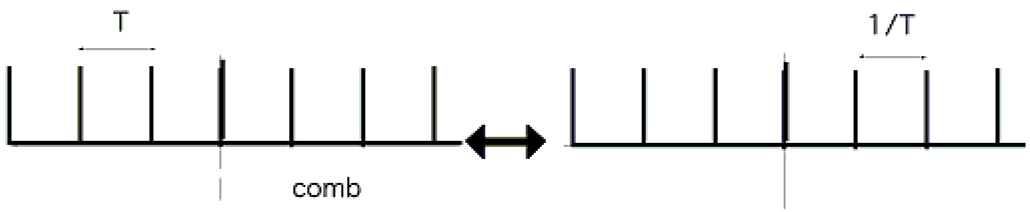
Common Fourier Transforms



Source: http://brokensymmetry.typepad.com/photos/uncategorized/2008/06/19/transform_pairs_3.gif

Sampling in the time domain – Spectral Leakage

Sampling in the time domain (multiplication by shah), implies convolution in the frequency domain by shah. The convolution causes periodic repetition of the spectrum. If sampling in time domain is coarse, then repeated spectra are close to each other, causing spectral leakage.



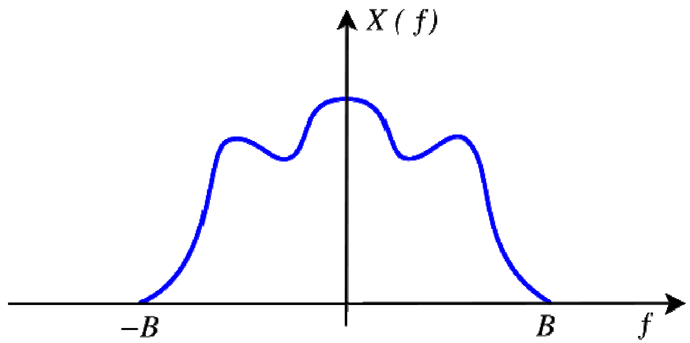
Nyquist-Shannon sampling theorem

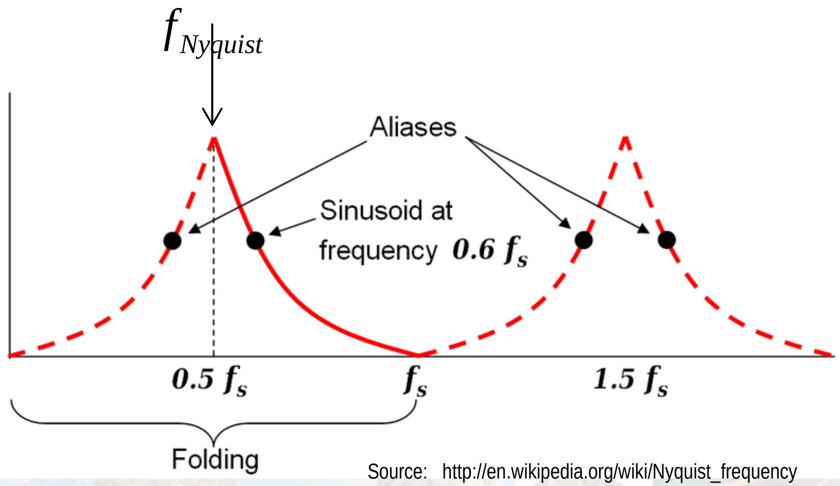
If a function x(t) contains no frequencies higher than B Hz, it is completely determined by giving its ordinates at a series of points spaced 1/(2B) seconds apart.

In other words, the sampling rate must be twice the highest frequency ($f_{Nyquist}$) in order to be able to reconstruct the signal.

$$f_{Nyquist} = \frac{1}{2} f_{sampling} = \frac{1}{2dt}$$

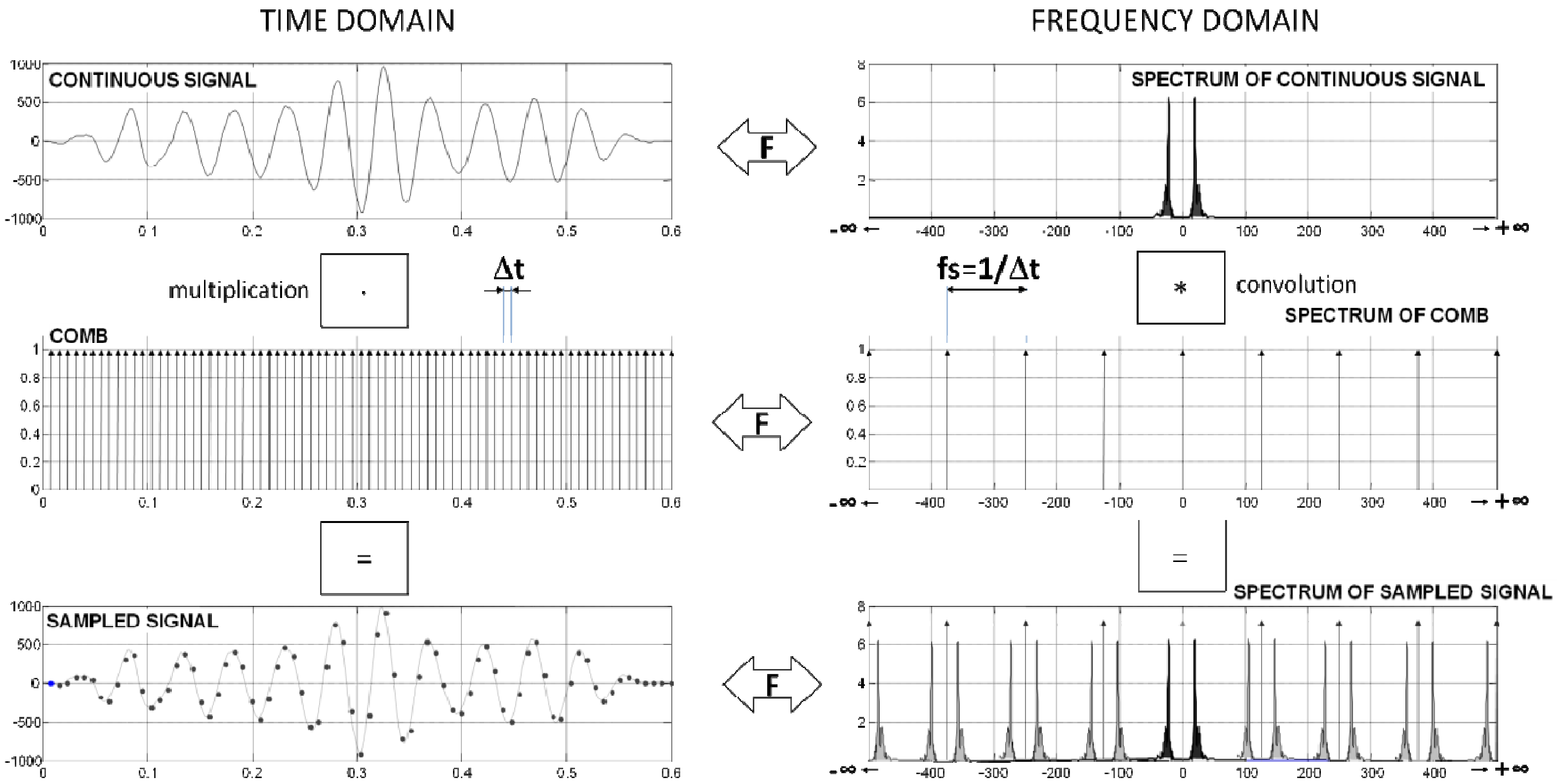
Signal frequencies higher than the Nyquist frequency will encounter a "folding" about the Nyquist frequency, back into lower frequencies. Ex. Sample rate 20Hz, Nyquist is 10Hz, then 11Hz signal will fold to 9Hz, similarly, a 9Hz can fold to 11 Hz.





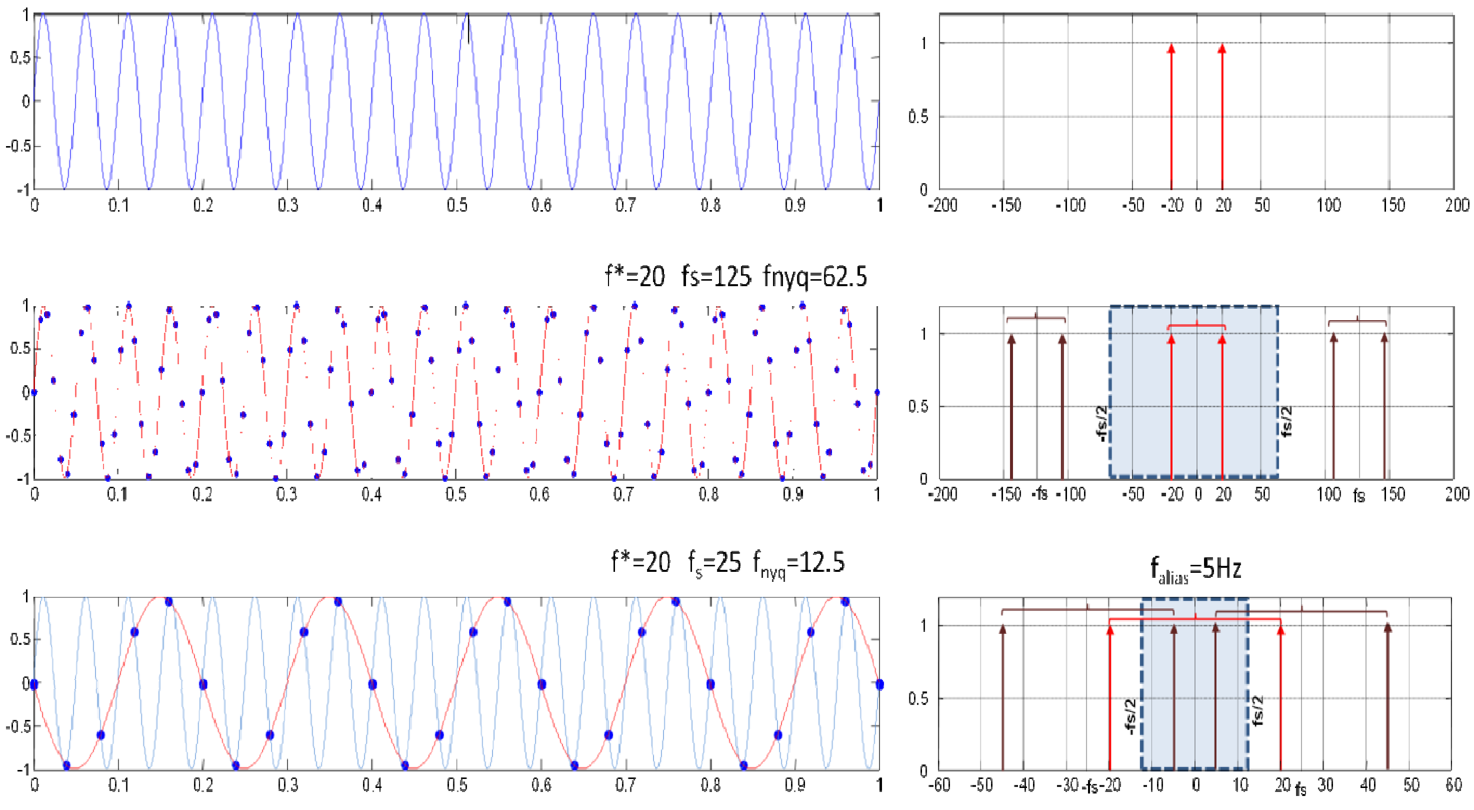
Source: http://en.wikipedia.org/wiki/Nyquist_frequency

Sampling in the time domain – Spectral Leakage



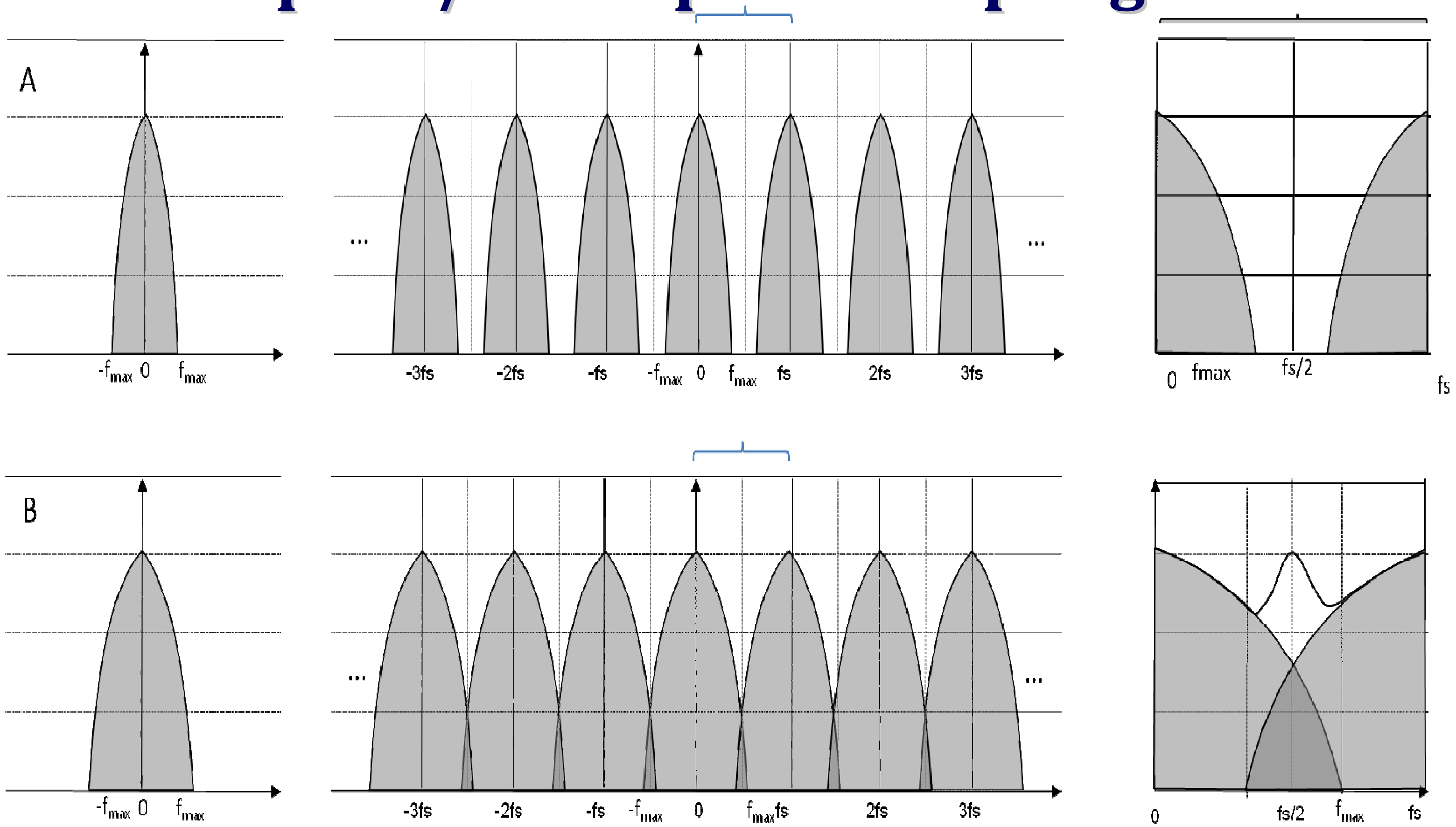
Source: Foti et al., 2013

Nyquist frequency: adequate/inadequate sampling



Source: Foti et al., 2013

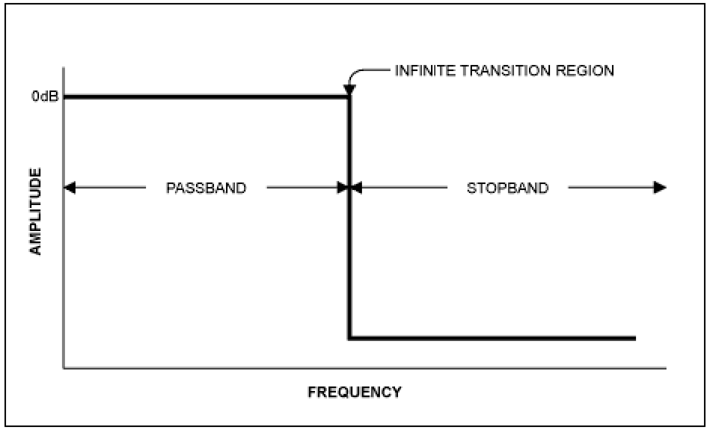
Nyquist frequency: adequate/inadequate sampling

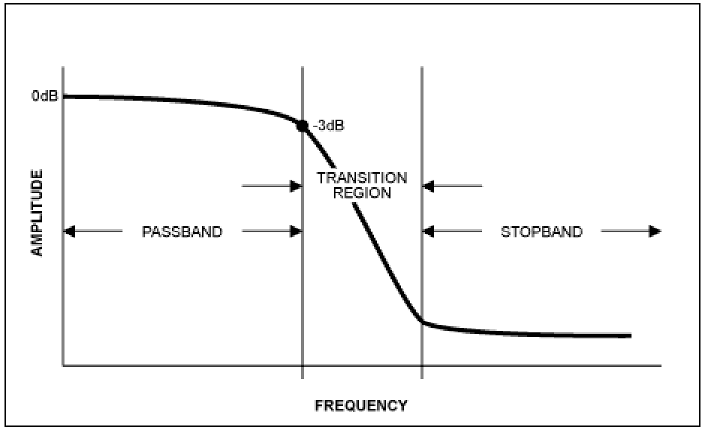


Source: Foti et al., 2013

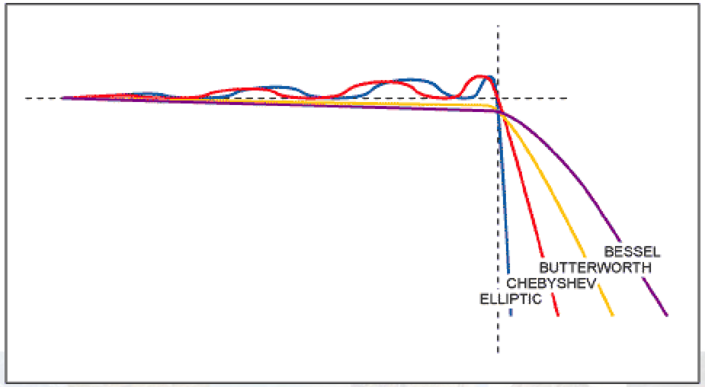
Anti-alias filtering

An **anti-aliasing filter** is a filter used before a signal sampler, to restrict the bandwidth of a signal to approximately satisfy the sampling theorem. Often, an anti-aliasing filter is a low-pass filter.



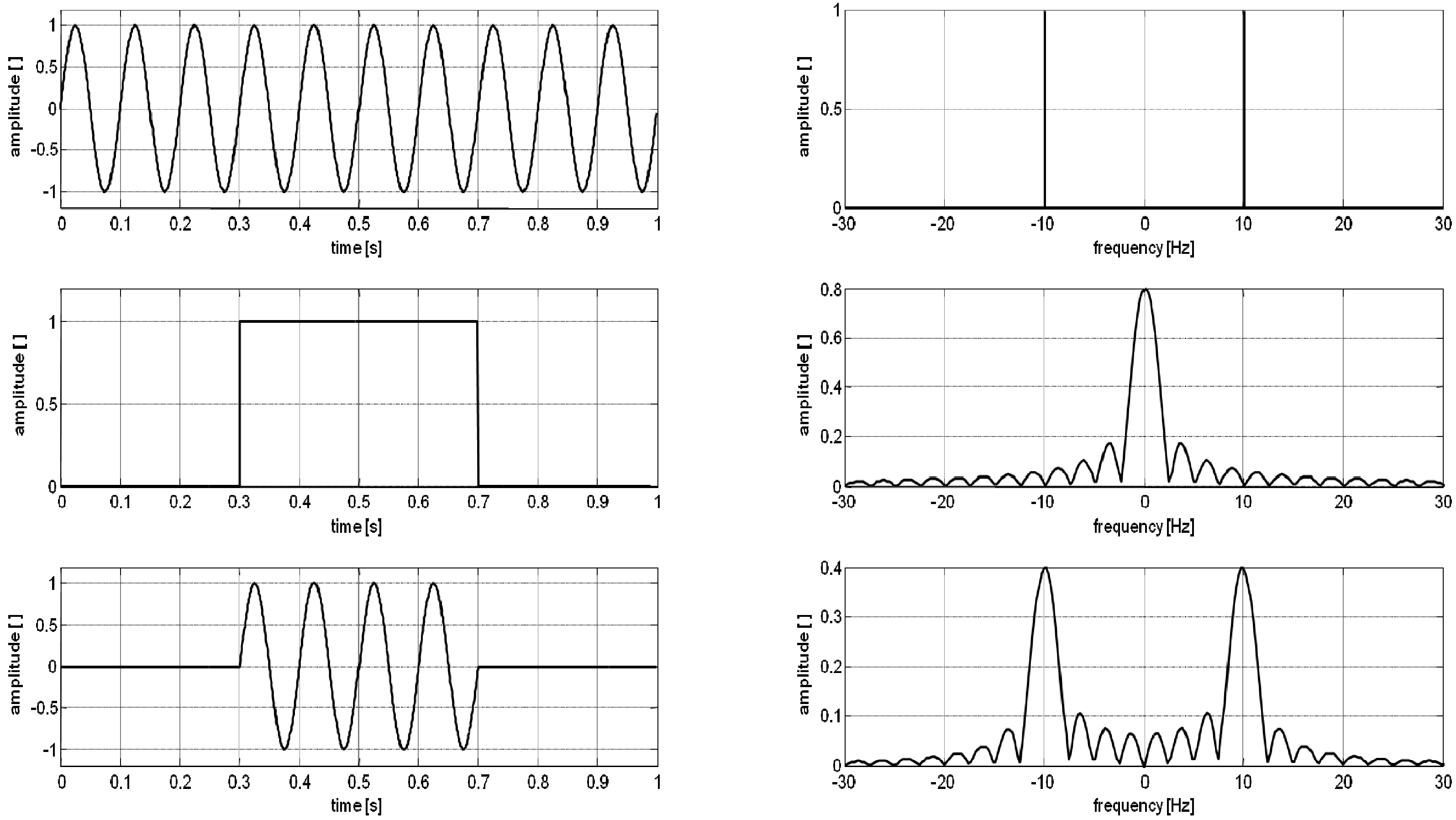


Brick-wall filter is unstable,
Instead choose filter with transition region:



http://www.maximintegrated.com/app-notes/index.mvp/id/928

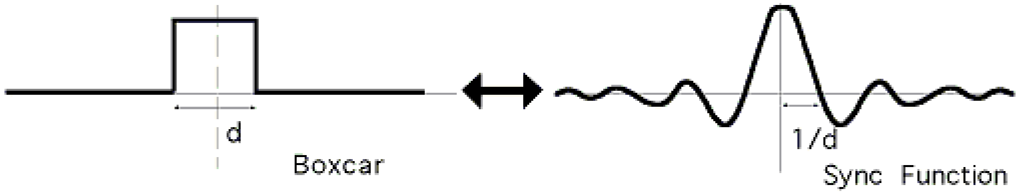
Effect of windowing



Source: Foti et al., 2013

Tapering

All signals which are finite in the time domain, imply multiplication by a boxcar in the time domain, which corresponds to convolution by sync in the frequency domain.

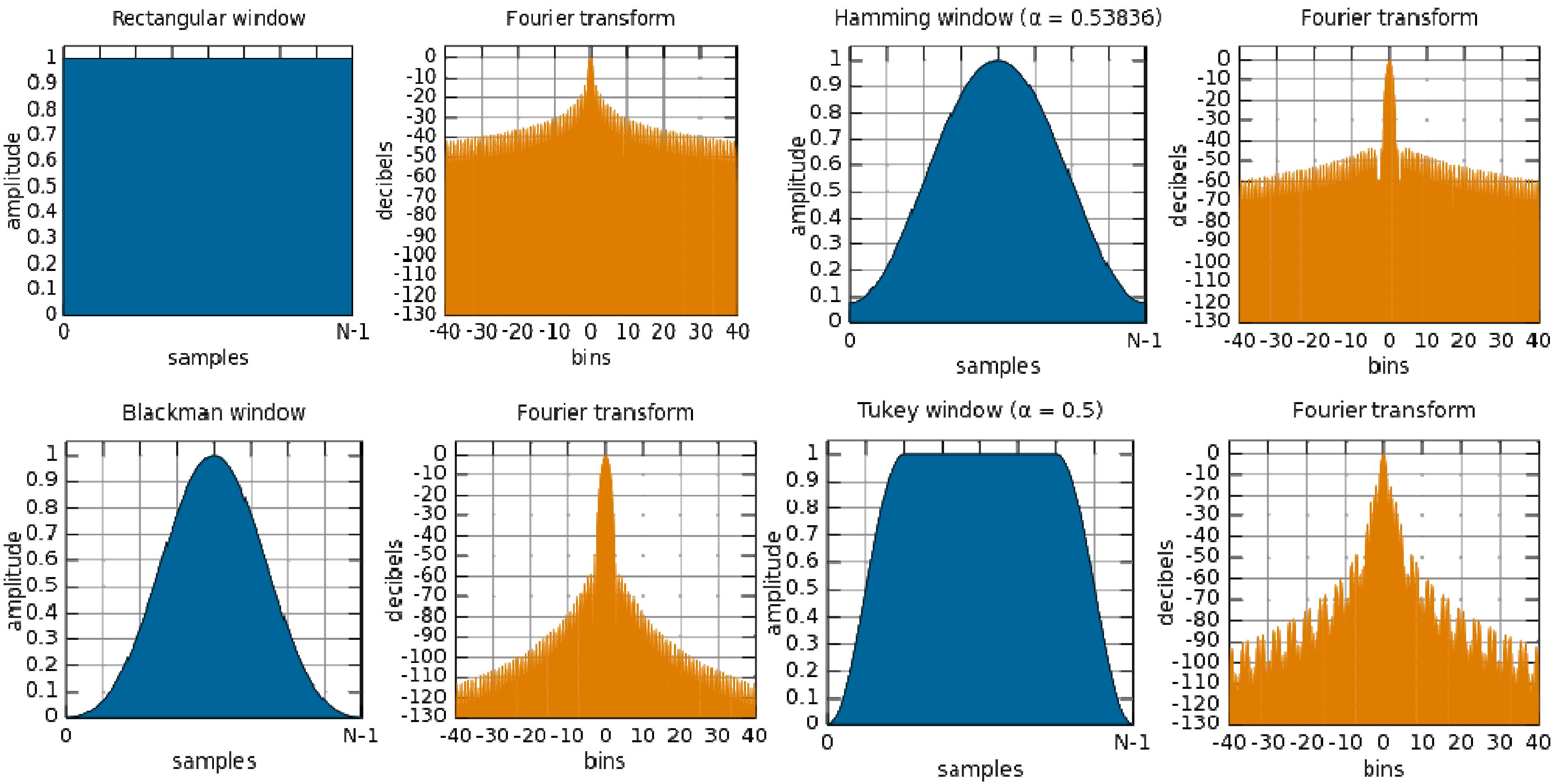


Convolution by sync causes ‘smearing’, which is a relatively localized spreading of frequency components. The shorter the boxcar, the longer the sync function, and the worse is the smearing effect.

To reduce leakage, it is advised to have longer recordings in time, and/or to use tapering windows in the time domain, which behave better in the frequency domain. Examples of taper window could be Hamming, Hann, Blackman, Gaussian, Tukey, Kaiser window, etc.

Source: http://brokensymmetry.typepad.com/photos/uncategorized/2008/06/19/transform_pairs_3.gif

Tapering: Examples



Source: http://en.wikipedia.org/wiki/Window_function

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Dynamic Range

Dynamic range is the ratio between the largest and smallest possible values of the signal. Usually measured as a base-10 (decibel, dB) or base-2 logarithmic (stops) value.

$$L_{\mathrm{dB}} = 10 \log_{10}\left(\frac{P_1}{P_0}\right)$$

Signal to Noise Ratio (S/N)

S/N is a measure of the desired signal to the level of background noise.

$$S/N = \frac{P_{signal}}{P_{noise}} = \left(\frac{A_{signal}}{A_{noise}}\right)^2$$

Alternatively, can define it based on ratio of mean to standard deviation of the signal:

$$S/N = \frac{\mu}{\sigma}$$

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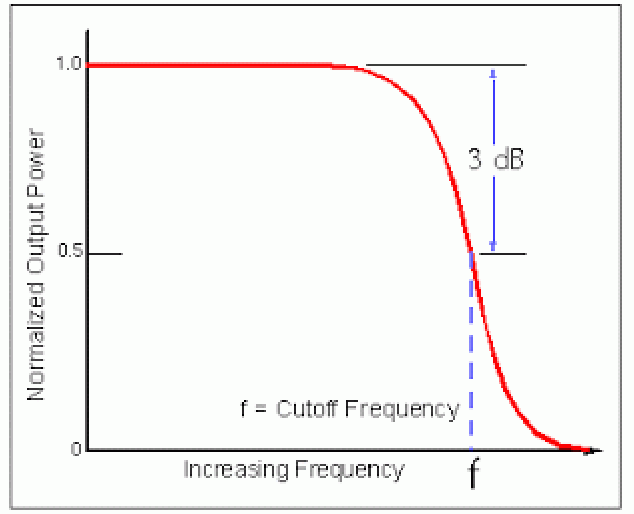
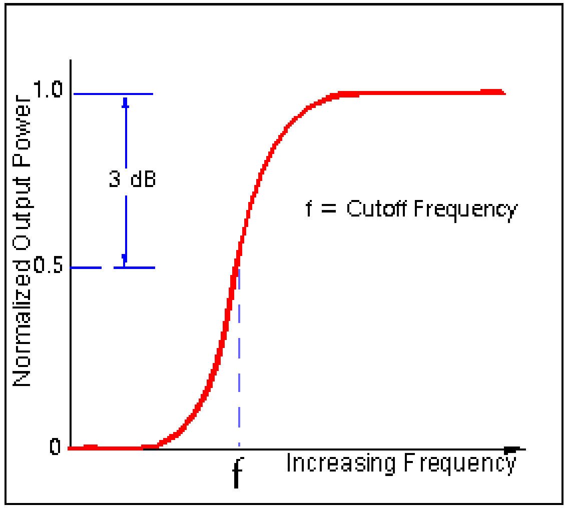
Filtering

Depending on the application, a filter may be applied equivalently as multiplication in the frequency domain, or a convolution in the time domain.

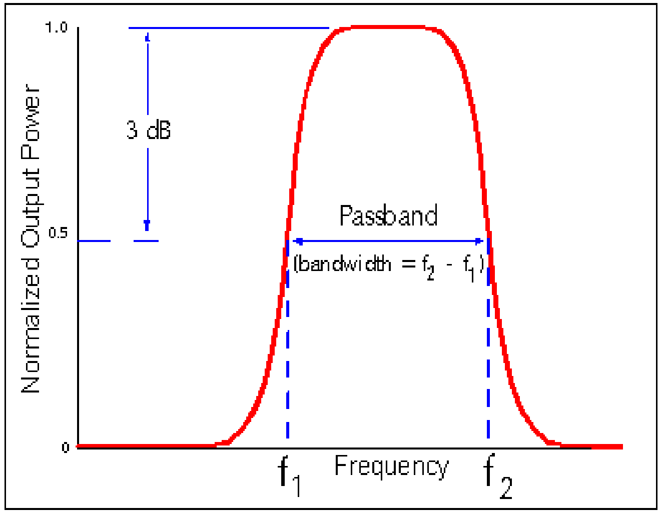
Common filters:
Low-pass (high-cut), high-pass (low-cut), bandpass filter



low pass filter



high-pass filter

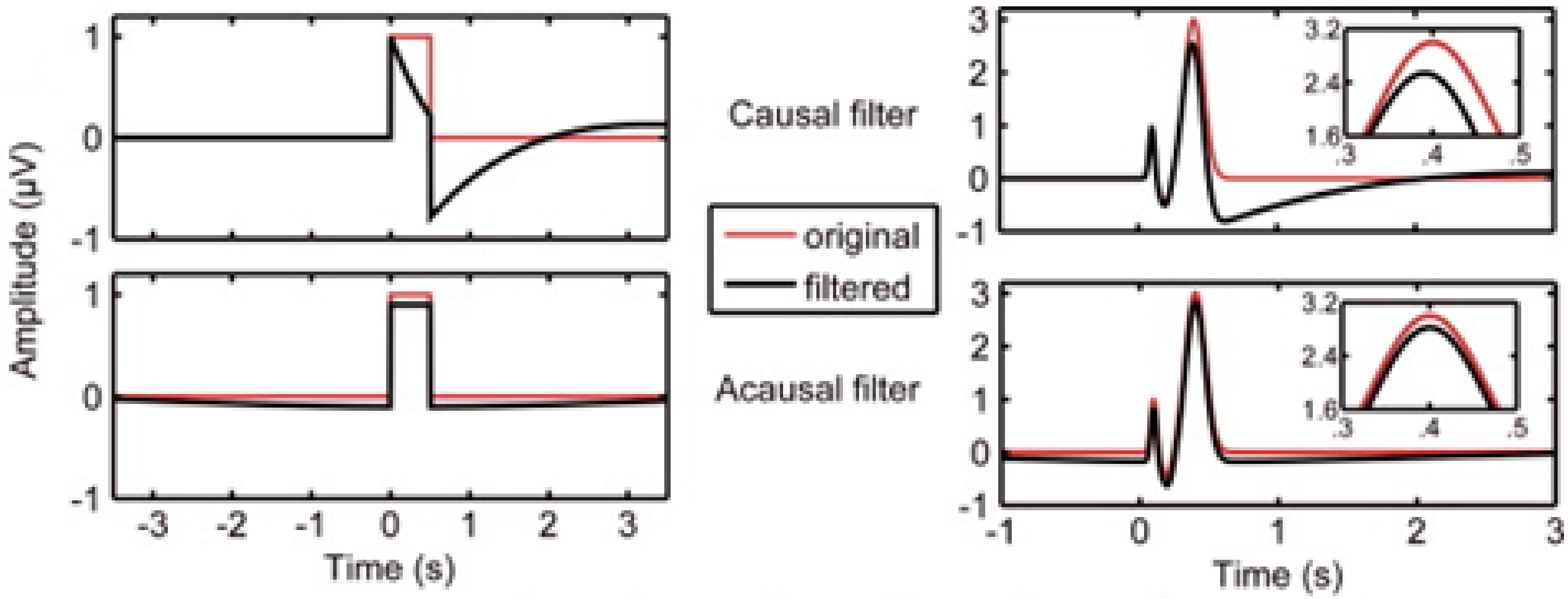


bandpass filter

Source: http://syahit-at-utm.blogspot.gr/2012/11/30-oct-2012.html#!/2012/11/30-oct-2012.html
http://www.songsofthecosmos.com/images/bandpass_filter.gif

Filtering

An acausal filter is a filter which does not change the phase of the signal, whereas a causal filter may change the phase of the signal.



Source: http://www.sciencedirect.com/science/article/pii/S0165027012002361

Uncertainty Principle (Heisenberg-Gabor limit)

A function cannot be band-limited both in time and frequency.

Stated alternatively, "one cannot simultaneously localize a signal (function) in both the time domain (*f*) and frequency domain (Fourier transform)".

When applied to filters, the result is that one cannot achieve high temporal resolution and frequency resolution at the same time; a concrete example are the resolution issues of the short-time Fourier transform– if one uses a wide window, one achieves good frequency resolution at the cost of temporal resolution, while a narrow window has the opposite trade-off.

Let T be the duration of the signal, and B be its badwidth, then: $$TB \geq 1$$

Signal Transformations

Why should we transform from time and space to frequency and wavenumber??

Wavefield transformations allow the separation and identification of different seismic events.

Common wavefield transformations, which involve Fourier transform: *f-k*, τ*-p*, ω*-p*

By means of wavefield transformations we can separate different types of waves (P- S- and surface waves). Hence, f-k transformation is a powefull tool to isolate, and filter out unwanted signals (noise).

Wave Equation

A particularly useful class of solutions:

$$u(x,t) = Ae^{i(\omega t \pm kx)} = A\left[\cos(\omega t \pm kx) + i\sin(\omega t \pm kx)\right]$$

Where k is defined as the wavenumber: $k = \dfrac{\omega}{v}$

In fact, k could be a vector in 3D.

This constitutes a function of both wavenumber and frequency. To analyze it we need to introduce 2D Fourier transforms.

Fourier Transform: 2D

Single component: ex. particle velocity as a function of time: V(t) from which we can, using Fourier transform, obtain the magnitude and phase spectra.

Multi-component: Use **2D Fourier transform** theory. From t-x to f-k domain.

g(x,t) ⊃ *G(k,f)*

$$G[k,\, f\,] = \frac{1}{\sqrt{MN}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g[m,n] e^{-j2\pi\left(\frac{mk}{M}+\frac{nf}{N}\right)}$$

$$g[m,n] = \frac{1}{\sqrt{MN}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} G[m,n] e^{-j2\pi\left(\frac{mk}{M}+\frac{nf}{N}\right)}$$

$$0 \le m,k \le M-1, 0 \le n,l \le N-1$$

Amplitude: $|\,G(u,v)\,| = \sqrt{G_r(u,v)^2 + G_i(u,v)^2}$

Phase: $\phi(u,v) = a\tan\left[\frac{G_i(u,v)}{G_r(u,v)}\right]$

Forward and Inverse 2D DFT

DFT

$$F[k,l] = \frac{1}{\sqrt{MN}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[m,n] e^{-j2\pi\left(\frac{mk}{M}+\frac{nl}{N}\right)}$$

IDFT

$$f[m,n] = \frac{1}{\sqrt{MN}} \sum_{l=0}^{N-1} \sum_{k=0}^{M-1} F[k,l] e^{j2\pi\left(\frac{mk}{M}+\frac{nl}{N}\right)}$$

$$(0 \le m,k \le M-1, \quad 0 \le n,l \le N-1)$$

$$\left\{ \begin{array}{c} F[k,l] \\ f[m,n] \end{array} \right\}$$

For 2D DFT, *F* and *f* are matrices of size M x N
(For 1D DFT we only had vectors!)

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Physical Meaning of 2D DFT

$$F[k,l] = \frac{1}{\sqrt{MN}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[m,n] e^{-j2\pi\left(\frac{mk}{M}+\frac{nl}{N}\right)}$$

$$f[m,n] = \frac{1}{\sqrt{MN}} \sum_{l=0}^{N-1} \sum_{k=0}^{M-1} F[k,l] e^{j2\pi\left(\frac{mk}{M}+\frac{nl}{N}\right)}$$

f(m,n) is a linear combination of complex exponentials

$$\left\{ e^{-j2\pi\frac{mk}{M}}, e^{-j2\pi\frac{nl}{N}} \right\}$$

with complex weights:

$$F(k,l)$$

Physical Meaning of 2D DFT

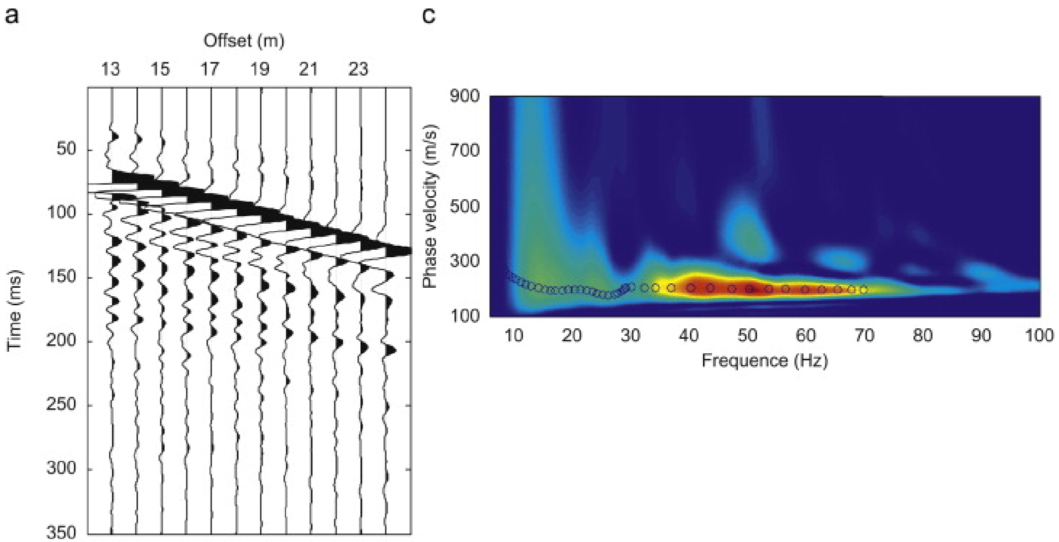
Complex weights can be written in polar form as:

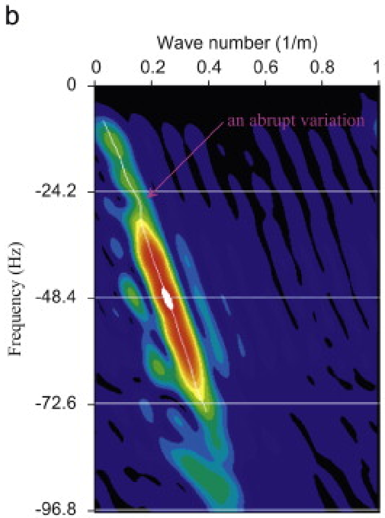
$$F(u,v) = F_r(u,v) + jF_i(u,v) = |F(u,v)|e^{j\angle F(u,v)}$$

Amplitude: $|F(u,v)| \triangleq \sqrt{F_r(u,v)^2 + F_i(u,v)^2}$

Phase: $\angle F(u,v) \triangleq tan^{-1}[F_i(u,v)/F_r(u,v)]$

Fourier Transform: 2D





Benefits of 2D Fourier transform (examples):

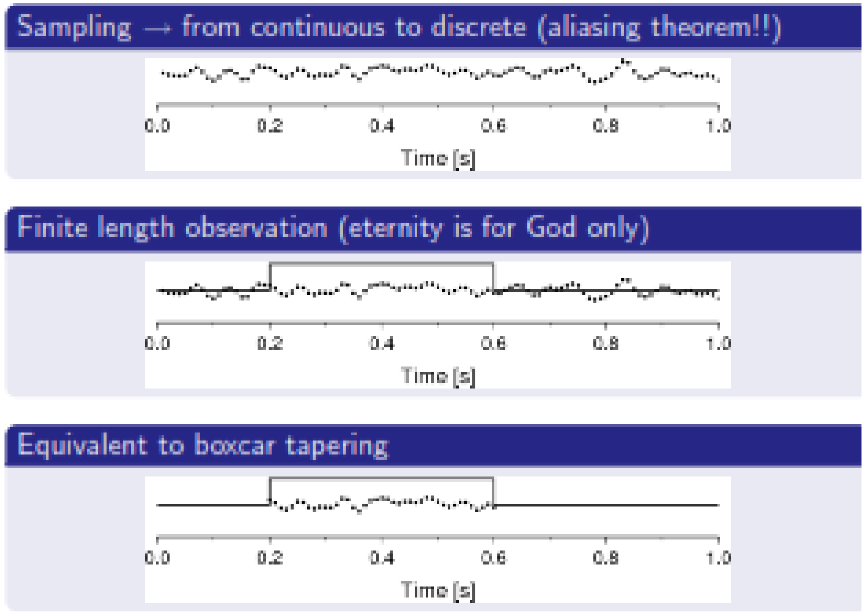
Filtering: May separate several phases (i.e. ground roll, seismic interference) in the f-k which cannot be separated in the t-x doman. Hence, the idea is to do a Fourier 2D transform, apply an f-k filter, and apply an inverse Fourier 2D transform, thereby eliminating undesired phases.

Surface wave dispersion analysis: selecting the dispersion curve used in shear-wave velocity inversion

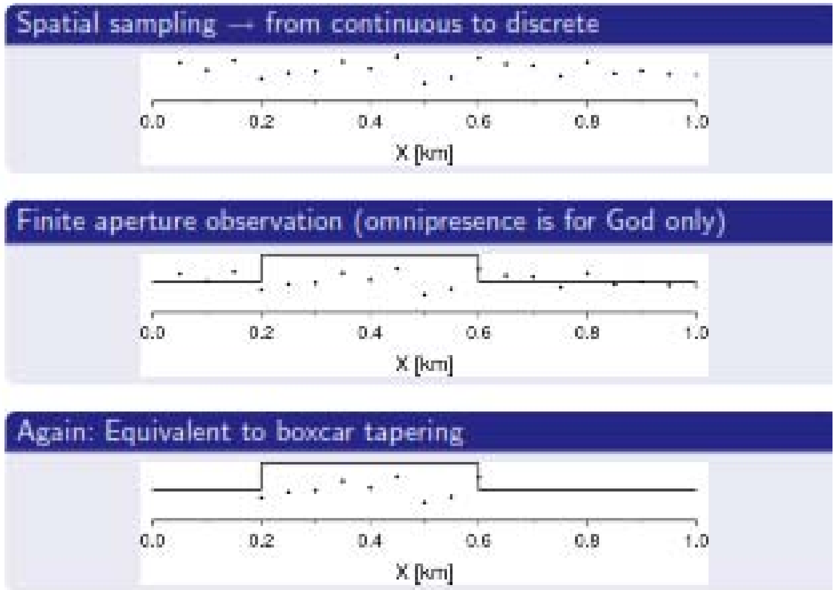
Source: Song et al, 2009.

Analogue:

Recording finite and discrete signal in time:



Recording finite and discrete signal in space:



Source: Geopsy H/V tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

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Linear Array: finite and discrete

Array measurements can be seen as a discrete spatial sampling (receiver locations) of a continuous process (seismic wavefield)

For 1D linear arrays with equidistant spacing, the equivalence to time series sampling is straightforward:

Time Domain	Spatial Domain
$\Delta T < T_{\min} / 2$	$\Delta x < \lambda^{*}_{\min} / 2$
$\Delta\omega = 2\pi / ((N-1)\Delta T)$	$\Delta k = 2\pi / ((N-1)d_{\min}) = 2\pi / D_{\max}$

* apparent

Source: Geopsy H/V tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

Linear Array: finite and discrete

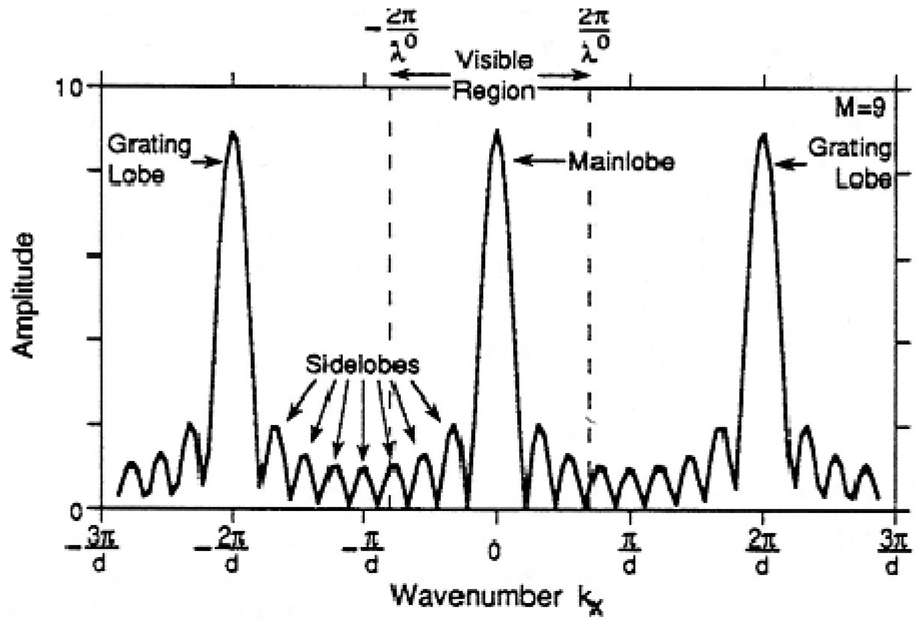
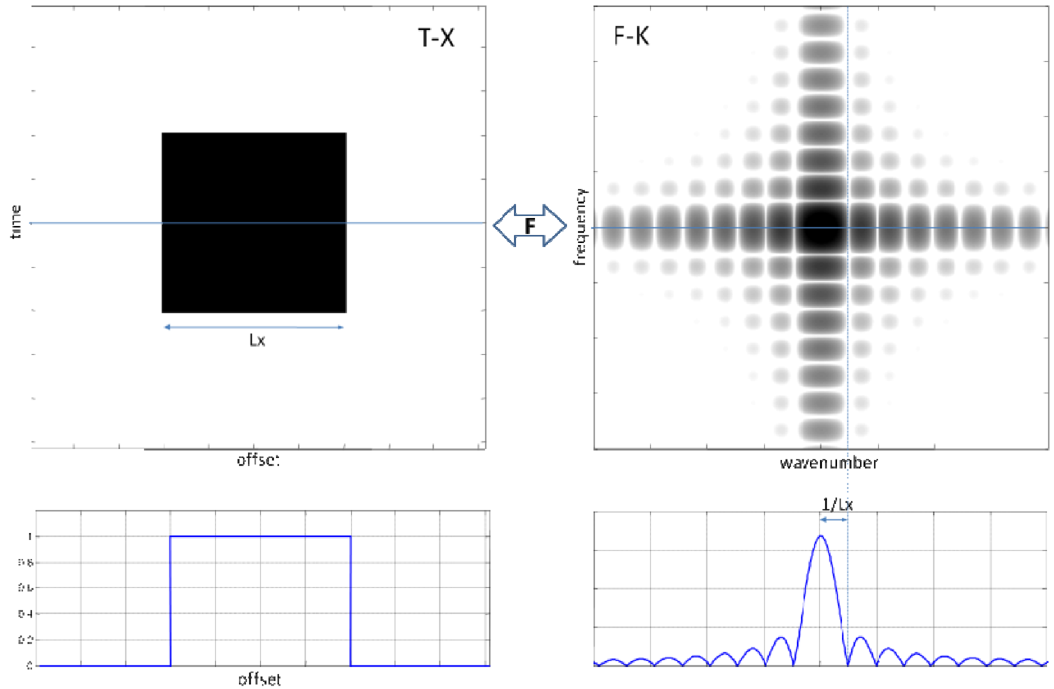


Figure 3.20 The aperture smoothing function magnitude $|W(k)|$ for uniform shading is plotted for a nine-sensor regular linear array. This spatial spectrum has period $k = 2\pi/d$. The visible region of the aperture smoothing function is that part for which $-2\pi/\lambda^o \leq k_x^o \leq 2\pi/\lambda^o$. What might be called secondary mainlobes—those not located at the origin—are termed grating lobes.

Source: Geopsy H/V tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

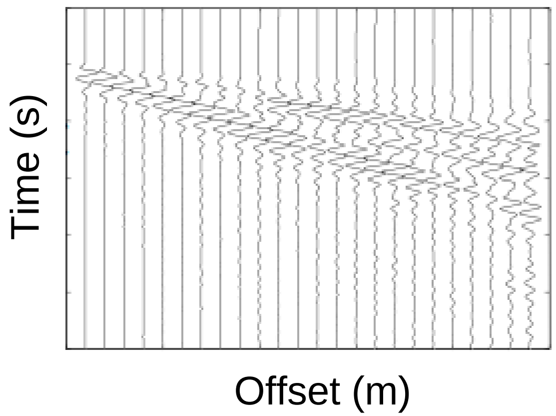
2D Windowing



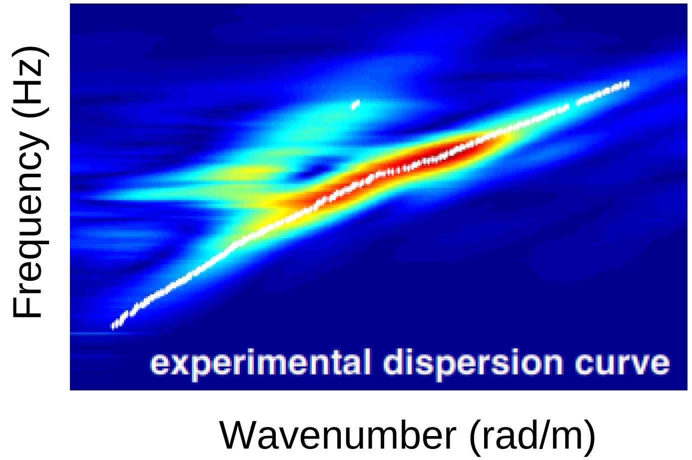
Time-offset box-car window: top) t-x and f-k domains. bottom) Slides at a constant time and at a constant frequency

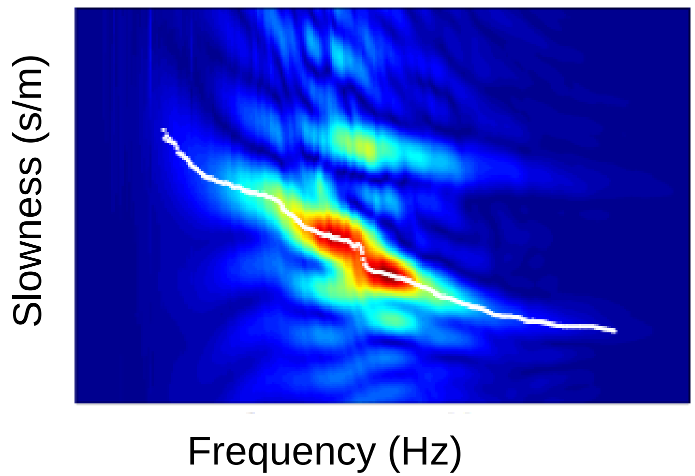
Source: Foti et al., 2013

Linear Array: from shot gather to f-k



Based on:
f-k transform
or τ-*p* transform



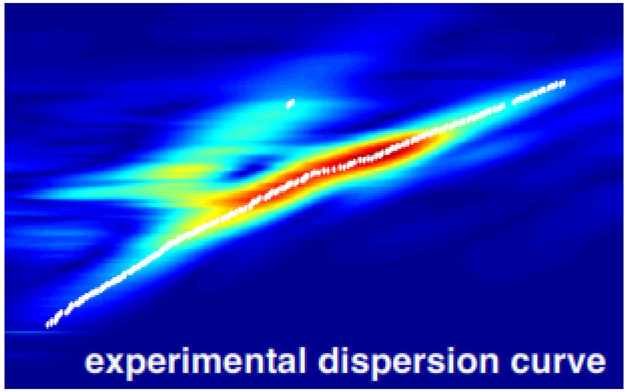


Source: Foti (2012)

f-k spectrum (1D array) to dispersion curve

f-k spectrum
1D array

Frequency (Hz)

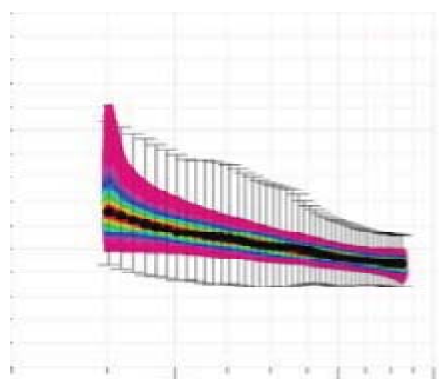


Wavenumber (rad/m)

$$k = \frac{2\pi f}{V}$$

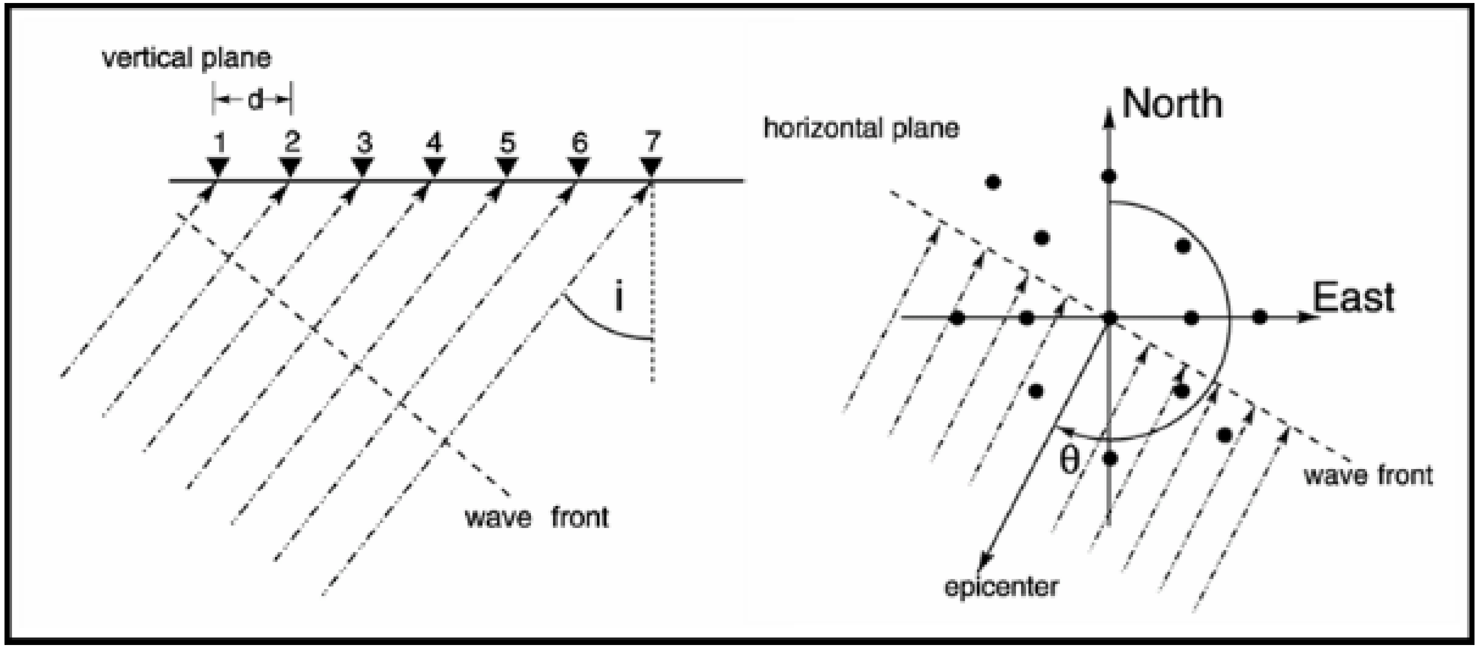
Dispersion curve

Phase Velocity (m/s)



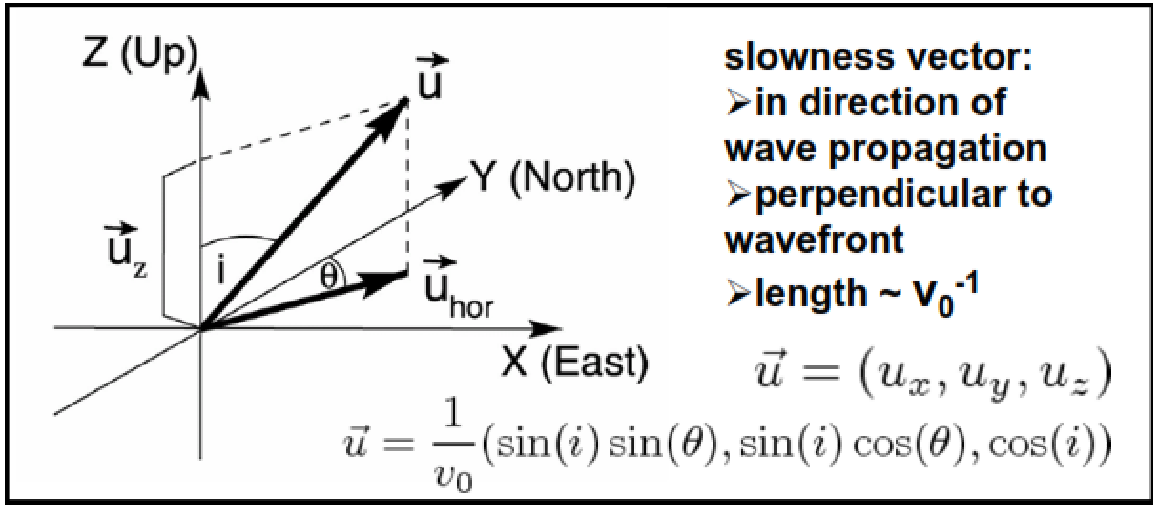
Frequency (Hz)

Slowness



Geometry of plane wave: parameters of wave propagation

Slowness



The slowness vector: $\vec{u} = u_{hor}(\sin(\theta), \cos(\theta), \frac{1}{\tan(i)})$

Horizontal slowness: $p = |\vec{u}_{hor}| = \frac{\sin i}{V_o}$

Source: Geopsy H/V tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

Transform-based methods: *τ-p vs. f-k*

τ-p Method

$$s(x,t) \supset S(\omega, x)$$

1D Fourier Transform

$$U(\omega, p) = \int_{0}^{+\infty} x dx J_o(\omega p x) S(\omega, x)$$

Hankel Transform

$$U(\omega, p) \supset R(\tau, p)$$

1D Inverse Fourier Transform

f-k Method

$$G[k, f] = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{t=0}^{N-1} g(x,t) e^{-i2\pi\left(\frac{xk}{M}+\frac{tf}{N}\right)}$$

$$0 \le x, k \le M-1, 0 \le t, f \le N-1$$

2D Fourier Transform

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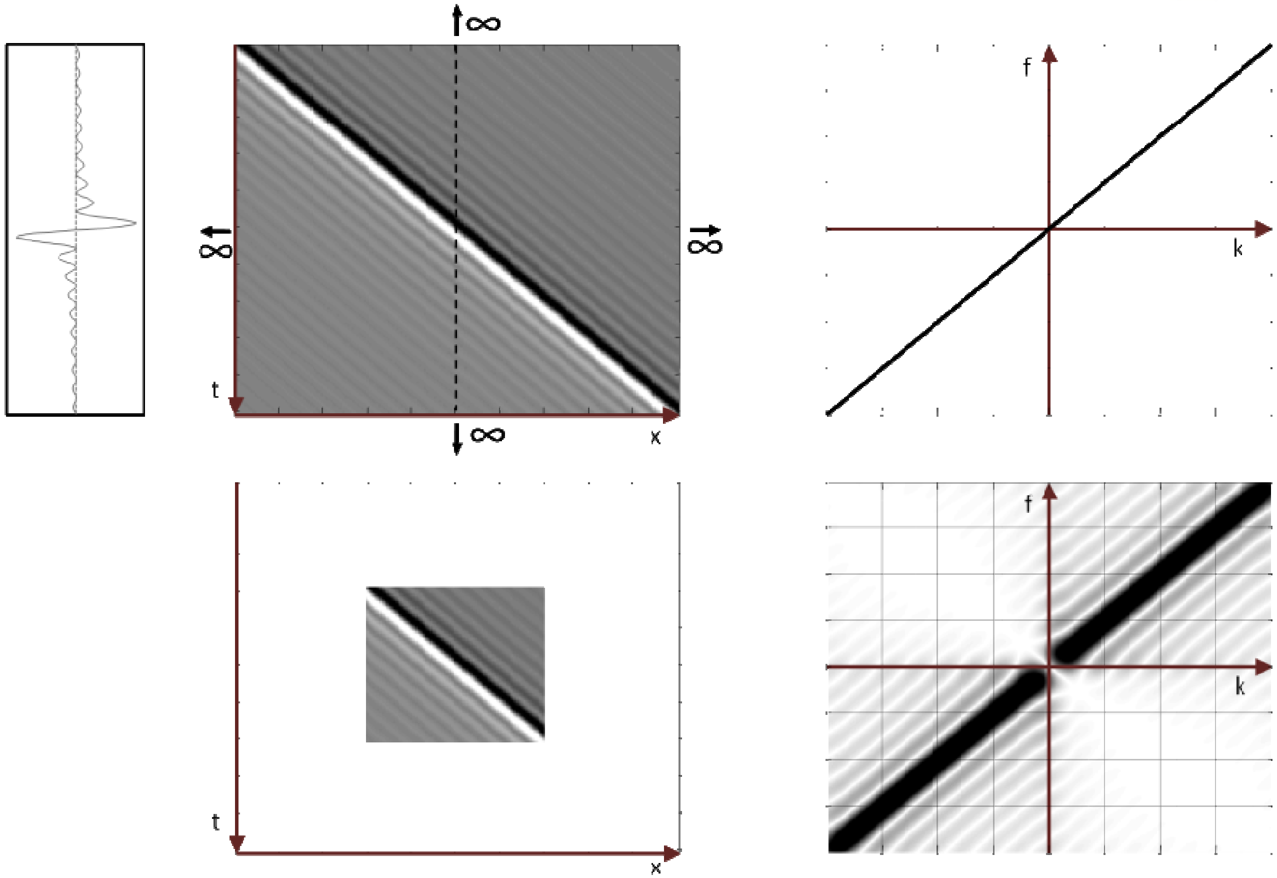
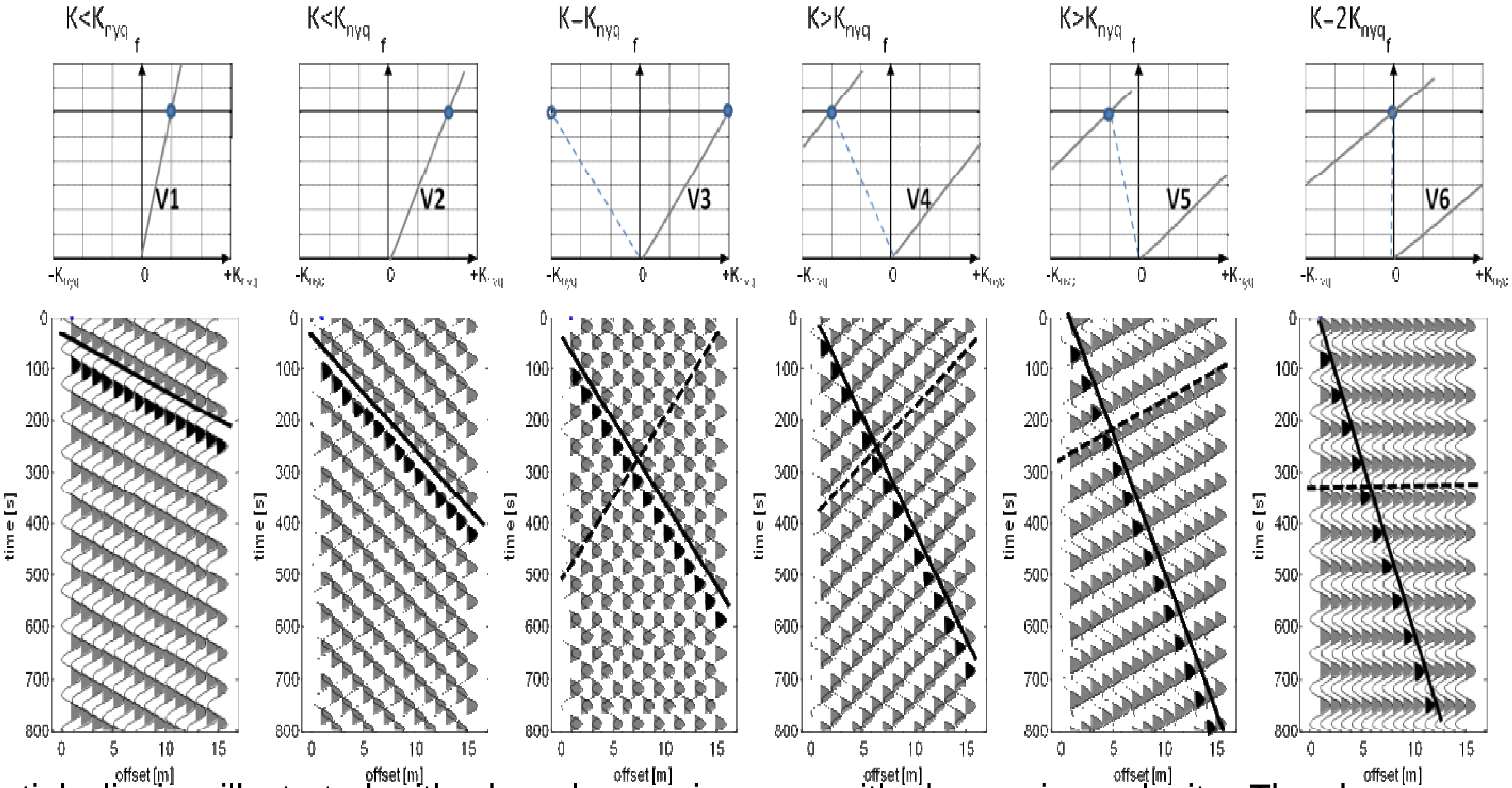
Linear Arrays



Effect of windowing on a broadband wave: the wavelet has a flat spectrum and is not dispersive. The spectral leakage creates a main lobe and side lobes.

Source: Foti et al., 2013

Linear Arrays



Spatial aliasing illustrated with plane harmonic wave with decreasing velocity. The decrease of the velocity produces a higher and higher wavenumber, that in the third panel reached the nyquist wavenumber for the used spatial sampling. Spatial aliasing occurs, and apparent velocities are negative in the panels on the right. The continuous line in the seismic gather represents the actual phase velocity, the dotted line the apparent phase velocity.

Source: Foti et al., 2013

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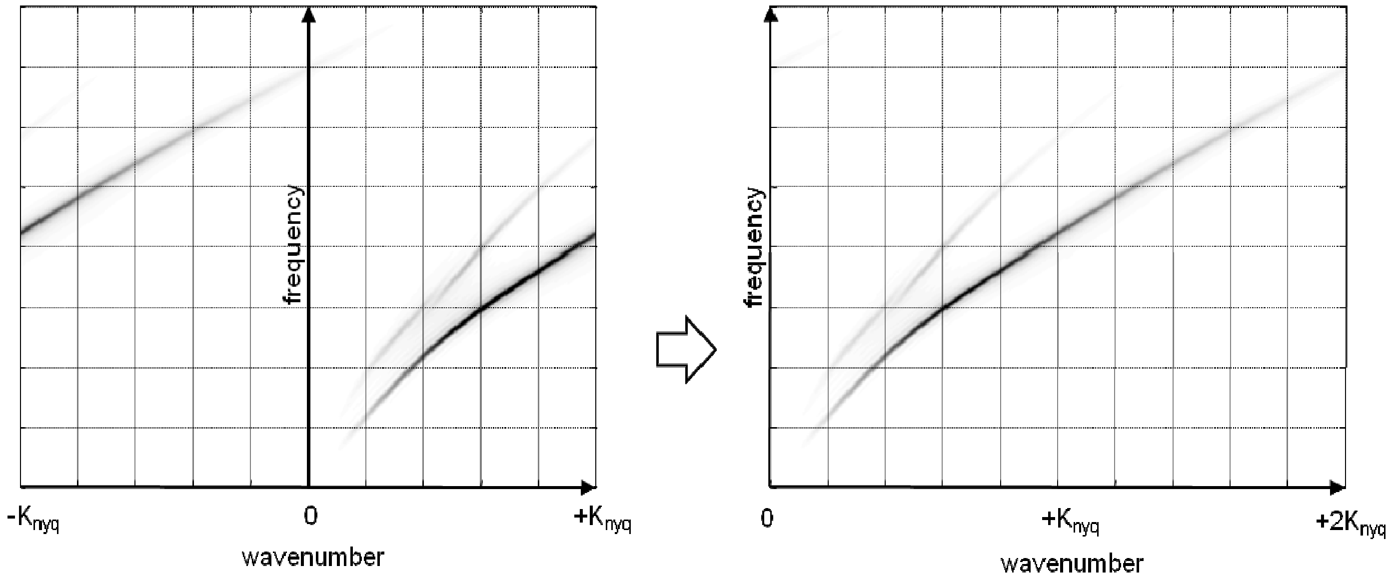
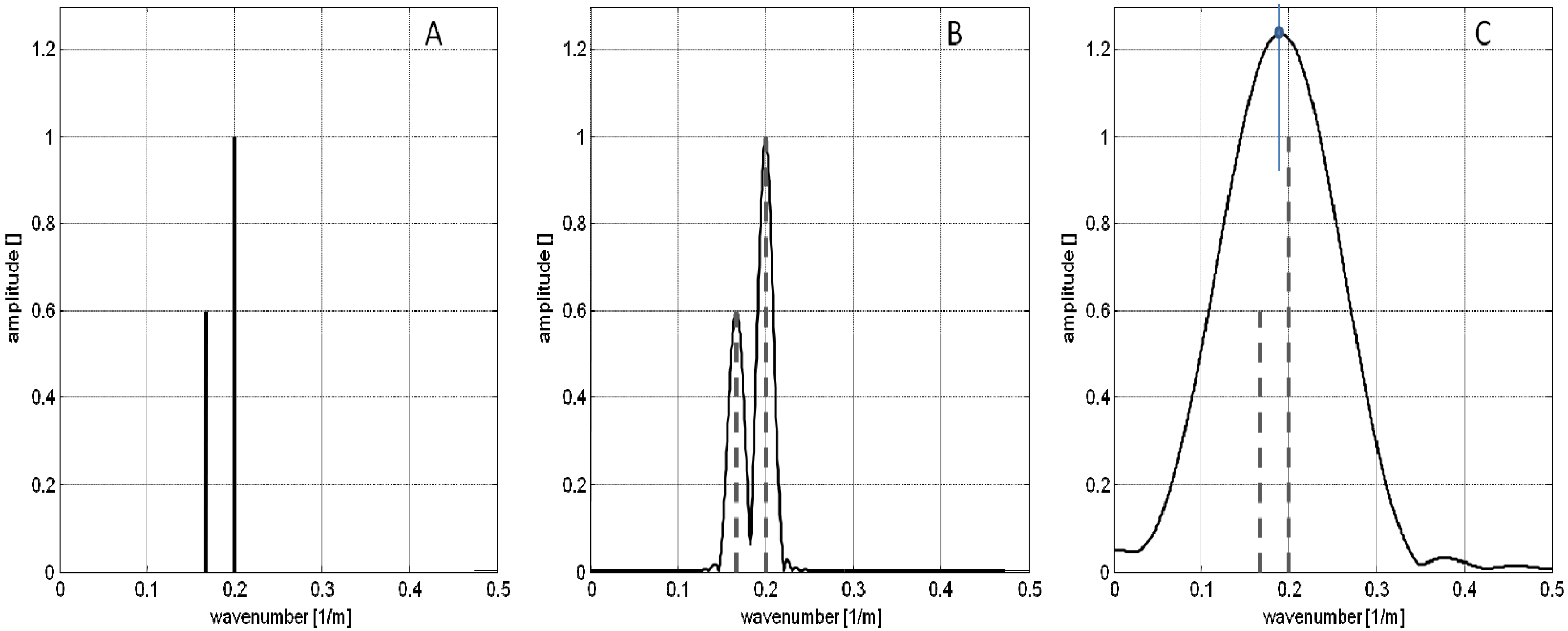
Linear Arrays



Unwrapping the negative quadrant of the f-k spectrum in off-end gathers

Source: Foti et al., 2013

Linear Arrays



Spectral resolution at a constant frequency: the possibility of resolving two modes with arrays of different lengths

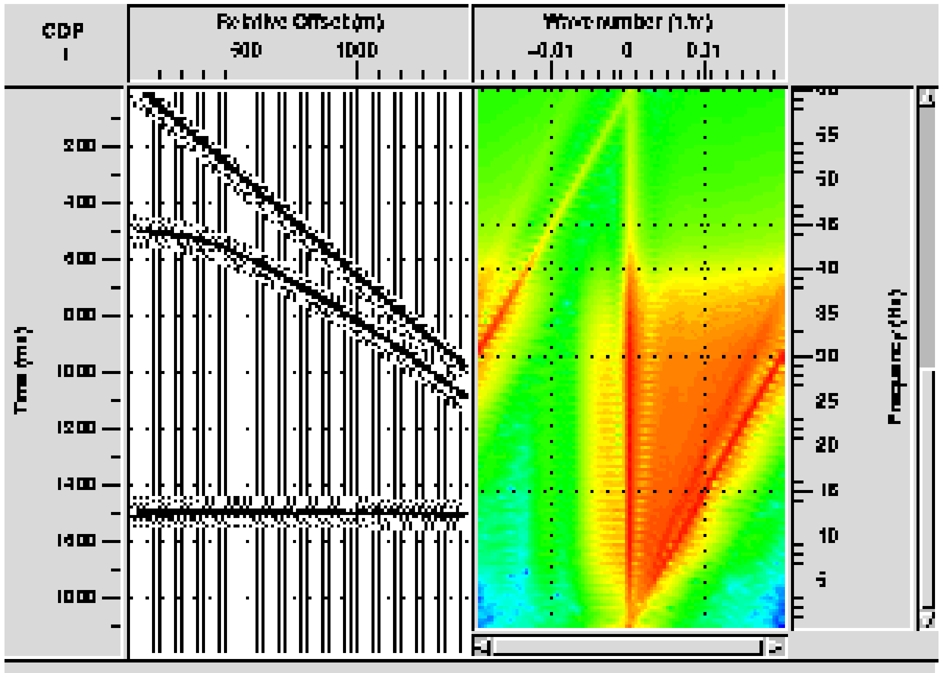
Source: Foti et al., 2013

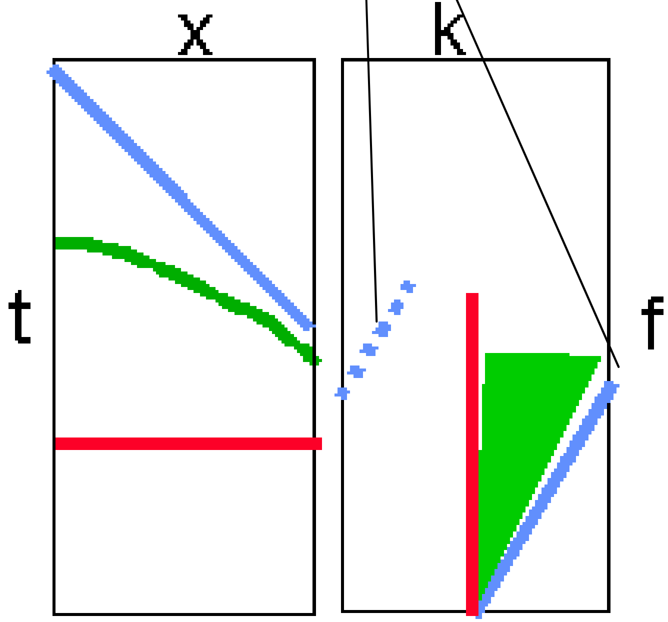
Aliasing (example CMP gather)

FK TRANSFORM

Blue event aliased above 30 Hz

TIME DOMAIN   F-K DOMAIN



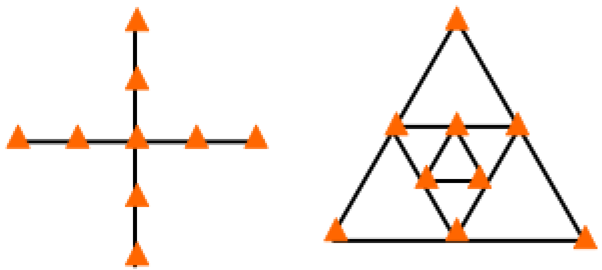


Source: http://www.xsgeo.com/course/basic.htm

2D array transformations

Extension to 2D arrays: finite and discrete arrays

similar story as for 1D-layouts,
BUT parametrization more difficult



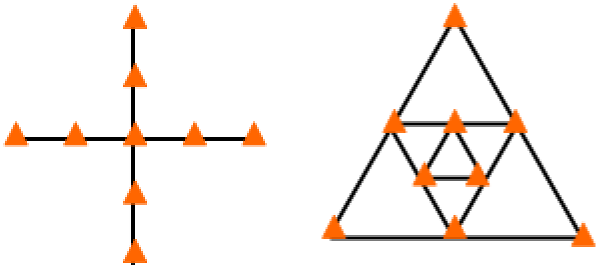
d_{min}, N, D_{max} (aperture)

Source: Geopsy H/V tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

Extension to 2D arrays: finite and discrete arrays

N clear

BUT: d_{min} and D_{max} show directional dependence



especially there will always be some direction, in which d_{min} is vanishing!

limits of array geometry: $\lambda_{min} > 2d_{min}$, $\lambda_{max} \sim 3D_{max}$

Source: Geopsy H/V tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

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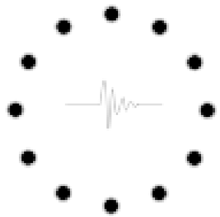
‘Advanced’ Beamforming

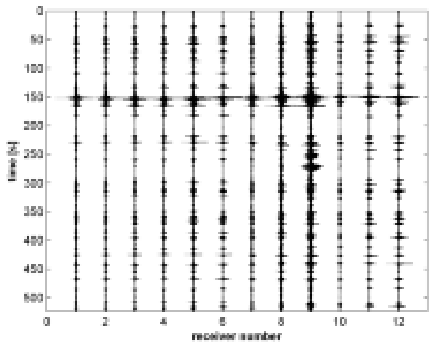
Problem with traditional beamforming:
what happens if one trace is contaminated by strong noise??

Solution:
Use spectral based beamforming (Barlett, Capon, MUSIC)
First Fourier transform time to frequency.
Then compute spatial covariance matrix.
Idea is to apply complex weights to sensors equivalent to spatial tapering, and to compute optimum weights
Weights are not computed explicitly, but contained in covariance matrix.

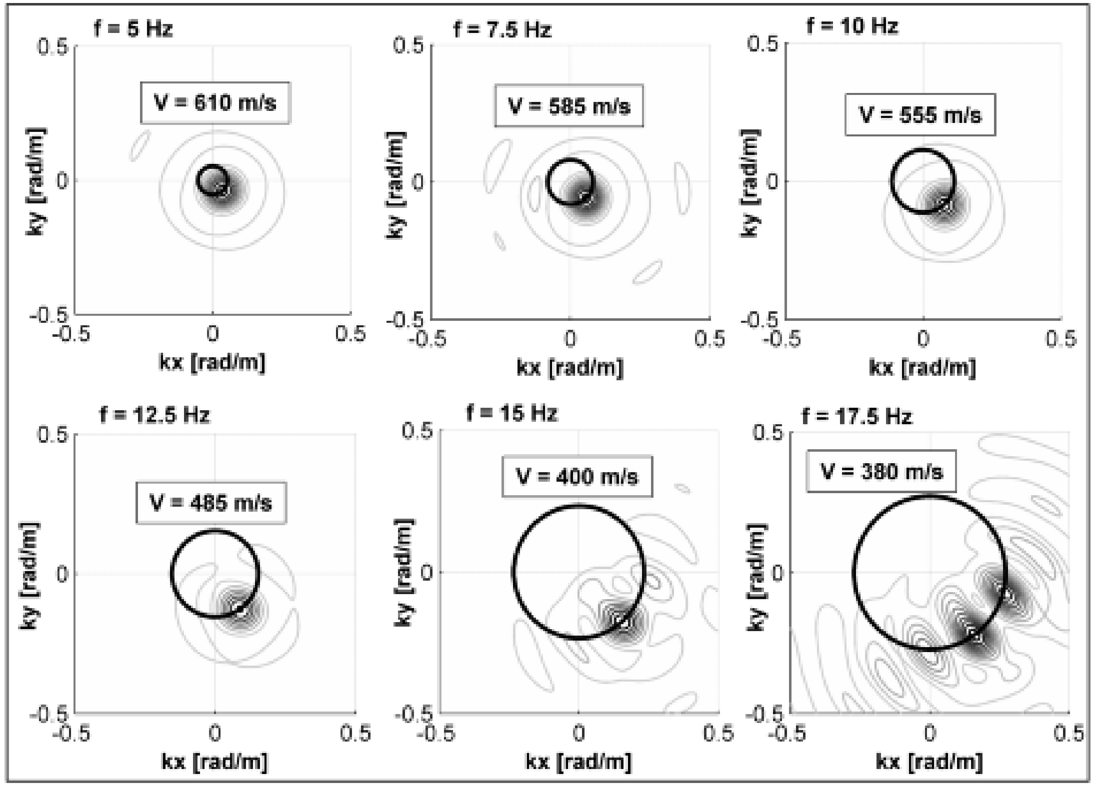
Other example of advanced beamforming are parametric beamformers which include penalty functions used to the spatial covariance matrix.

Product of beamforming is estimate for **f-k spectrum**



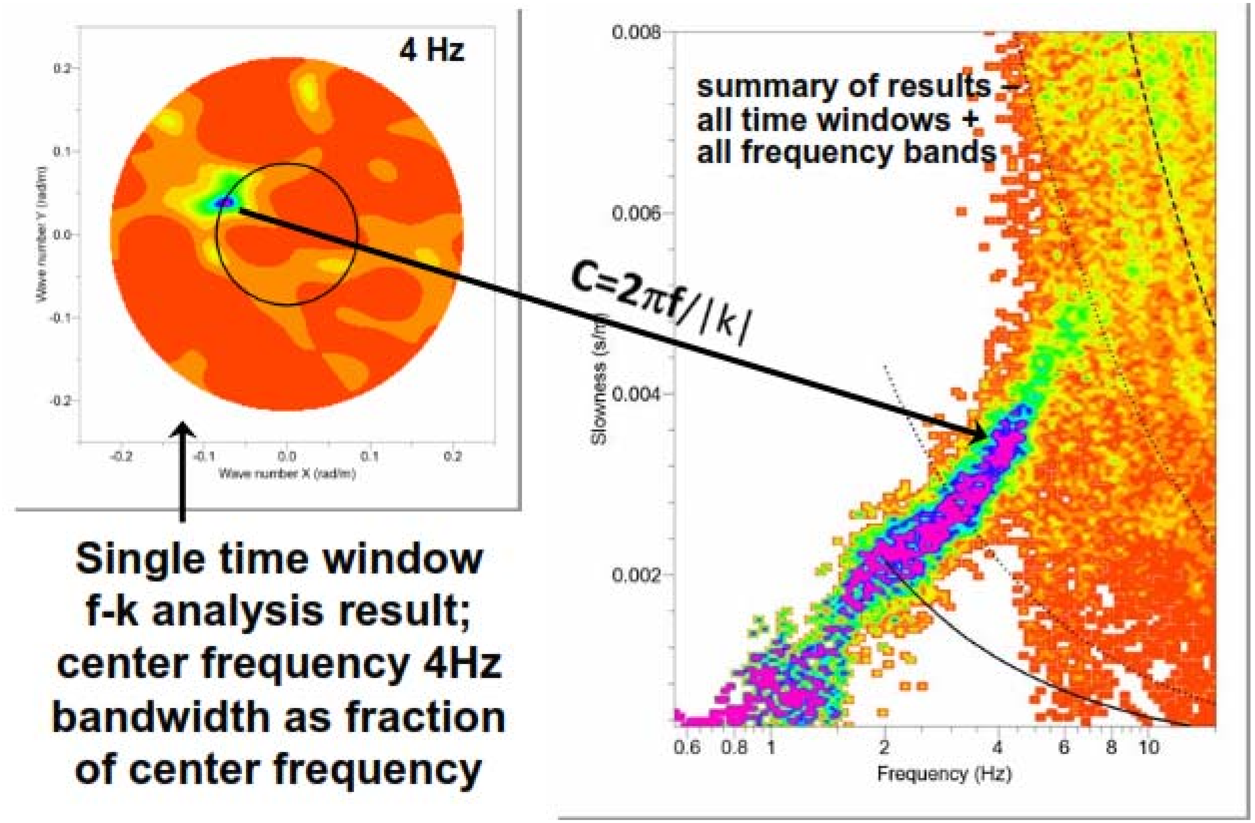


Frequency Domain Beamformer



(Foti et al., 2007)

From f-k analysis to dispersion curve



Source: Geopsy FK tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

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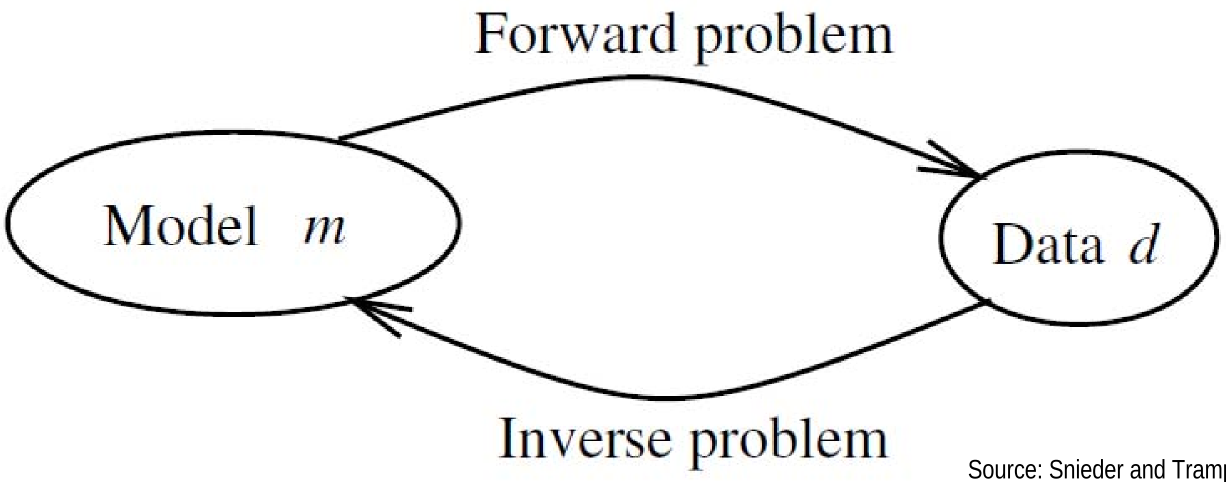
Forward and inverse modeling

Acknowledgements:

The content and index for a large part of the inversion notes was provided from Dr. Juan Luis Fernandez-Martinez, Professor in Applied Mathematics at the University of Oviedo, Spain. These notes were unpublished and were provided through personal communication.

Also these notes borrow heavily from Lai et al. (1998) and the Inversion Lecture notes from the Sessarray 2010 seminar held in Thessaloniki, Greece.

Traditional definition



Source: Snieder and Trampert, 1999

Mathematically, F[m]=d

F is the function of the direct problem (ex. geophysical problem), d is the data (given), and m is the searched model.

Forward

	Input		Operator		Output
	Model parameters	→	Numerical representation of system dynamics	→	Computed Responses

Inverse

	Input		Operator		Output
	Observation data	→	Mathematical Tools	→	Estimate of system parameters

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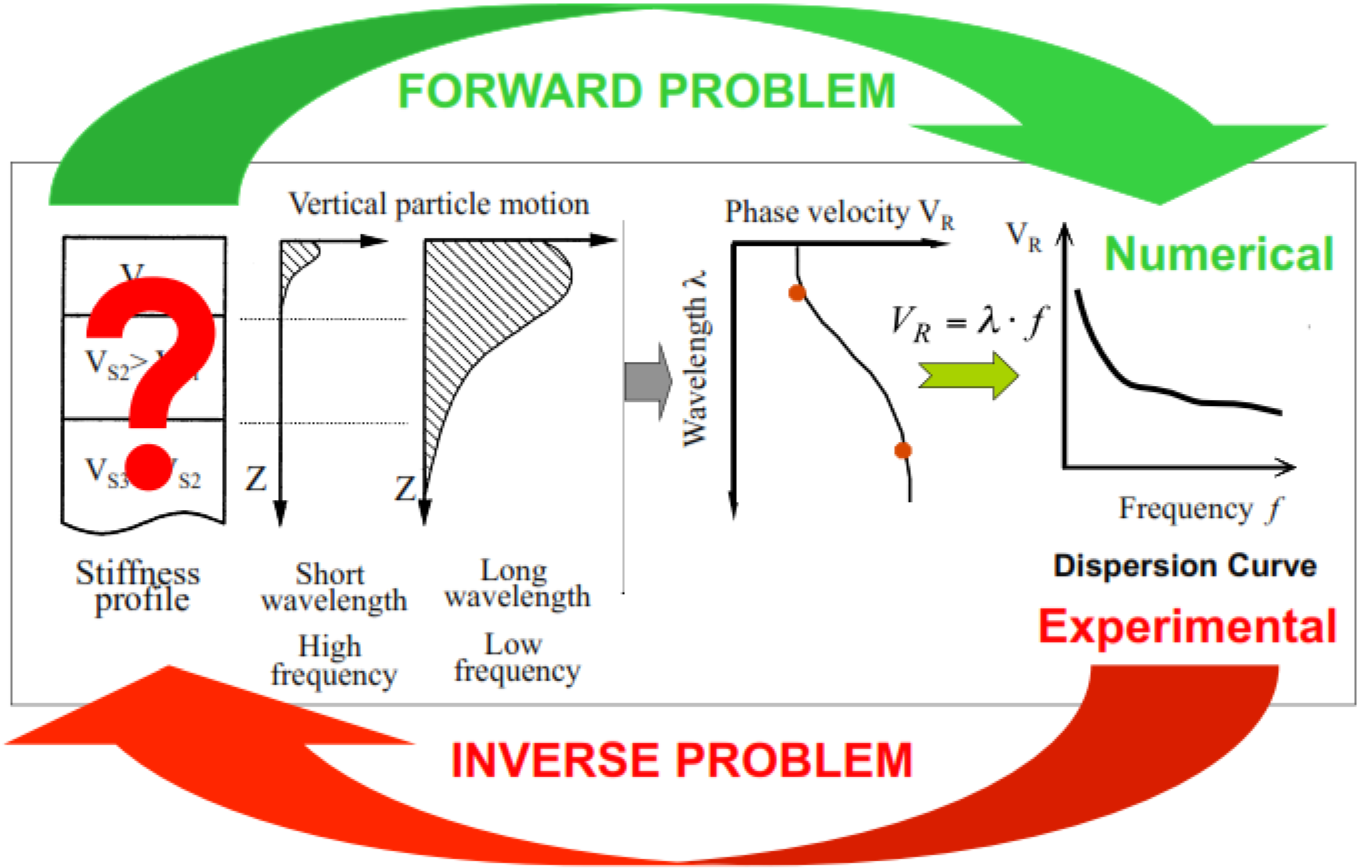
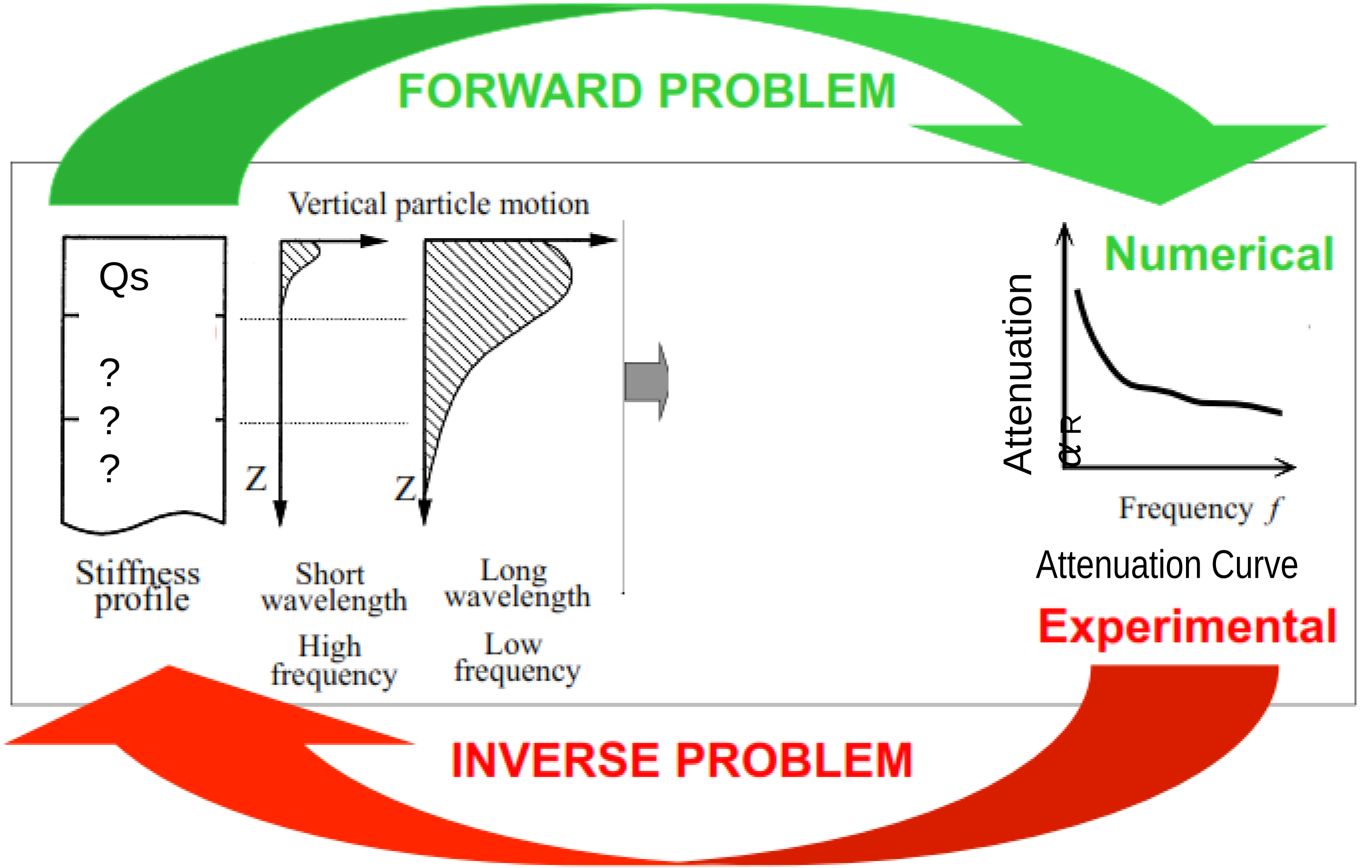
Determining Vs

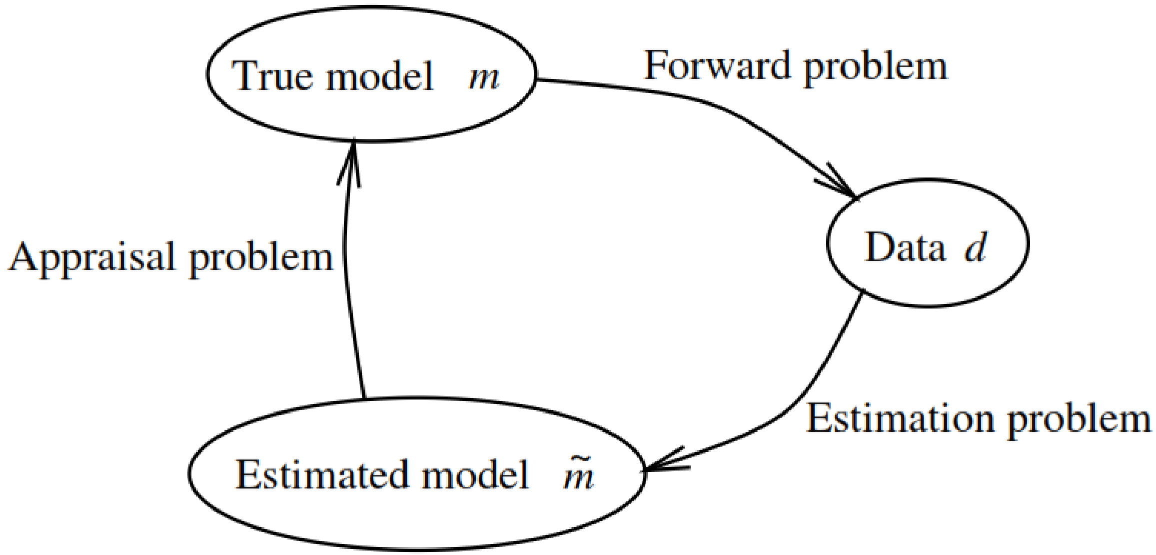


Source: Foti, 2012

Determining Qs



Source: adapted from Foti, 2012



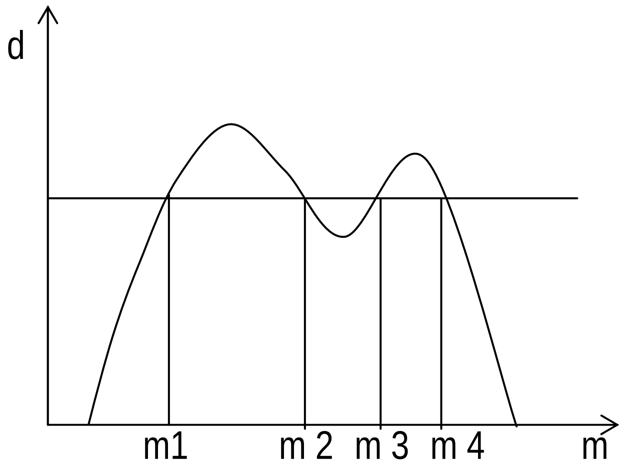
In general, inversion = estimation + appraisal

<u>But, why is $\tilde{m}$ different than m?</u>

1.non-uniqueness

2.real data is “contaminated’ with errors

Non-uniqueness



Which model?

Lack of stability

$$F[m]=d$$

$$F[m+\Delta m] = d + \Delta d$$

measurement errors

If $\Delta d \to 0$ then $\Delta m \not\to 0$

Small perturbations in the data may cause “grand” perturbations in the parameters!!

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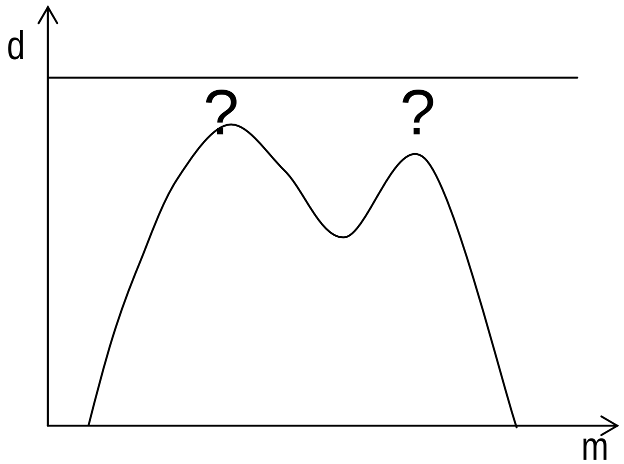
Non existence



$$\nexists m \in M : F[m] = d$$

Ill-posed problems

The mathematical formulation of inverse problems leads to models that typically are *ill-posed: According to Hadamard, a mathematical problem is called*
well-posed if:

1. for all admissible data, a solution exists,
2. for all admissible data, the solution is unique and
3. the solution depends continuously on the data.

If one of these properties is violated, the problem is called ill-posed.

Problems of non-uniqueness, non-existence or solution stability mean that the problem is "*ILL-POSED*"

Ill-posed problems

Neither existence nor uniqueness of a solution to an inverse problem are guaranteed.

In practical applications, one never has exact data, but only data perturbed by noise are available due to errors in the measurements or also due to inaccuracies the model itself.

Even if their deviation from the exact data is small, algorithms developed for well-posed problems then fail in case of a violation of the third Hadamard condition (*3. the solution depends continuously on the data*) if they do not address the instability, since data as well as round-off errors may then be amplified by an arbitrarily large factor.

In order to overcome these instabilities one has to use *regularization methods,* which in general terms replace an ill-posed problem by a family of neighboring well-posed problems.

Ill-posed problems

If the solution does not exist, or is not unique, or is not stable WE MUST CONSTRUCT IT!!

-Redefine solution with a priori information
-Stabilize the solution with regularization methods

Parametrization

Parametrization involved will influence the way the inverse problem is posed and its solution.

If fewer parameters than field data…….*OVERDETERMINED*
If parameters > field data………………..UNDERDETERMINED
If parameters ~ field data………………..EVENLY DETERMINED

For computational simplicity, we use finite and minimum set of parameters

Distribution of physical properties is uniquely determined if measurements span the observational bandwidth [0 ∞]

But, due to technical limitations, field observations are contained in a finite observation interval, so field data is incomplete

Linear vs. Nonlinear inversion problems

Linear problem:

Tx = y ex. Earth's gravitational field

where *T denotes a bounded linear operator acting between Hilbert spaces X* and *Y .*

Nonlinear problem: ex. Seismic tomography

F(x) = y

where *F acts between two Hilbert spaces X and Y . The basic assumptions for* a reasonable theory are that *F is continuous and is weakly sequentially closed,* i.e., for any sequence $x_n \to X, F(x_n) \to Y$

imply that $x \in D$ and $F(x) = y$

As opposed to the linear case, F is usually not explicitly given, but represents the operator describing the direct (also sometimes called "forward") problem.

Source: Engl and Kugler,

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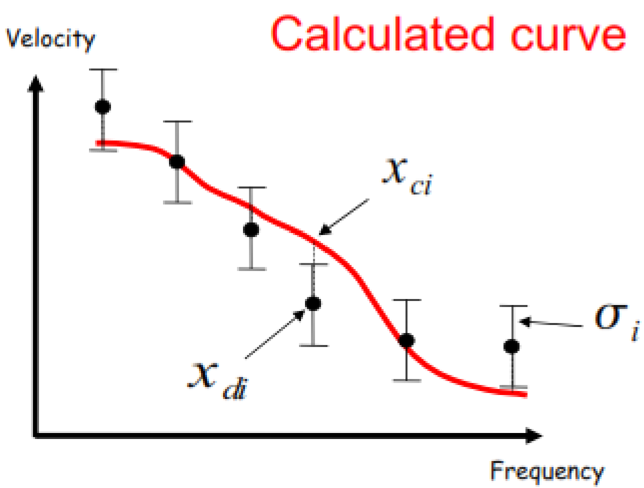
Misfit Definition

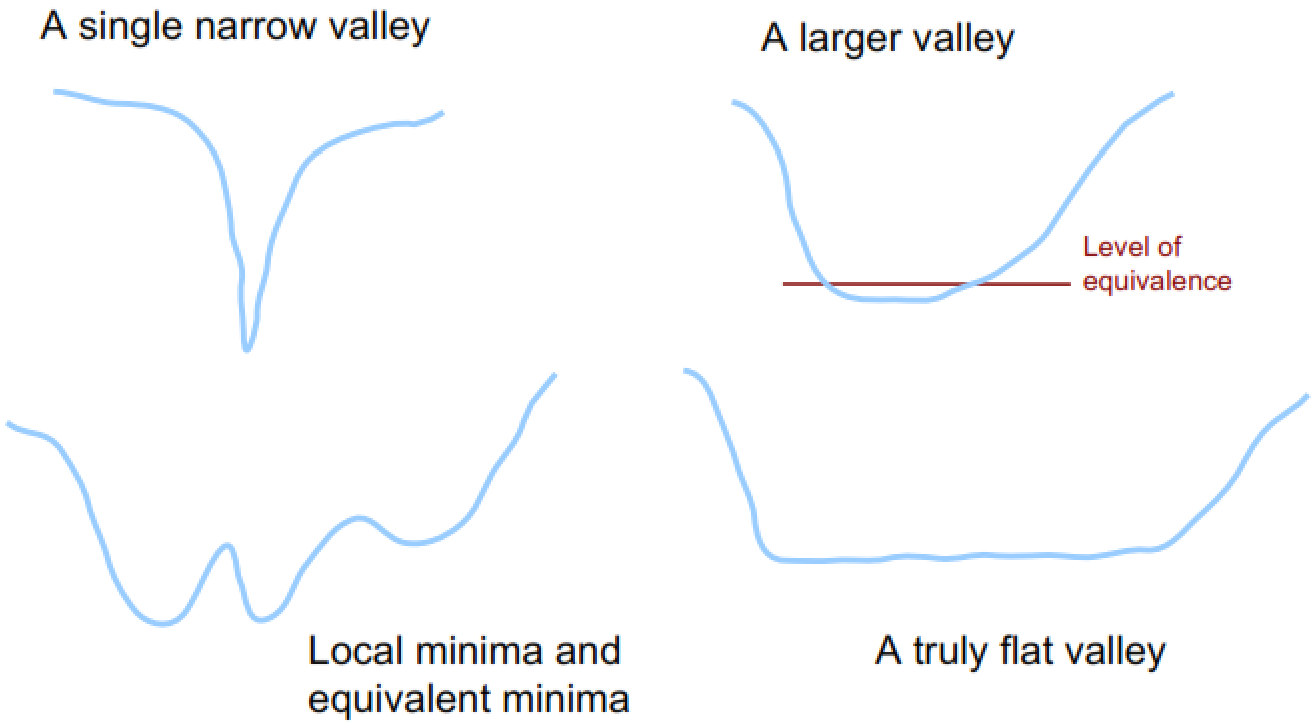


$$\text{Misfit} = \sqrt{\sum_{i=1}^{n_F} \frac{(x_{di} - x_{ci})^2}{\sigma_i^2 n_F}}$$

n_F number of frequency samples

Source: Geopsy Inversion tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

Possible shapes for misfit function



Source: Geopsy Inversion tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

Linear vs. Nonlinear inversion problems

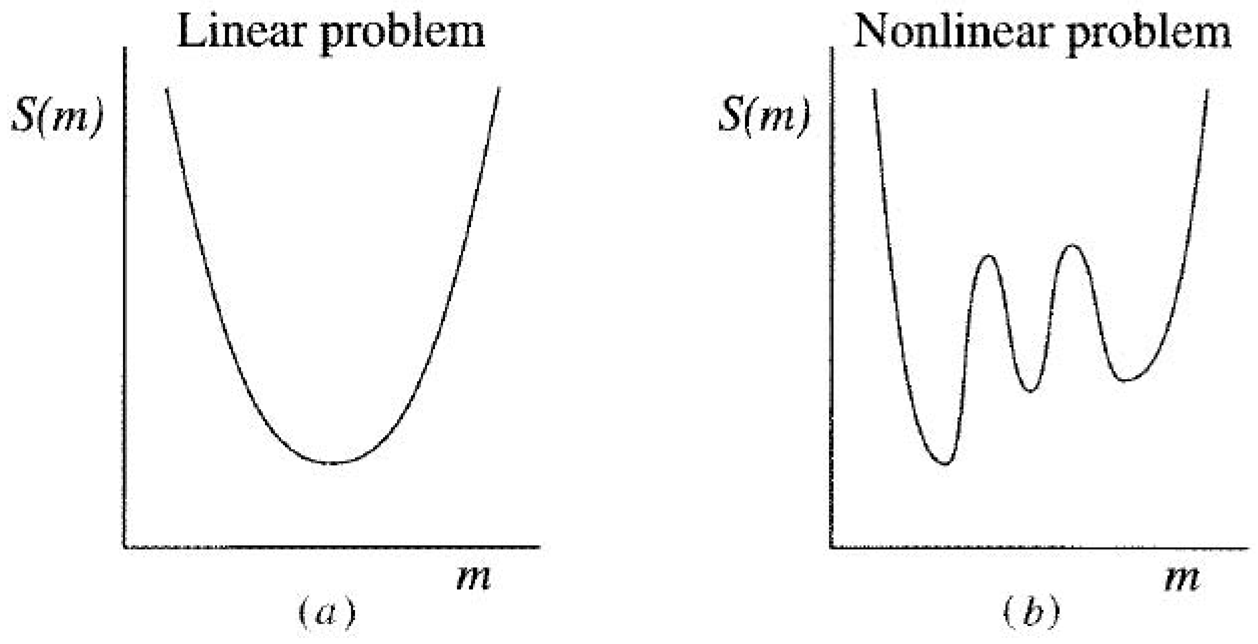


Figure 3. (*a*) The least-squares misfit function for a linear problem. (*b*) The conventional view of the misfit function for a nonlinear inverse problem.

Source: Snieder, 1998

Discrete vs. continuous problems

Discrete models (finite number of degrees of freedom) vs. Continuous models (infinitely many degrees of freedom)

In many practical inverse problems, one aims to retrieve a model that has infinitely many degrees of freedom from a finite amount of data. It follows from a simple variable count that this cannot be done in a unique way.

For practical purposes, we model the problem as having a finite number of parameters (in geophysics, this is usually NOT true!!)

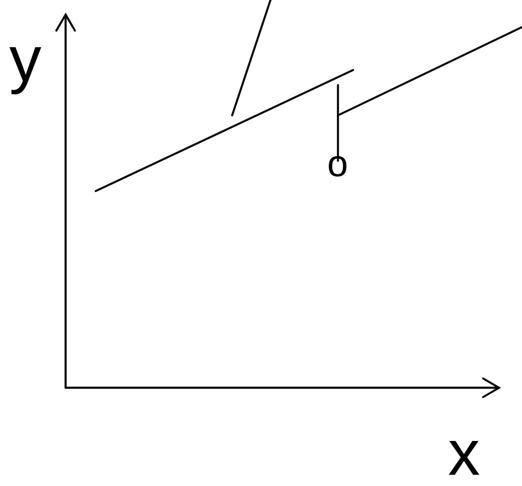
Simple Linear Regression

n data pairs $\{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}$

Regression line: $y = a + bx$

$$y_i = a + bx_i + e_i$$

vertical distance between ith data point and regression line



Least Squares Method

Minimize error such that we minimize:

$$S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - abx_i)^2$$

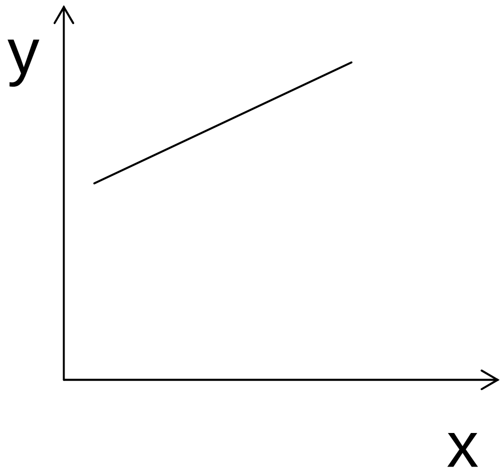
Least Squares Method

Minimize: $S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - abx_i)^2$

$$\left.\begin{aligned} \frac{\partial S}{\partial a} &= 0 \\ \frac{\partial S}{\partial b} &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} &\sum_{i=1}^{n} 2(y_i - a - bx_i)(-1) = 0 \\ &\sum_{i=1}^{n} 2(y_i - a - bx_i)(-x_i) = 0 \end{aligned}\right\} \quad \begin{aligned} &\sum_{i=1}^{n} a + \sum_{i=1}^{n} bx_i = \sum_{i=1}^{n} y_i \\ &\sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} bx_i^2 = \sum_{i=1}^{n} x_i y_i \end{aligned}$$

Slope of fitted line: $b = \dfrac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$

$$a = \frac{\sum y - b\sum x}{n}$$



If more than 2 model parameters: *MULTIPLE REGRESSION ANALYSIS*

Matrix Formulation

$d = Gm, m = G^{-1}d$ *For perfect data…*

But, usually we have…

$d = Gm + e_i$

$$Minimize: \sum_{i=1}^{n}(d_i - \sum_{j=1}^{p} G_{ij}m_j)^2 = (d - GM)^{\mathrm{T}}(d - GM) = d^{\mathrm{T}}d - d^{\mathrm{T}}Gm - m^{\mathrm{T}}G^{\mathrm{T}}d + m^{\mathrm{T}}G^{\mathrm{T}}Gm)$$

$$= -d^{\mathrm{T}}G - G^{\mathrm{T}}d + G^{\mathrm{T}}Gm + m^{\mathrm{T}}G^{\mathrm{T}}G = 0$$

$\boxed{2G^{\mathrm{T}}Gm = 2G^{\mathrm{T}}d}$ *Normal equations*

$m = [G^{\mathrm{T}}G]^{-1}G^{\mathrm{T}}d$ *For imperfect data!*

Finding inverse

$$m = [G^{\mathrm{T}}G]^{-1}G^{\mathrm{T}}d$$

How do we solve for the inverse?

Examples:

- Cramer's Rule
- Gauss Elimination Method
- Gauss-Jordan Elimination Method
- Lu or Triangular Decomposition Method
- Singular Value Decomposition

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Constrained Linear Least Squares Inversion

This is inversion with **a priori information**.

First, why do we add a priori information?

Because it helps single out a unique solution out of the infinitely many plausible solutions to the problem if there are observational errors and uncertainties.

Constrained Linear Least Squares Inversion

$Dm = h$ We bias m toward h.

diagonal

$$\phi = (d - Gm)^{\mathrm{T}}(d - Gm) + \beta^2 (Dm - h)^{\mathrm{T}}(Dm - h)$$

$$Need\ \frac{\partial \phi}{\partial m_j} = 0$$

$$So, 2G^{\mathrm{T}}Gm - 2G^{\mathrm{T}}d + 2\beta^2 D^{\mathrm{T}}Dm - 2\beta^2 D^{\mathrm{T}}h = 0$$

$$(G^{\mathrm{T}}G + \beta^2 D^{\mathrm{T}}D)m = G^{\mathrm{T}}d + \beta^2 D^{\mathrm{T}}h$$ *"Normal Equations"*

$$D = I$$

$$(G^{\mathrm{T}}G + \beta^2 I)m = G^{\mathrm{T}}d + \beta^2 h$$

$$m_{CONSTRAINED} = (G^{\mathrm{T}}G + \beta^2 I)^{-1}(G^{\mathrm{T}}d + \beta^2 h)$$

β is an auxiliary parameter, which chosen by trial and error. It is called the Lagrange multiplier.

Inversion with Smoothness Measures

Question: Can a solution be stabilized if no prior estimate is available? Yes, inversion with smoothness measures.

This is another remedy for non-uniqueness… when in doubt, SMOOTH!!

Minimize (m_1-m_2), (m_2-m_3), (m_p-m_{p-1})

$$\begin{vmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ldots\ldots & & \\ & & & 1 & -1 \end{vmatrix} \begin{vmatrix} m1 \\ m2 \\ m3 \\ \ldots \\ mp \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{vmatrix}$$

Also called the Marquardt damped solution.

Error Analysis – Observation Error

How do the experimental errors translate into errors in model estimates?

If we know the observational errors, we can incorporate them in the problem formulation
Hence, we will be getting a weighted solution

Usual assumption: Standard data errors σ i are of a Gaussian distrubution with zero mean.

We essentially weigh each datum by its associated observational error: $\frac{di}{\sigma i}$

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Uncertainty in forward model

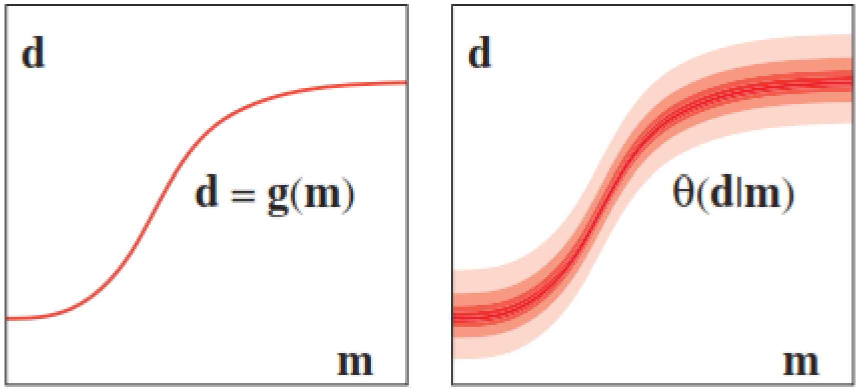


Figure 1.4. a) *If uncertainties in the forward modelization can be neglected, a functional relationship* $\mathbf{d} = \mathbf{g}(\mathbf{m})$ *gives, for each model* $\mathbf{m}$, *the predicted (or calculated) data values* $\mathbf{d}$. b) *If forward-modeling uncertainties cannot be neglected, they can be described, giving, for each value of* $\mathbf{m}$, *a probability density for* $\mathbf{d}$ *that we may denote* $\theta(\mathbf{d}|\mathbf{m})$. *Roughly speaking, this corresponds to putting vertical uncertainty bars on the theoretical relation* $\mathbf{d} = \mathbf{g}(\mathbf{m})$.

Source: Tarantola, 2005

Assessing the quality of the solution

Goodness-of-fit: the commonly used model acceptance criterion.
Assumptions: di is normally distributed with known σ i

Statistical parameter q

$$q = \sum_{i=1}^{n} \frac{(d_{iobs} - G_{ij} m_j)^2}{\sigma_i^{\,2}}, \; j = 1, p$$

In practive, a model with $n - p \le q \le n + \sqrt{2n}$ is acceptable,

If q>> n the model underfits the data
If q<<n the model overfits the data (contains computed artifacts).

Root-Mean-Square

$$RMS = \frac{1}{n} \sum_{i=1}^{n} \frac{(d_{iobs} - G_{ij} m_j)^2}{\sigma^2}$$

Assessing the quality of the solution (cont.)

-Parameter Resolution Matrix R

1.Unconstrained solution:

$$m = \{(G^{\mathrm{T}}G)^{-1}G^{\mathrm{T}}\}d$$

$$R = HG = (G^{\mathrm{T}}G)^{-1}G^{\mathrm{T}}G = I$$

R has dimension p x r where p are the parameters and r the number of non-zero eigenvalues

If R = I then each parameter is uniquely defined, and the resolution is perfect!

2.Marquardt damped solution (smoothing)

$$R = (G^{\mathrm{T}}G + \beta I)^{-1}G^{\mathrm{T}}G = I + \frac{G^{\mathrm{T}}G}{\beta I}$$

Solution is not perfect (R is not I)

3.Inversion with a priori data (constrained)

$$R = (G^{\mathrm{T}}G + \beta^2 D^{\mathrm{T}}D)^{-1}G^{\mathrm{T}}G + (G^{\mathrm{T}}G\beta^2 D^{\mathrm{T}}D)^{-1}\beta D^{\mathrm{T}}\beta D$$

$$= (G^{\mathrm{T}}G + \beta^2 D^{\mathrm{T}}D)^{-1}(G^{\mathrm{T}}G + \beta^2 D^{\mathrm{T}}D) = I$$

Constrained solution with a priori parameters has perfect resolution

Assessing the quality of the solution (cont.)

-Parameter Resolution Matrix R

Comment: Resolution matrix is only an experimental design guide. A perfect resolution does not imply an accurate or reliable model! In general, it is overstated in geophysical inversion literature.

If $R=I$ this means the solution may be found

If $R \neq I$ then true solution may not be found

Inversion Techniques

Forward problem:
Analytic or numerical processing
Only one solution

Inverse problem:
Trial and error to adjust parameters of the model
Simplex downhill method
Brute force uniform search (gridding)
Least square methods (based on derivatives)
Brute force Monte Carlo sampling
Simulated Annealing
Genetic Algorithm
Neighborhood Algorithm
Generally not only one solution

Source: Geopsy Inversion tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

Inversion Techniques

How do we search the model space: Local vs. global search algorithms

Local search algorithms can be used on problems that can be formulated as finding a solution maximizing a criterion among a number of candidate solutions. Local search algorithms move from solution to solution in the space of candidate solutions (the *search space*) by applying local changes, until a solution deemed optimal is found or a time bound is elapsed.

Local algorithms result in ONE solution.

Inversion Techniques

How do we search the model space: Local vs. global search algorithms

<u>Global search</u> algorithms sample a large portion of the model space, and detect several local maxima.

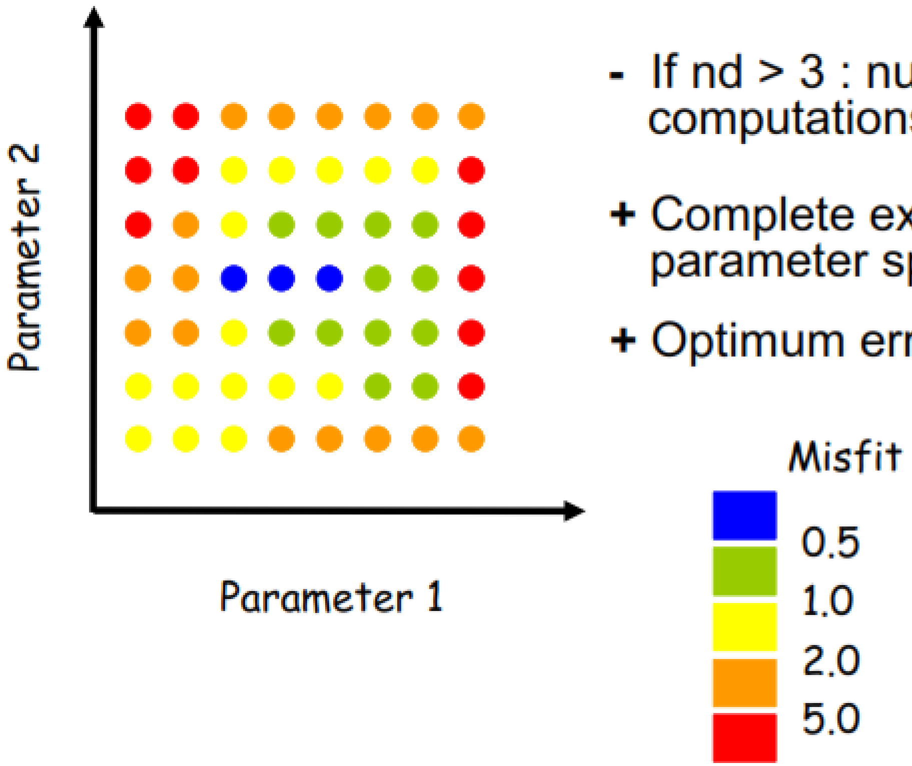
The strategy adopted in a global search method varies according to different philosophies, some of which include genetic algorithms, fractal inversion, neural network inversion, enumerative methods, and Monte Carlo simulation.

Global-search procedures are in general more expensive than local-search procedures, both in terms of time and computer resources.

However, they are more robust and reliable compared to the latter.

Global search algorithms provide several solutions.

A. Uniform search (gridding)



- If nd > 3 : number of forward computations are prohibitive

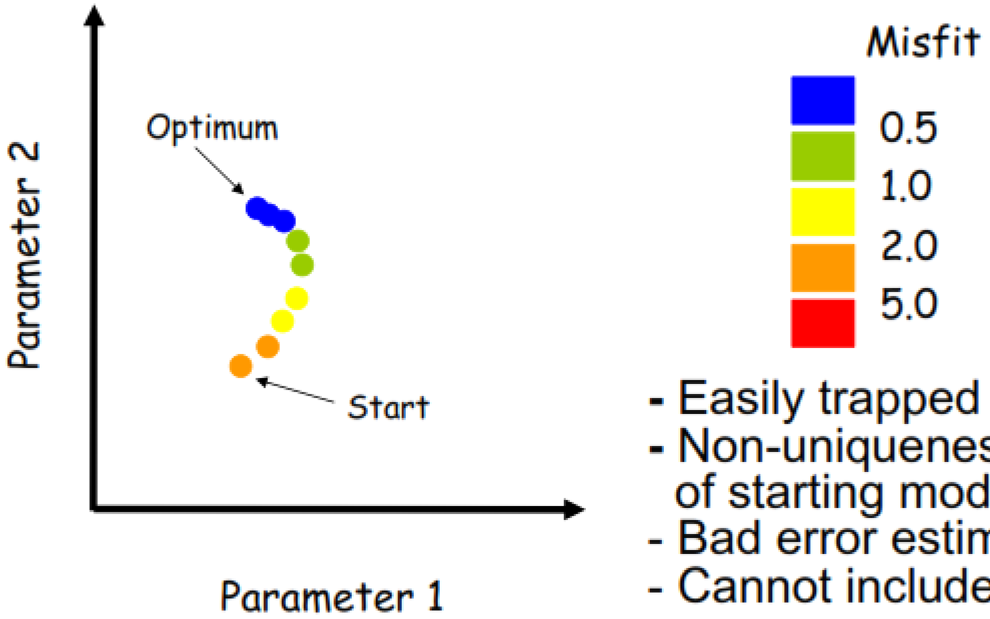
+ Complete exploration of the parameter space

+ Optimum error estimates

Source: Geopsy Inversion tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

B. Iterative methods (optimization)
Least Square, Simplex, Gradient Methods...

Least Square, Simplex, Gradient methods, ...

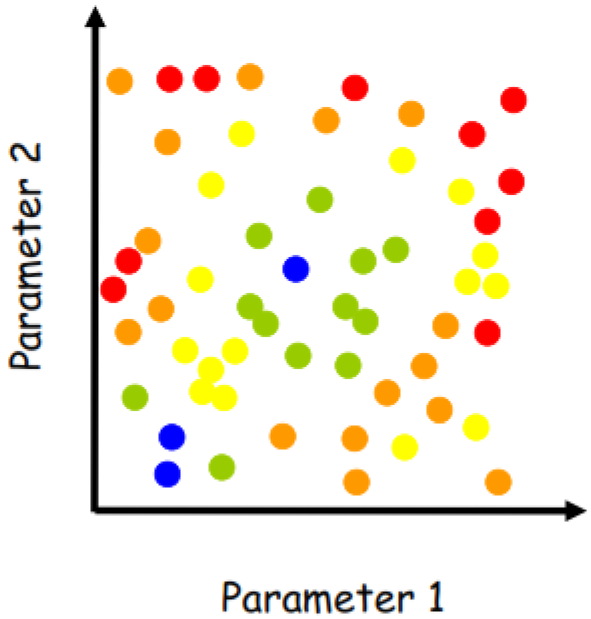


- Easily trapped in local minima
- Non-uniqueness <=> choice of starting model
- Bad error estimates
- Cannot include prior information

+ High dimensionality
+ Few forward computations

Source: Geopsy Inversion tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

C. Random Search (Monte Carlo)

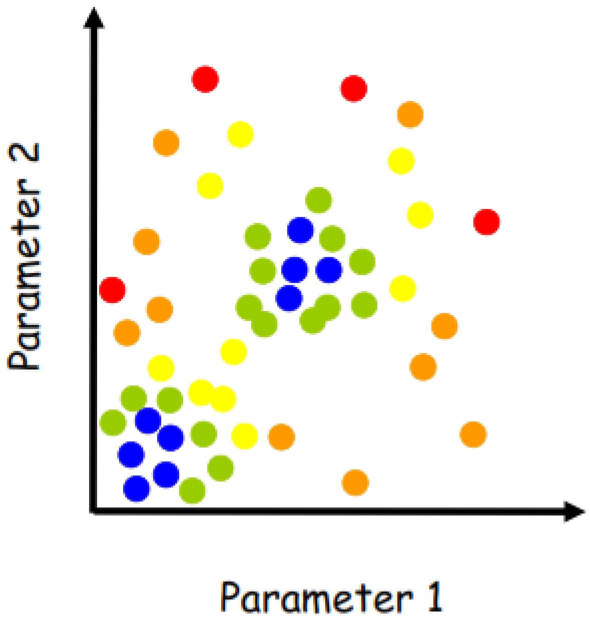


- Requires lot of forward computations
+ Not too bad exploration of the parameter space
+ Good error estimates

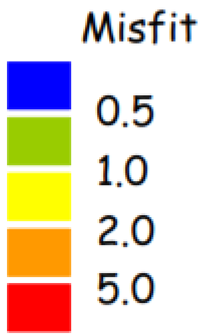
Source: Geopsy Inversion tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

D. Oriented random search

(Simulated Annealing, Genetic Algorithms, and Neighbourhood Algorithm)



- \+ Requires less of forward computations than MC
- \- Max nd ~ 25-50
- \+ Not too bad exploration of the parameter space
- \+ Good error estimates



Source: Geopsy Inversion tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

Solution of the Rayleigh Inverse Problem

non-linear linear

FORWARD →

medium parameters
$\{\rho(z), V^*p(z), V^*s(z)\}$

dispersion and attenuation
curves $V_R(\omega)$ and $\alpha_R(\omega)$

← INVERSE

Well-posedness must observe three conditions (Tikhonov and Arsenin, 1977; Engl, 1993):

a. For all admissible data, a solution exists.

b. For all admissible data, the solution is unique.

c. The solution depends continuously on the data.

(b and c:) usually violated

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Solution of the Rayleigh Inverse Problem

Nonuniqueness either because lack of sufficient information to constrain solution OR information available may not be independent

Two remedies are a priori information, and smoothness and regularity
Simple for ideal error-free observation. More complex for data containing bias and random errors.

Source Lai et al., 1998

Solution of the Rayleigh Inverse Problem

Violation of continuity: solution is very sensitive to perturbations in the data

A stability analysis by means of singular-value expansion method (Menke, 1989; Engl, 1993) shows the smallest singular value which controls amplification of the measurement errors. The rate of decay of the singular values arranged in order of decreasing magnitude is used as a measure to quantify the degree of instability of inverse problem.

For very unstable problems there are mathematical techniques, called regularization methods, that approximate the ill-posed problem with a parameter-dependent family of neighboring well-posed problems (Tikhonov and Arsenin, 1977; Engl, 1993).

Because some of these regularization methods can also be applied to non-linear inverse problems given that they admit a varriational formulation where the objective is the minimization of appropriate functionals.

Source Lai et al., 1998

Coupled vs. Uncoupled Analysis

Coupled analysis
(*dispersion* and the attenuation curves are inverted simultaneously)
Uncoupled analysis
(curves are inverted separately)

Coupled analysis is more stable than uncoupled analysis!

Also uncoupled analysis is restricted by the assumption of weak dissipation, whereas coupled is not!

Source Lai et al., 1998

Coupled vs. Uncoupled Analysis

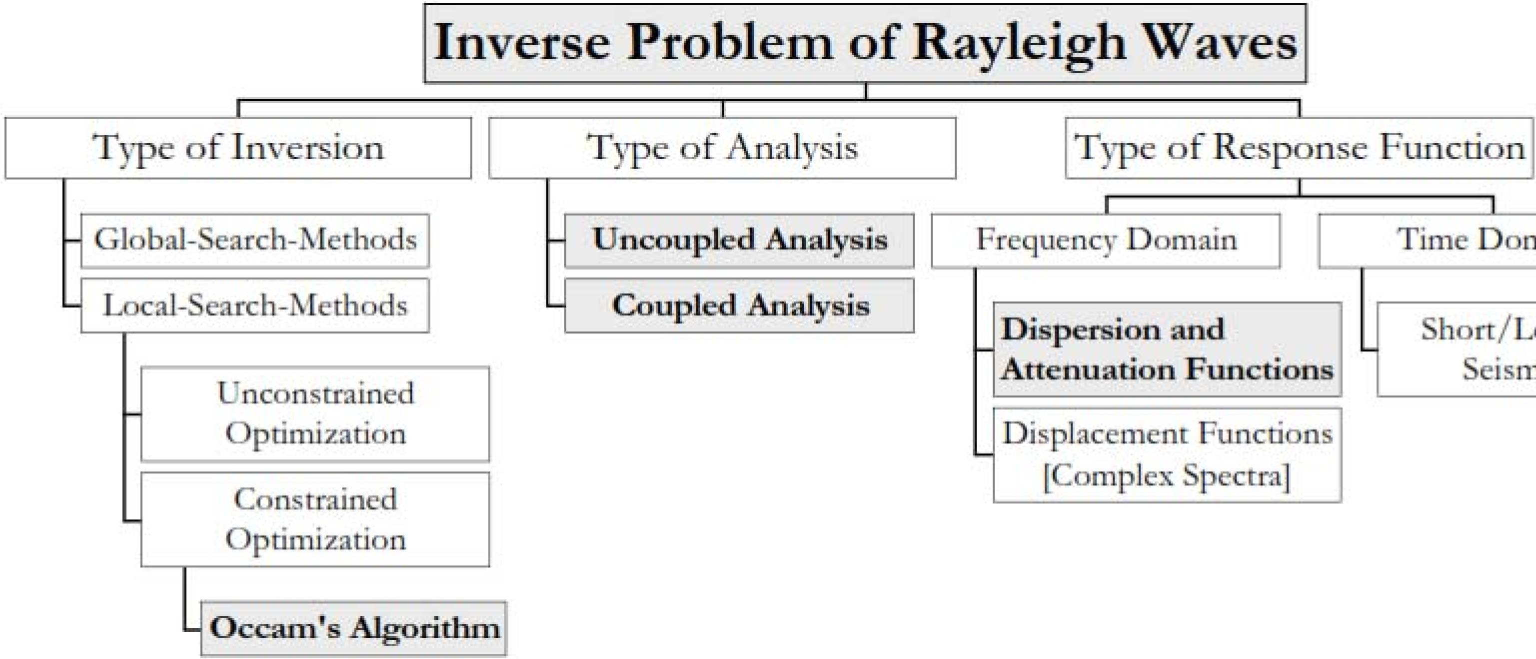
Coupled analysis is more stable than uncoupled analysis! Why?

-In uncoupled inversion, the solution is not independent because errors from dispersion inversion carry over to attenuation inversion.

-In coupled no negative coupling effect because both sets of experimental data are inverted simultaneously in a single, complex-valued, inversion.

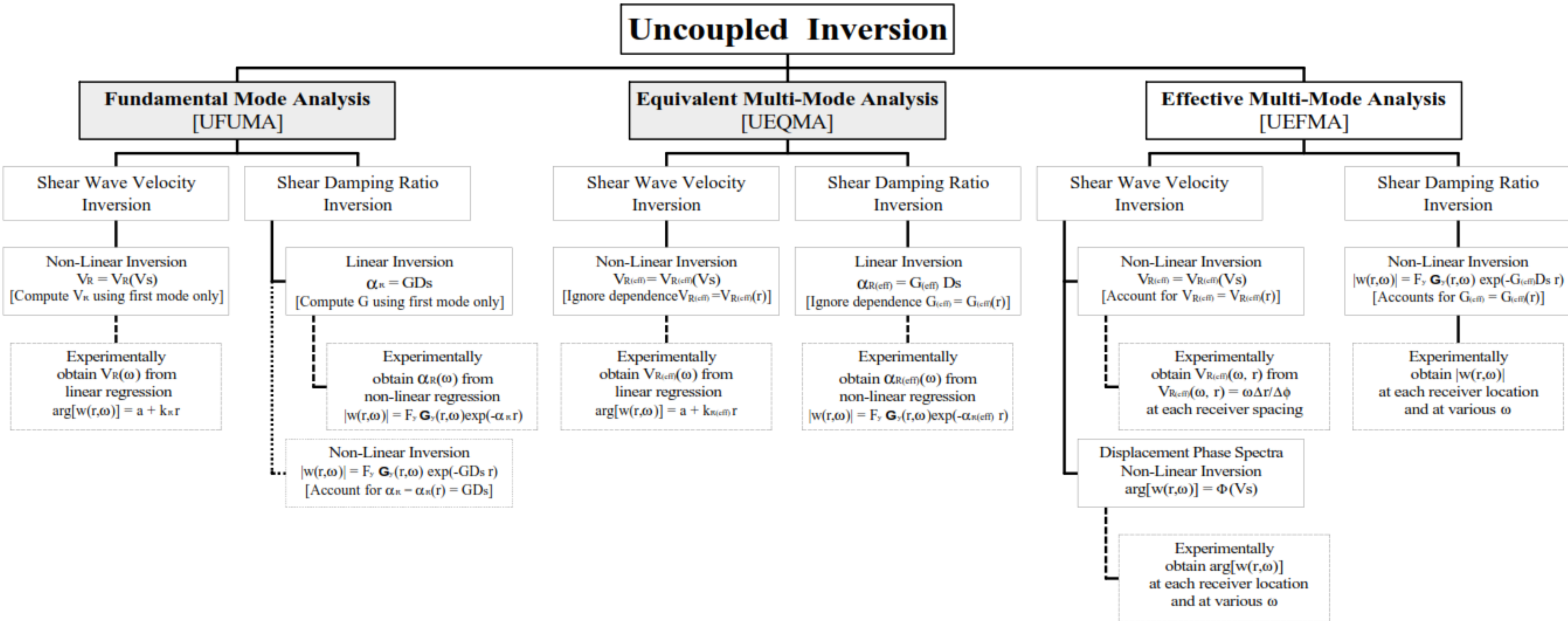
-extra *internal constraint that is embedded in the* formalism of the complex inversion. The intimate connection between the real and the imaginary parts of the variables involved in the simultaneous inversion adds a built-in constraint that makes the coupled inversion a better-posed problem.

Source Lai et al., 1998

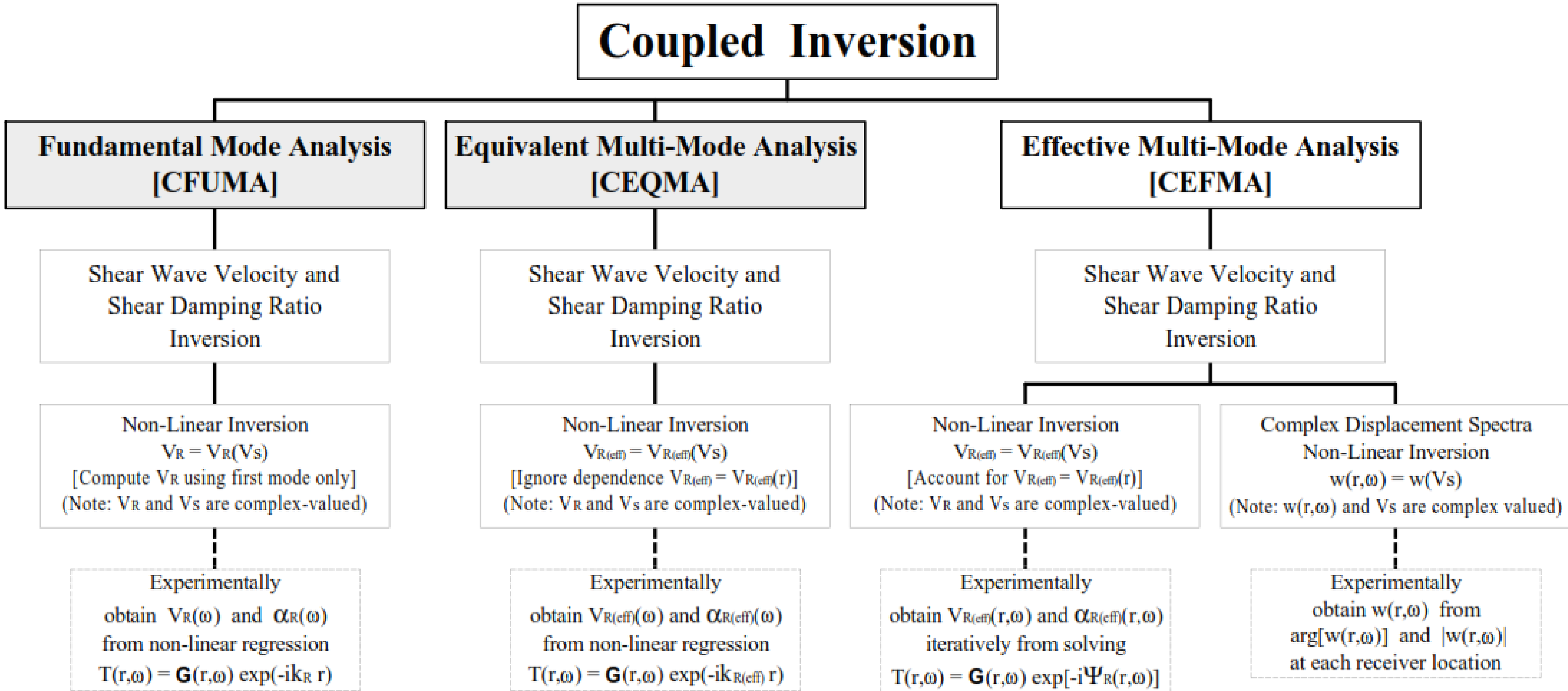


Algorithms for the Solution of the Rayleigh Inverse Problem

Source Lai et al., 1998



Source Lai et al., 1998



Source Lai et al., 1998

Two selected algorithms:
Occam's and Neighborhood Algorithm

Occam's Algorithm

-References: Constable et al., 1987; Parker, 1994

-Programmed into SWAN (http://www.geoastier.it/)

-Local search algorithm

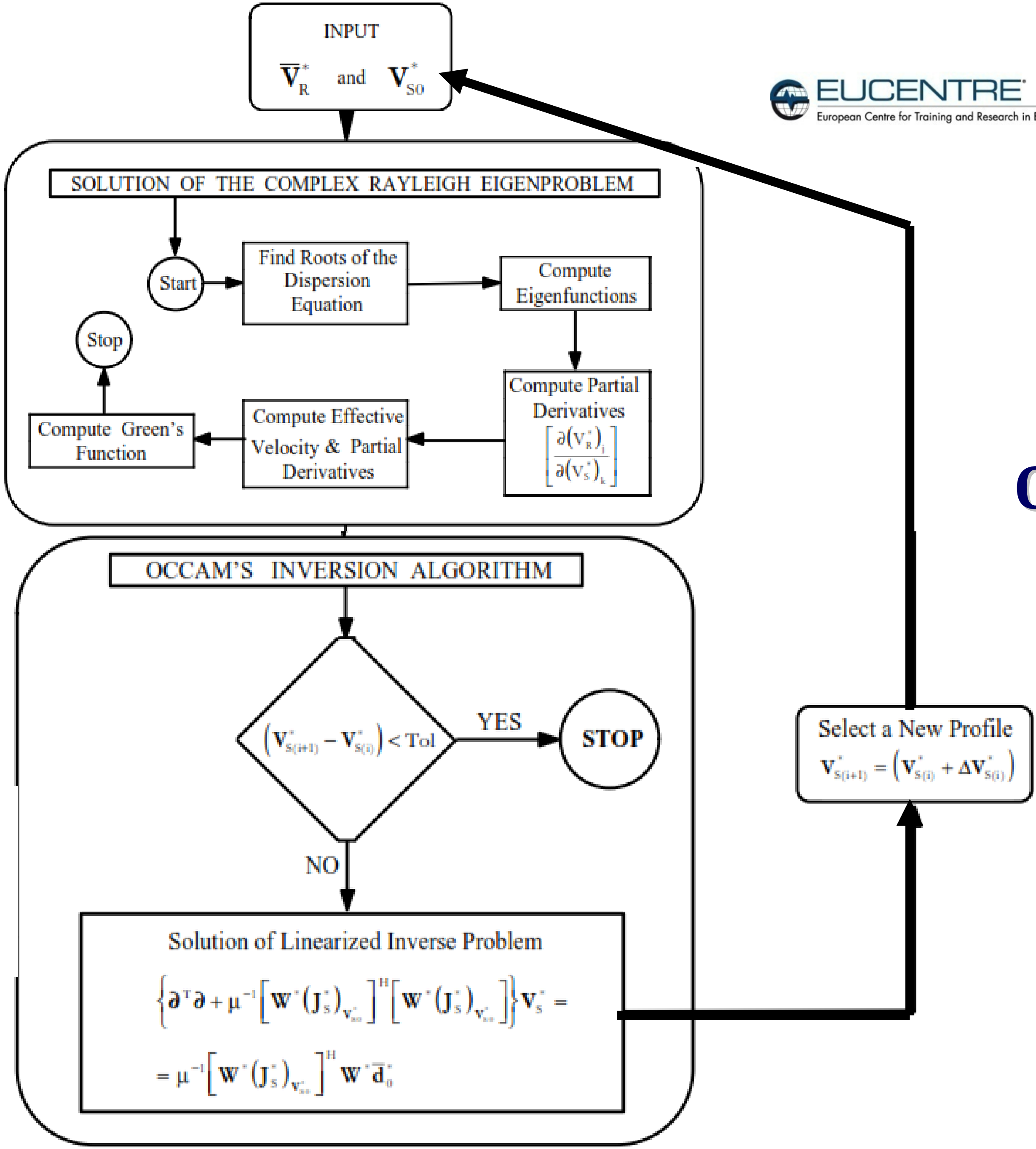
-Summary: given a set of experimental data and their associated uncertainties, find the smoothest profile of model parameters subject to the constraint of a specified misfit between observed and predicted data.

-The development of this class of algorithms was motivated by the following observations:

..The solution of a parameter identification problem relies on the ability to synthetically reproduce a set of experimental data by means of a mathematical model describing a particular physical problem.

.. In discrete inverse theory, the mathematical model is assumed to depend on a certain number of unknown model parameters, whose determination is the objective of the inversion algorithm.

Occam's Algorithm



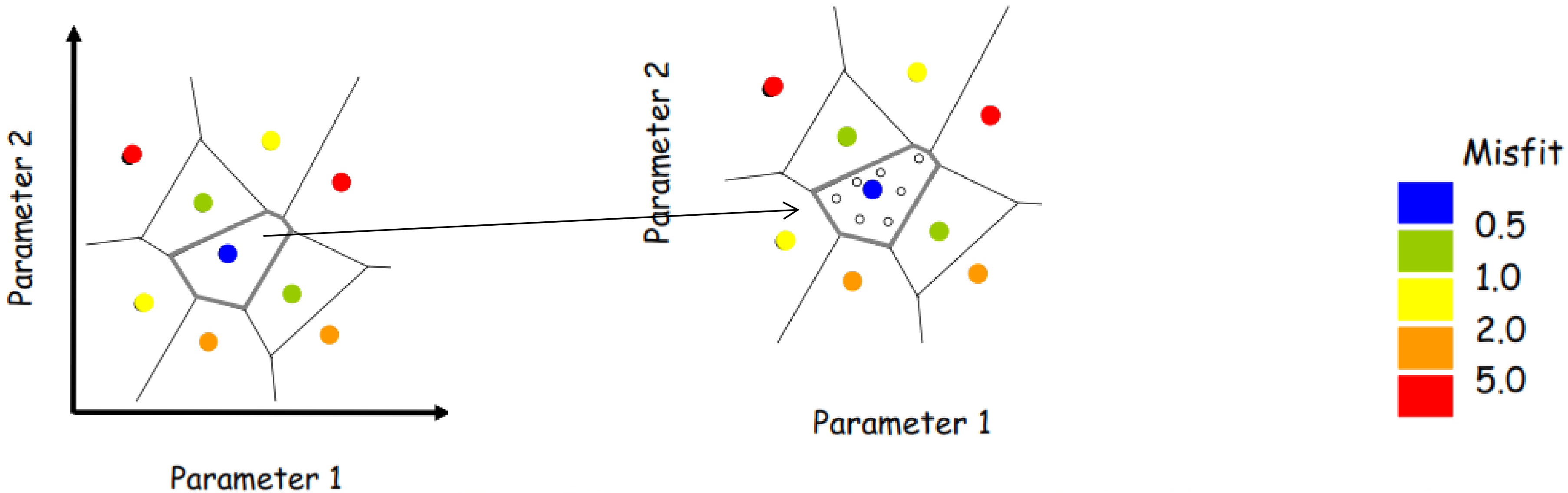
Neighborhood Algorithm

-References: Sambridge, 1999; Wathelet, 2008

-Programmed into geopsy (http://www.geopsy.org/

-Global search algorithm

-Search globally then refine search.



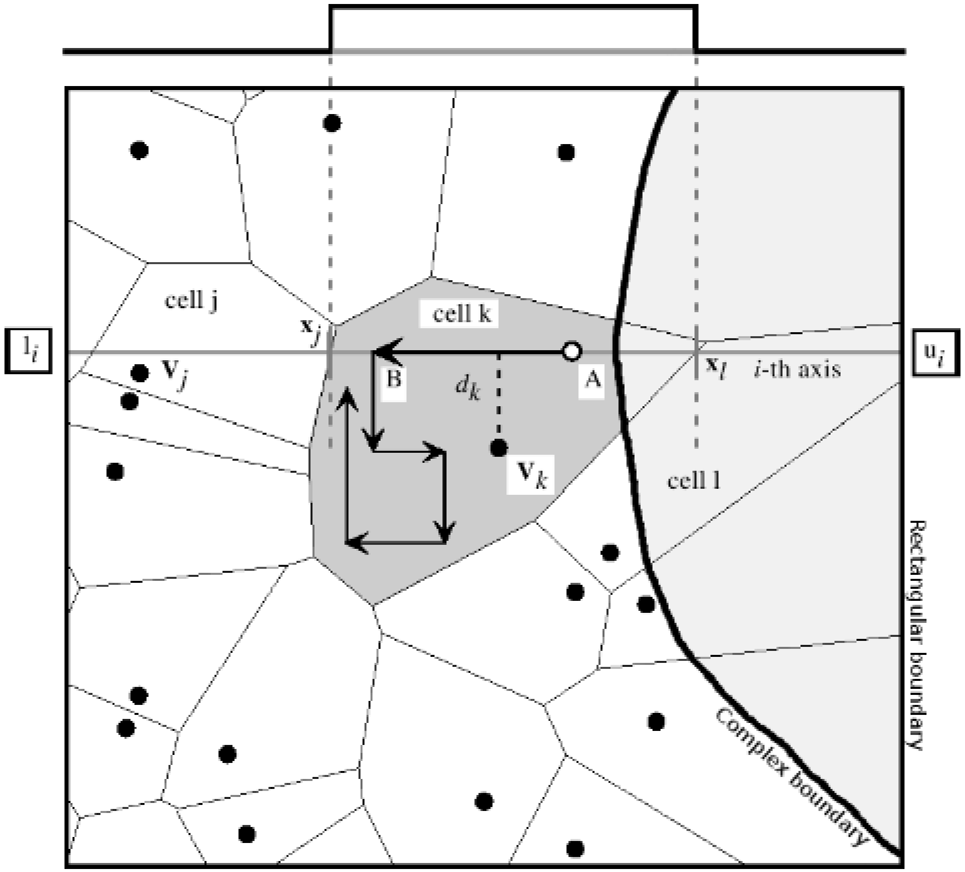
Source: Geopsy Inversion tutorial seminar, Thessaloniki, 2010 organized by LGIT, ITSAK, Universitat Postdam, IRD

Neighborhood Algorithm

A modified Neighborhood kernel:
irregular parameter boundaries

From model A
add “valid” random
perturbations so
that model B stays
in cell k

Loop over all axes



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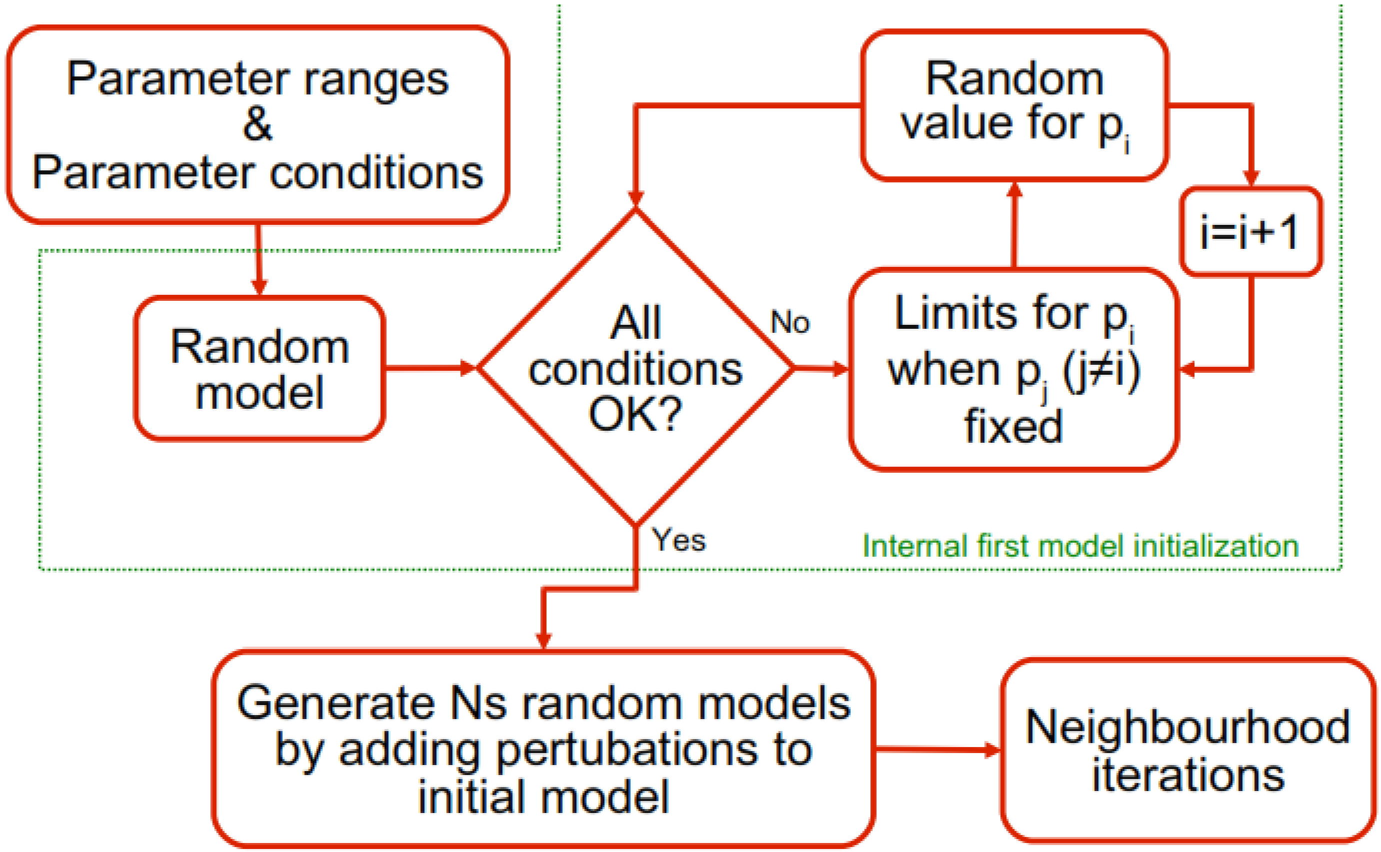
Conditions in Neighborhood Algorithm



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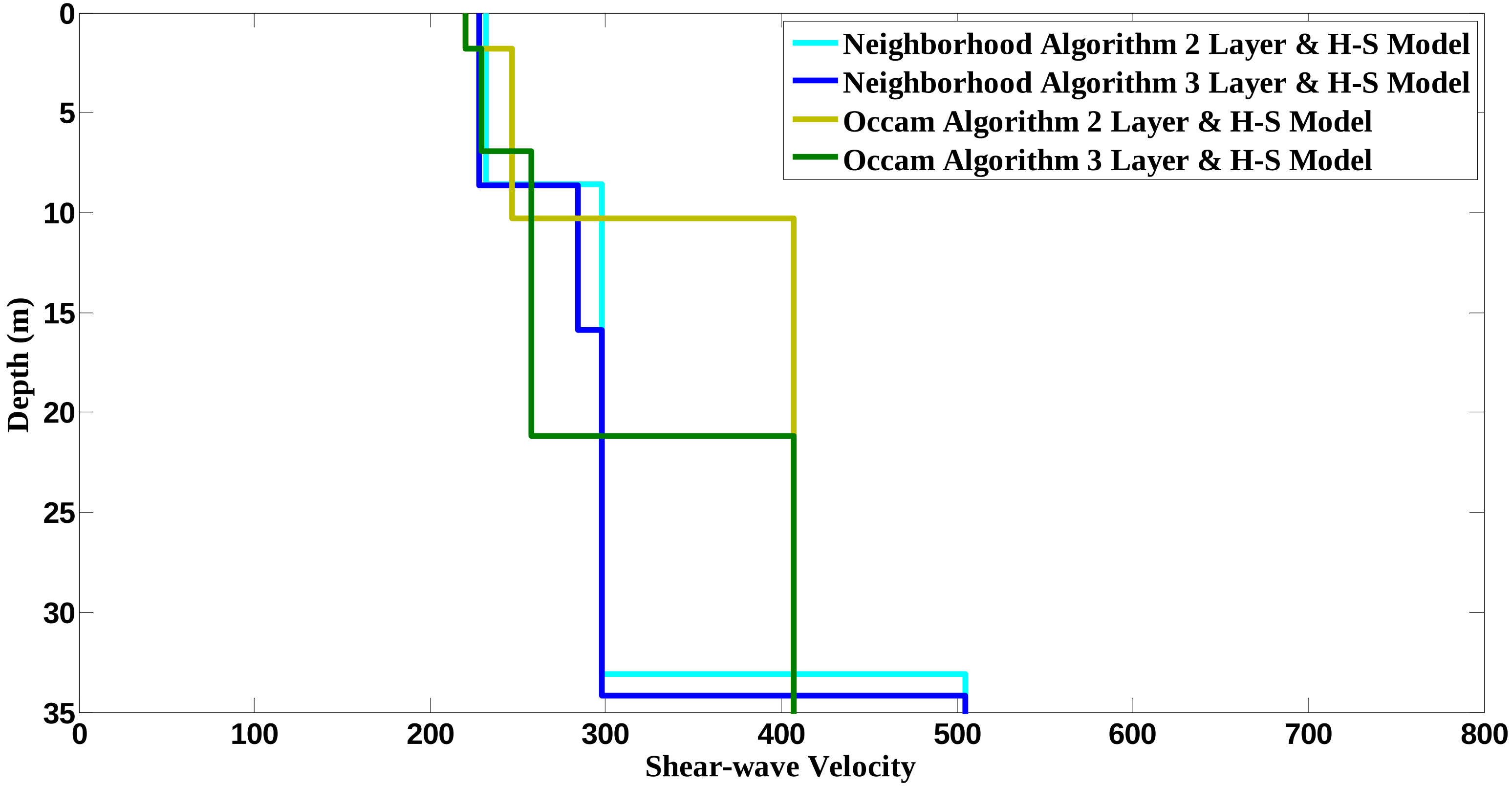
Strategy: Why not compare if we can?



Example: Hospital complex, Iseo, Italy.