SASPARM

Support Action for Strengthening Palestinian-administrated Areas capabilities for seismic Risk Mitigation

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MODULE 3 : GROUND RESPONSE ANALYSES AND NEAR-SURFACE SITE CHARACTERIZATION

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FUNDAMENTALS OF WAVE PROPAGATION

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Outline

- *<u>Propagation of waves in unbounded elastic, homogeneous continua</u>*
  - Derivation of basic equations of elastodynamics
  - Solution of 1-D wave equation
  - Longitudinal and transversal waves
  - Harmonic waves and stationary oscillations

- *<u>Propagation of waves in elastic, inhomogeneous, dissipative continua</u>*
  - Free surface effect for normal incidence
  - Reflection and transmission through a welded interface
  - Propagation of waves in viscoelastic continua
  - Plane waves, Fermat's principle, Snell's law, mode conversion
  - Surface wave propagation, Rayleigh and Love waves

1-D wave propagation in unbounded elastic homogeneous continua

Stress-strain relation (Hooke's law)

If small strain theory is adopted, stress-strain relation for a homogeneous, isotropic and linear elastic medium can be stated by making use of Hooke's law:

in indicial notation:

$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \delta_{ij}\nu\sigma\right]$$

$$\sigma = \sigma_x + \sigma_y + \sigma_z$$

in explicit form:

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{\partial v}{\partial y} = \dfrac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \\ \varepsilon_z = \dfrac{\partial w}{\partial z} = \dfrac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \end{cases}$$

Lamé constants

$$G = \mu = \frac{E}{2\,(1+\nu)}$$

$$\lambda = \frac{E\nu}{(1+\nu)\,(1-2\nu)}$$

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Stress-strain relation (Hooke's law)

If $\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \delta_{ij}\nu\sigma\right]$ is resolved in terms of stress components, then:

in indicial notation:

$$\sigma_{ij} = \lambda\vartheta\delta_{ij} + 2\mu\varepsilon_{ij} \qquad \vartheta = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

kinematic equations

If a displacement field is assumed to have only one component such that $s=[u(x,t),0,0]^T$, then the only non zero component of the strain tensor ε_{ij} is $\varepsilon_{x.}$ such that:

1D deformation

$\varepsilon_x = \frac{\partial u}{\partial x}$ and $\varepsilon_y = \varepsilon_z = 0$ from the application of Hooke's law: $\sigma_y = \sigma_z = \frac{\nu}{1-\nu}\sigma_x$

from which:

$$\varepsilon_x = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\sigma_x$$

$$\sigma_x = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_x$$

$$\sigma_x = (\lambda + 2\mu)\,\varepsilon_x = (\lambda + 2\mu)\frac{\partial u}{\partial x}$$

stress-strain relation for 1-D deformation field

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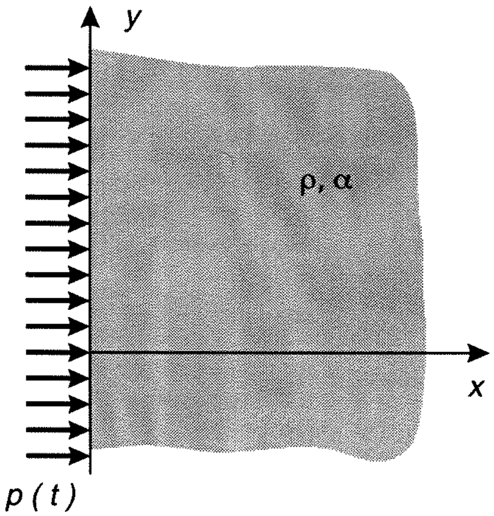
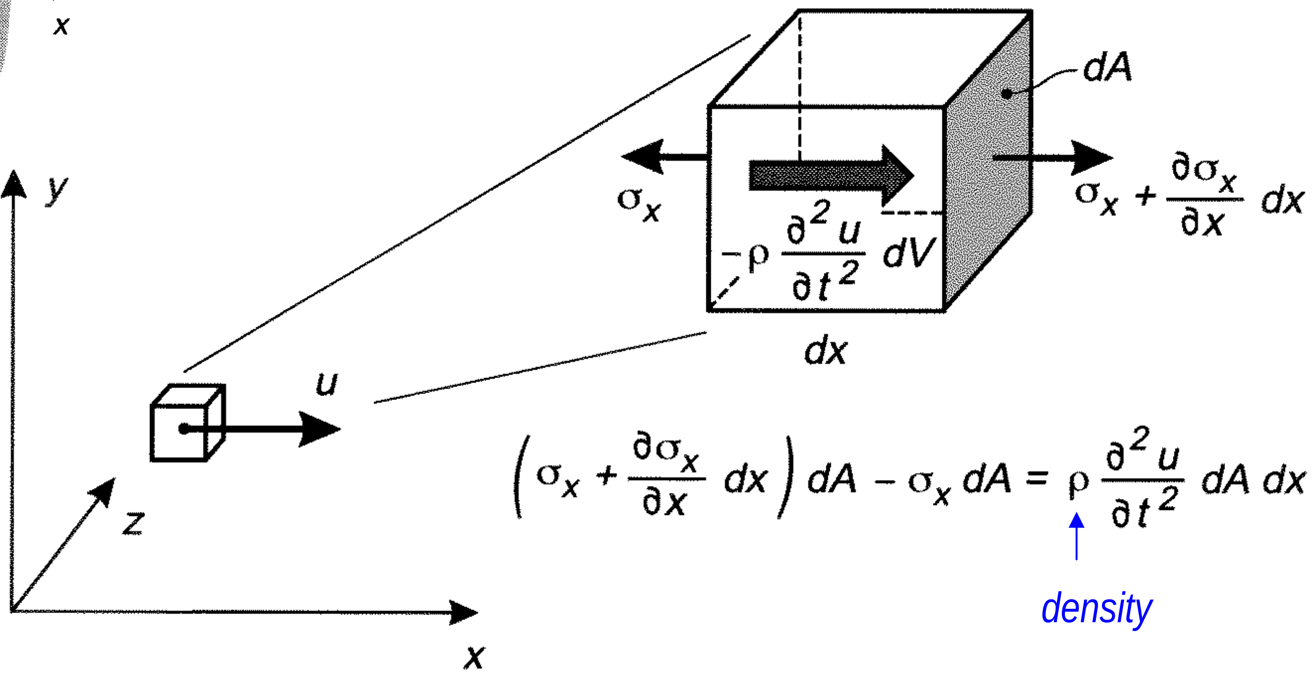
Equations of motion

Assuming only u(x,t) component, the equation of motion can be obtained as follows:



$$\varepsilon_x = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\sigma_x$$

$$\begin{cases} \sigma_x = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_x \\ \sigma_x = (\lambda+2\mu)\,\varepsilon_x = (\lambda+2\mu)\,\dfrac{\partial u}{\partial x} \end{cases}$$



$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\,dx\right) dA - \sigma_x\, dA = \rho\,\frac{\partial^2 u}{\partial t^2}\, dA\, dx$$

density

(from *Faccioli, 2005*)

Equations of motion

from which $$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$$ *since* $$\sigma_x = (\lambda + 2\mu)\,\varepsilon_x = (\lambda + 2\mu)\frac{\partial u}{\partial x}$$

yields $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}$$

<u>1-D wave propagation equation</u>

where $$\alpha = \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}}$$

speed of propagation (depends on density and elastic moduli)

SOLUTION

can be shown

$$u(x,t) = f_1(x - \alpha t) + f_2(x + \alpha t)$$

$$u(x,t) = f_1\left(t - \frac{x}{\alpha}\right) + f_2\left(t + \frac{x}{\alpha}\right)$$

f_1 and f_2 are arbitrary functions that are determined from initial conditions

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1-D wave equation

Assuming the argument of f_1(phase) $x - \alpha t = const \Rightarrow$ also $f_1(x - \alpha t) = const$

In order that the phase remains constant, if t is incremented of Δt, x must increase of $\Delta x = \alpha \Delta t$, in fact:

$$t_1 = t + \Delta t \quad \textit{and} \quad x_1 = x + \alpha \Delta t \quad \Rightarrow \quad x_1 - \alpha t_1 = x + \alpha \Delta t - \alpha t - \alpha \Delta t = x - \alpha t$$

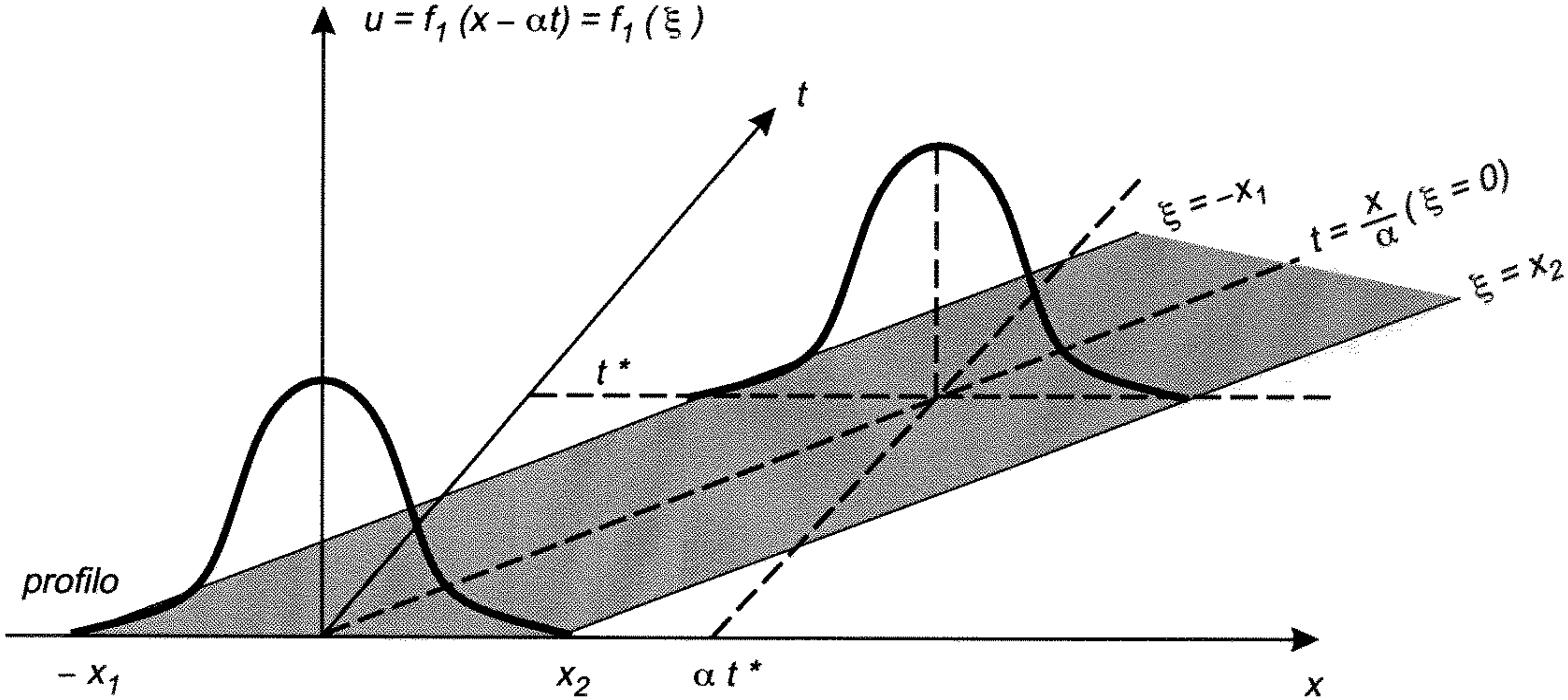
from which $$f_1(x_1 - \alpha t_1) = f_1(x - \alpha t) = f_1(x)$$

In other words, the displacement profile of $f_1(x - \alpha t)$ represents a propagating perturbation which appears to be <u>STATIONARY</u> for an observer moving with constant velocity α along the +x axis. Therefore:

- α can be interpreted as the speed of propagation of the profile
- $f_1(x - \alpha t)$ represents a wave propagating forward (along +x direction)
- $f_2(x + \alpha t)$ is represents wave propagating backward (along -x direction)

1-D wave equation

Displacement profile of 1-D wave



(from *Faccioli, 2005*)

In the propagation the signal moves without DISTORTION !

Longitudinal waves

Parameter α (or V_P) represents the velocity of propagation of longitudinal waves. Velocity of propagation should not be confused with $\dot{u}(x,t)$ which represents the particle motion, which instead is a function of position and time instant under consideration.

The values of α in near-surface geological materials can be measured experimentally (as it will be shown later) by means of in situ and laboratory tests. Their range of variation is rather wide. At depths of few kilometers from the Earth surface the values of α are typically in the range of 6.0 to 7.0 km/s.

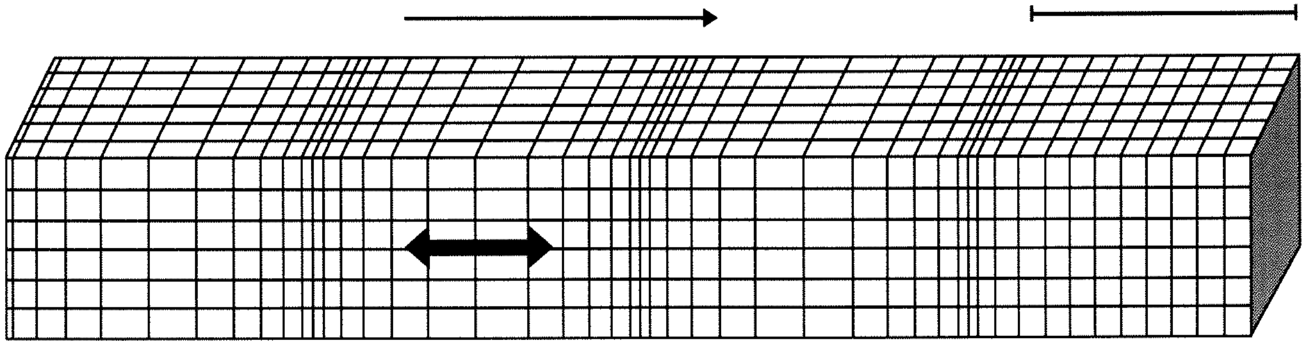
Geomaterials	α ***or*** V_P ***(km/s)***
Alluvium (clays, silts, sands)	0.5 − 2.0(*)
Soft rock, dense gravel	2.0 − 3.0
Calcareous rock, dolomite	3.0 − 5.0
Crystalline rock	4.0 − 6.5

(*) lower bound values for alluvial sediments are for dry geomaterials (above water table)

(from *Faccioli, 2005*)

Longitudinal waves

P-waves

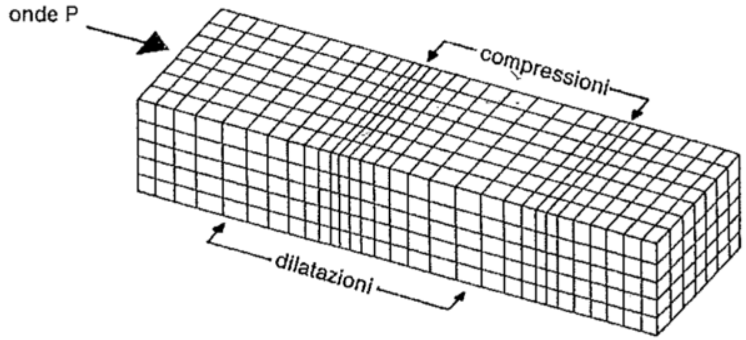


→ direction of propagation

|—| unperturbed rock

⟷ direction of particle motion (vibrations)

Characteristics of P-waves



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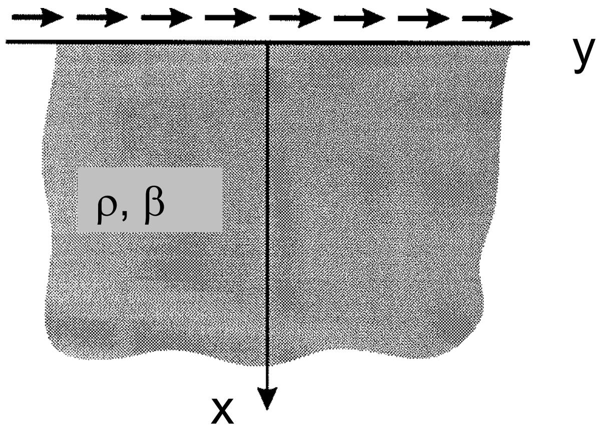
Transversal waves

Transversal motion

Consider now the case in which the elastic medium is excited by a dynamic perturbation that still propagates in the x direction, but gives rise to a displacement field that acts only in the y direction, and is independent from the y and z coordinates, namely **s** = $[0, v(x,t), 0]^T$.

The derivation of the equation of motion that must be satisfied by v(x,t) is left as an easy but useful exercise (together with the demonstration that the shear stress τ_{xy} does not vary with y).



it is found

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\beta^2} \frac{\partial^2 v}{\partial t^2}$$

where

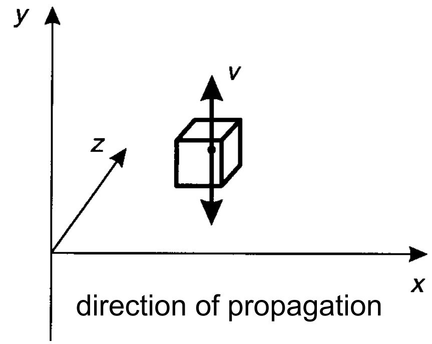
$$\beta = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} = \left(\frac{G}{\rho}\right)^{\frac{1}{2}}$$

velocity of propagation of transversal (shear) waves (depends on density and elastic shear modulus)

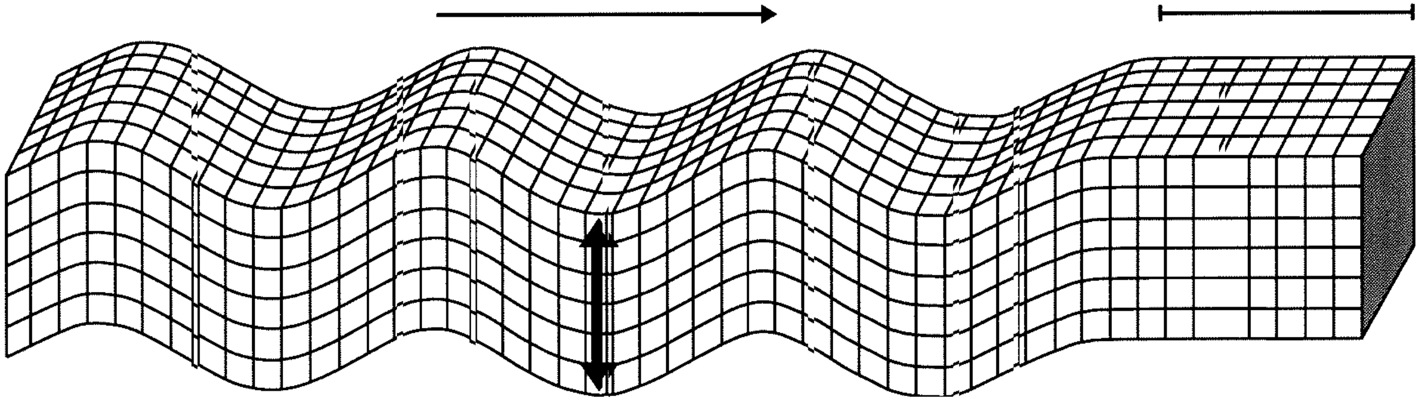
Transversal waves

Due to assumed displacement field, the only non-vanishing strain component is shear strain:

$$\gamma = \gamma_{xy} = \gamma_{yx} = \frac{\partial v}{\partial x} \qquad \textit{and thus} \qquad \Rightarrow \quad \tau = \tau_{xy} = \tau_{yx}$$

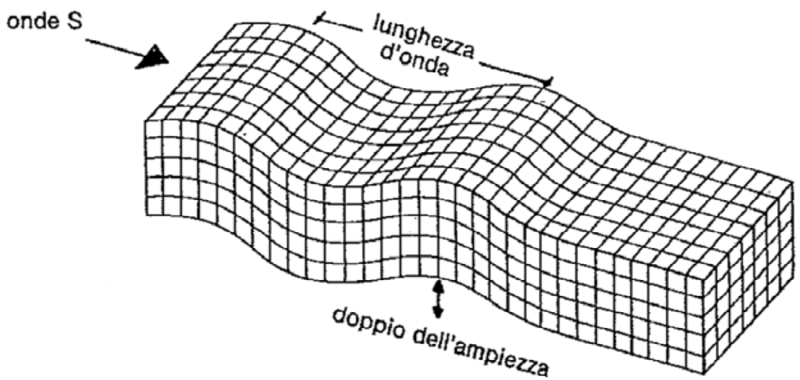


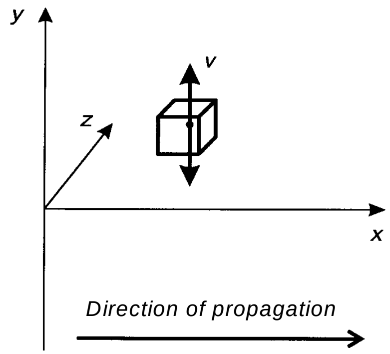
S-waves



→ direction of propagation |—| unperturbed rock   ↕ direction of particle motion (vibrations)

Transversal waves





Geomaterials	**β or V_s (m/s)**
Very soft clays with high water content (e.g. Mexico City)	40-80
Normally consolidated clays and silts	150-300
Medium to very dense sands	200-400
Gravel	400-800
Soft rocks	500-1000
Fractured limestone	700-1500
Crystalline rocks	2500-3500

(from *Faccioli, 2005*)

Longitudinal and transversal waves

The relation between α (V_p) and β (V_s) may be combined together to get:

$$(\alpha/\beta)^2 = \frac{\lambda + 2\mu}{\mu} = \frac{2(1-\nu)}{(1-2\nu)} > 1$$

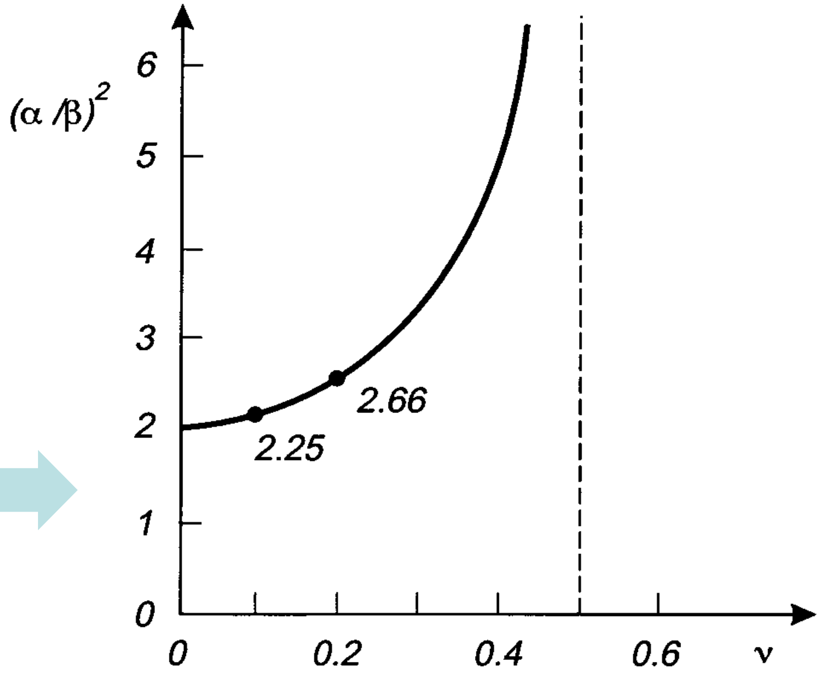
which shows that it is <u>always</u> $\beta < \alpha$!

for $\nu = 0.25 \Rightarrow \alpha = \sqrt{3}\,\beta$

In saturated porous media:

ν ---> 0.5 and α ---> ∞ (incompressible medium)

β is a fundamental soil parameter in geotechnical earthquake engineering !



(from *Faccioli, 2005*)

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Harmonic waves and stationary oscillations

Stationary oscillations

The elastic medium is subjected to a **stationary oscillation** or vibration, if the motion of each of its particles is proportional to some temporal function identical for all particles, and thus independent of x.

If we recall 1-D wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}$$

a stationary solution is obtained by setting:

$$u(x,t) = X(x)T(t)$$

by making use of separation of variables method

obtaining

$$\frac{1}{\alpha^2}\frac{d^2T}{dt^2}\frac{1}{T} = \frac{d^2X}{dx^2}\frac{1}{X} = -K^2$$

since LHS is only a function of "t", and RHS is only a function of "x"

$$\frac{d^2T}{d(\alpha t)^2} + K^2 T = 0$$

and

$$\frac{d^2X}{dx^2} + K^2 X = 0$$

whose integration yields

Stationary oscillations

$$\begin{cases} T = C_1 e^{iK\alpha t} + C_2 e^{-iK\alpha t} = (C_1 + C_2)\cos(K\alpha t) + i(C_1 - C_2)\sin(K\alpha t) \\ X = C_3 e^{iKx} + C_4 e^{-iKx} = (C_3 + C_4)\cos(Kx) + i(C_3 - C_4)\sin(Kx) \end{cases}$$

thus

$$u = \left(C_1 e^{iK\alpha t} + C_2 e^{-iK\alpha t}\right)\left(C_3 e^{iKx} + C_4 e^{-iKx}\right) =$$

$$= C_1 C_3 e^{iK(x+\alpha t)} + C_1 C_4 e^{-iK(x-\alpha t)} + C_2 C_3 e^{iK(x-\alpha t)} + C_2 C_4 e^{-iK(x+\alpha t)}$$

When only the real part is considered:

$$\mathrm{Re}[u] = (C_1 C_3 + C_2 C_4)\cos[K(x+\alpha t)] + (C_1 C_4 + C_2 C_3)\cos[K(x-\alpha t)] =$$

$$= A\cos[K(x+\alpha t)] + B\cos[K(x-\alpha t)].$$

the most general STATIONARY oscillation can be constructed by superimposing two sinusoidal waves having opposite direction of propagation.

$\Rightarrow$ **constructive and destructive interference**

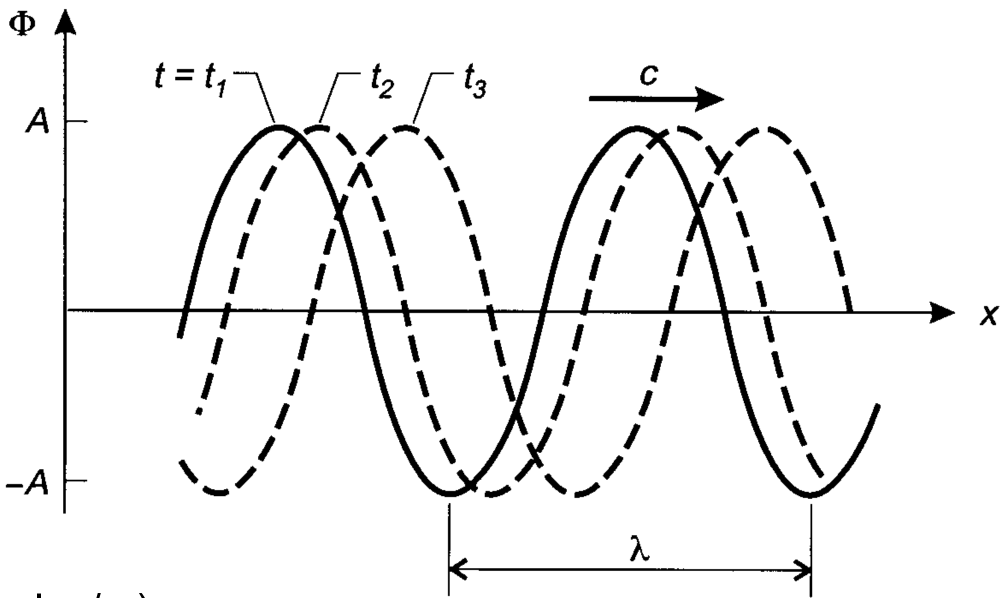
Harmonic waves

$$\mathrm{Re}[u] = (C_1C_3 + C_2C_4)\cos[K(x+\alpha t)] + (C_1C_4 + C_2C_3)\cos[K(x-\alpha t)] =$$

$$\Phi(x,t) = A\cos[K(x+\alpha t)] + B\cos[K(x-\alpha t)].$$

↑ *harmonic wave*

A = amplitude (any unit)



TEMPORAL PARAMETERS

- *T* = λ/α = *period (sec)*
- *f* = 1/T = α/λ = *temporal frequency (Hz)*
- ω = 2πf =2π/T = *circular frequency* (rad/s)

SPATIAL PARAMETERS

- $\lambda = 2\pi/K$ = *wavelength (m)*
- ν = 1/λ = spatial frequency, wavenumber (cycles/m)
- K = 2π/λ = 2$\pi\nu$ = ω/α = circular wavenumber (rad/m)
- α = velocity of propagation c = $\lambda/T = \omega/K = \lambda f = f/\nu$ (m/s)

(from *Faccioli, 2005*)

Harmonic waves

Monochromatic waves

Given two waves with <u>same frequency</u> propagating with the same speed in the same medium

$$\Phi_1(x,t) = A_1 \cos[2\pi f\ (x/c - t)] \qquad \Phi_2(x,t) = A_2 \cos[2\pi f\ (x/c - t) + \varepsilon]$$

phase difference/phase shift

If $\varepsilon = m\pi$ (m is *even*) $\Rightarrow$ phase shift is an *even* multiplier of λ
$\Rightarrow$ maxima of both waves coincide with each other

Two waves are IN PHASE

If $\varepsilon = m\pi$ (m is *odd*) $\Rightarrow$ phase shift is an *odd* multiplier of λ
$\Rightarrow$ minima of one wave coincide with maxima of other

Two waves are OUT of PHASE

Harmonic waves

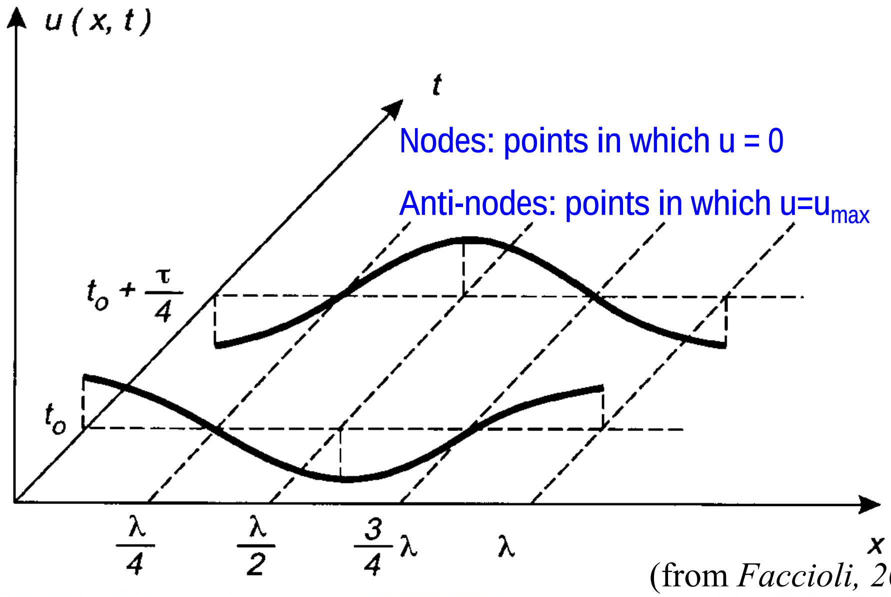
Stationary oscillations

$$\mathrm{Re}[u] = (C_1C_3 + C_2C_4)\cos[K(x+\alpha t)] + (C_1C_4 + C_2C_3)\cos[K(x-\alpha t)] =$$
$$= A\cos[K(x+\alpha t)] + B\cos[K(x-\alpha t)].$$

If both waves has the same amplitude A

$$\mathrm{Re}[u] = 2A\cos(Kx)\cos(2\pi f t)$$

- Generic point oscillates in time with a sinusoidal law, without propagation of motion
- The wavelength $\lambda = 2\pi/K$ of the resulting oscillation coincides with that of the component waves
- These characteristics of the stationary oscillation belong to what in mechanics is called a *mode of vibration* of a linear system



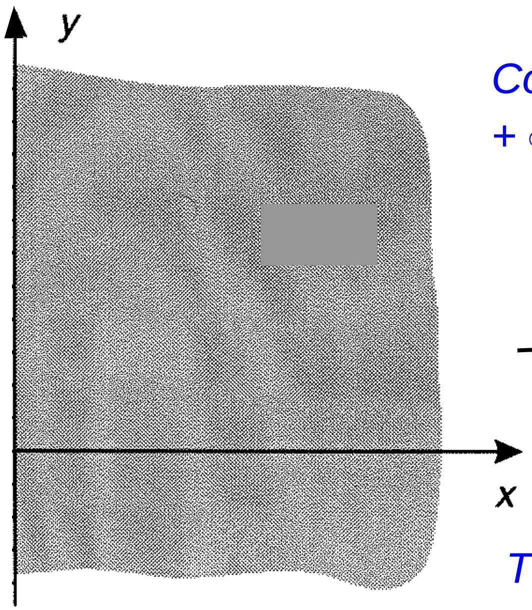
(from *Faccioli, 2005*)

Free surface effect for normal incidence

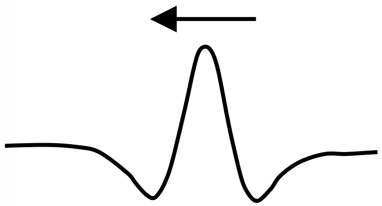
Free surface effect for normal incidence

1D wave propagation in an elastic half-space

Free surface effect



Consider an incident wave propagating with a velocity c from + ∞ toward the origin:



$$u^{i}(x,t) = f\left(t + \frac{x}{c}\right)$$

The stress-free condition along the surface of the half-space translates itself into the following condition (said free-surface B.C.):

$$\sigma_x = (\lambda + 2\mu)\,\varepsilon_x = (\lambda + 2\mu)\,\frac{\partial u}{\partial x} = 0$$

substituting this expression in the general solution of 1D wave equation:

Free surface effect for normal incidence

1D wave propagation in an elastic half-space

Free surface effect

$$u(x,t) = f\left(t + \frac{x}{\alpha}\right) + g\left(t - \frac{x}{\alpha}\right) \Rightarrow f'(t) - g'(t) = 0$$ *from which*

$$g(t) = f(t) + \text{const} \Rightarrow$$ *since* const = 0 $$\Rightarrow g(t - x/c) = f(t - x/c)$$

which represents wave u^r generated for total reflection of free surface propagating in the opposite direction to that of the incident wave. The total displacement at any point x is:

$$u(x,t) = u^i + u^r = f\left(t + \frac{x}{c}\right) + f\left(t - \frac{x}{c}\right)$$ *from which* $$u(0,t) = 2f(t) = 2\,u^i$$

$\Rightarrow$ **doubling of displacement amplitude due to stress-free B.C. !**

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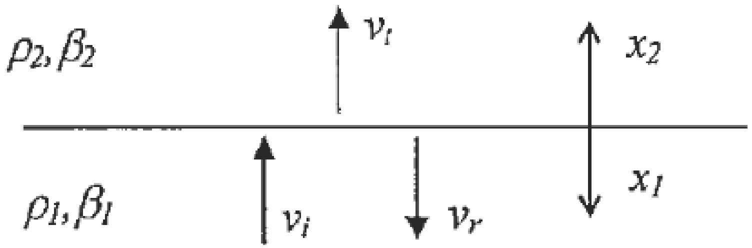
Reflection and transmission through a welded interface

Reflection and transmission coefficients

Consider an incoming harmonic displacement S-wave with a unit amplitude propagating through an interface, which separates media $1(\rho_1,\beta_1)$ and $2(\rho_2,\beta_2)$, with normal incidence.



(ρ: is mass density, and β: shear wave velocity)

In such situation, total displacement in medium 1 will be given by two contributions: the incident wave (v_i) plus the reflected (v_r) wave. In medium 2 only contribution is that of the transmitted wave (v_t) through the interface. These three waves may be written as follows:

incident wave: $v_i = \exp\left[i\omega(t + \frac{x_1}{\beta_1})\right]$ *Propagating in medium 1*

reflected wave: $v_r = c_r \exp\left[i\omega(t - \frac{x_1}{\beta_1})\right]$ *Propagating in medium 1*

transmitted wave: $v_t = c_t \exp\left[i\omega(t - \frac{x_2}{\beta_2})\right]$ *Propagating in medium 2*

Reflection and transmission coefficients

Reflection and transmission coefficients are determined by the continuity conditions at the interface between the two media:

i) continuity of displacement

$$v_i\big|_{x_1=0} + v_r\big|_{x_1=0} = v_t\big|_{x_2=0}$$

ii) continuity of stress

$$\rho_1\beta_1^2\left(\frac{\partial(v_i+v_r)}{\partial x_1}\right)_{x_1=0} = -\rho_2\beta_2^2\left(\frac{\partial v_t}{\partial x_2}\right)_{x_2=0}$$

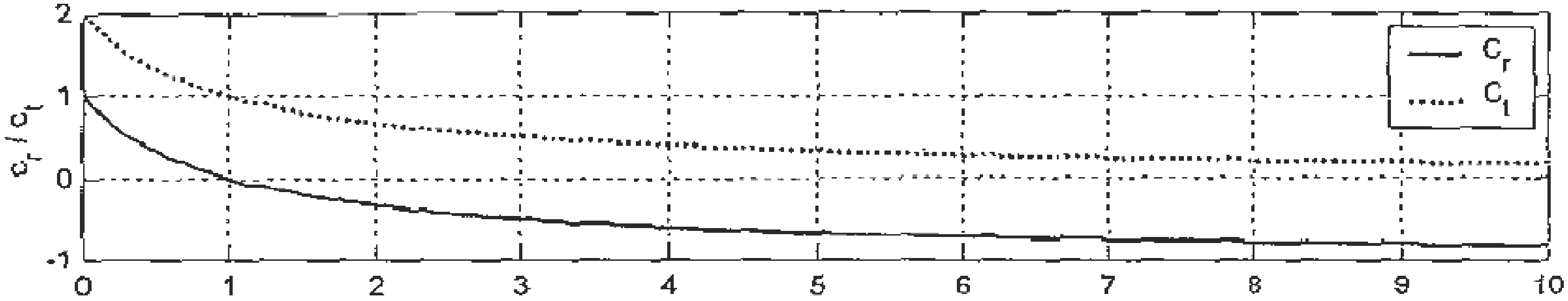
Boundary conditions (i) and (ii) yield:

$$\begin{cases} 1 + c_r = c_t \\ -1 + c_r = -\eta c_t \end{cases}$$ namely $$\begin{cases} c_r = \dfrac{1-\eta}{1+\eta} \\ c_t = \dfrac{2}{1+\eta} \end{cases}$$ where $$\eta = \frac{\rho_2\beta_2}{\rho_1\beta_1}$$ *impedance ratio*

Reflection and transmission coefficients

Dependence of reflection and transmission coefficients to impedance ratio is illustrated below:

Reflection and transmission coefficients



impedance ratio

(from *Faccioli, 2005*)

SPECIAL CASES:

rigid end (base) condition:

(*impedance ratio* = ∞)

c_r=-1

c_t=0

free end (surface) condition:

(*impedance ratio* = 0)

c_r=1

c_t=2

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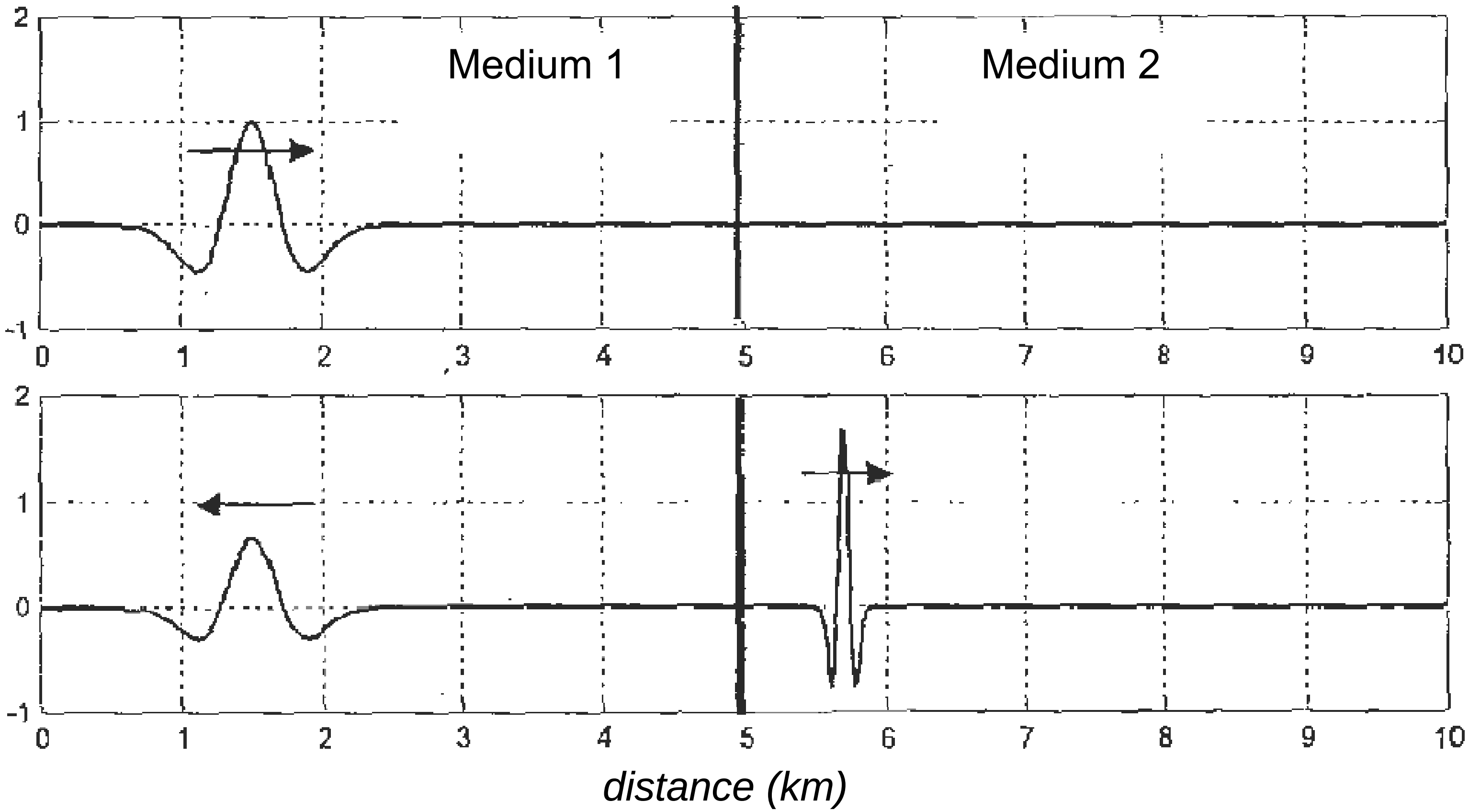
Reflection and transmission coefficients

Illustration of reflection and transmission phenomena in spatial domain:



Propagation in viscoelastic medium

Linear viscoelastic constitutive model

Assumption: One strain component: shear

Constitutive model: $\tau = \mu\gamma + \eta\dot{\gamma}$ with:

- $\gamma = \dfrac{\partial v}{\partial x}$
- η : viscosity constant, assumed x independent

Dynamic equilibrium eq.: $\dfrac{\partial \tau}{\partial x} = \rho \ddot{v}$ → $\dfrac{\partial \tau}{\partial x} = \mu \dfrac{\partial^2 v}{\partial x^2} + \eta \dfrac{\partial^3 v}{\partial x^2 \partial t} = \rho \ddot{v}$

Assumption:

Stationary solution: $v(x,t) = v(x) e^{i\omega t}$ → $(\mu + i\omega\eta)\dfrac{\partial^2 v}{\partial x^2} + \rho\omega^2 v = 0$

Introducing complex quantities:

- $\mu^* = \mu\left(1 + \dfrac{i\omega\eta}{\mu}\right)$
- $\beta^* = \left(\dfrac{\mu^*}{\rho}\right)^{0.5}$

→ $v'' \left(\dfrac{\omega}{\beta^*}\right)^2 v = 0$

Stationary Solution: $v(x) = Ae^{i\frac{\omega}{\beta^*}x} + Be^{-\frac{\omega}{\beta^*}x}$

$$v(x) = Ae^{i\omega\left(t + \frac{x}{\beta^*}\right)} + Be^{i\omega\left(t - \frac{x}{\beta^*}\right)}$$

Linear viscoelastic constitutive model

$$\bullet\ 2\zeta(\omega) = \frac{1}{Q(\omega)} = \frac{\omega\eta(\omega)}{\mu}$$

Internal damping factor — Quality factor

Two cases:

- $\eta = cost$: viscous damping; ζ and Q^{-1} linearly increase with frequency
- $\eta \propto \omega^{-1}$: hysteretic damping; ζ and Q^{-1} constant

Generally $Q >> 1 \rightarrow \zeta << 1$

$$\bullet\ \mu^* = \mu\left(1 + \frac{i\omega\eta}{\mu}\right) = \mu\left(1 + \frac{i}{Q}\right)$$

$$\bullet\ \beta^* = \beta\left(1 + \frac{i}{Q}\right)^{0.5} \cong \beta\left(1 + \frac{i}{2Q}\right) = \beta(1 + i\zeta) \Rightarrow \frac{x}{\beta^*} \cong \frac{x}{\beta(1+i\zeta)} = \frac{x}{\beta(1+\zeta^2)} - i\frac{x\zeta}{\beta(1+\zeta^2)} \cong \frac{x}{\beta} - i\frac{x}{2Q\beta}$$

$$v(x) = Ae^{i\omega\left(t + \frac{x}{\beta^*}\right)} + Be^{i\omega\left(t - \frac{x}{\beta^*}\right)}$$

$$v(x) = Ae^{i\omega\left(t + \frac{x}{\beta}\right)}e^{\left(\frac{\omega x}{2Q\beta}\right)} + Be^{i\omega\left(t - \frac{x}{\beta}\right)}e^{\left(-\frac{\omega x}{2Q\beta}\right)}$$

Wave with backward propagation — Wave with forward propagation

Linear viscoelastic constitutive model

- Hysteretic damping; $Q = cost$ → Factor of attenuation $\frac{\omega x}{2Q\beta}$ proportional to frequency

↓

Higher damping at higher frequencies

- Viscous damping; Q^{-1} linearly increase with frequency → factor of attenuation dependence from ω^2

↓

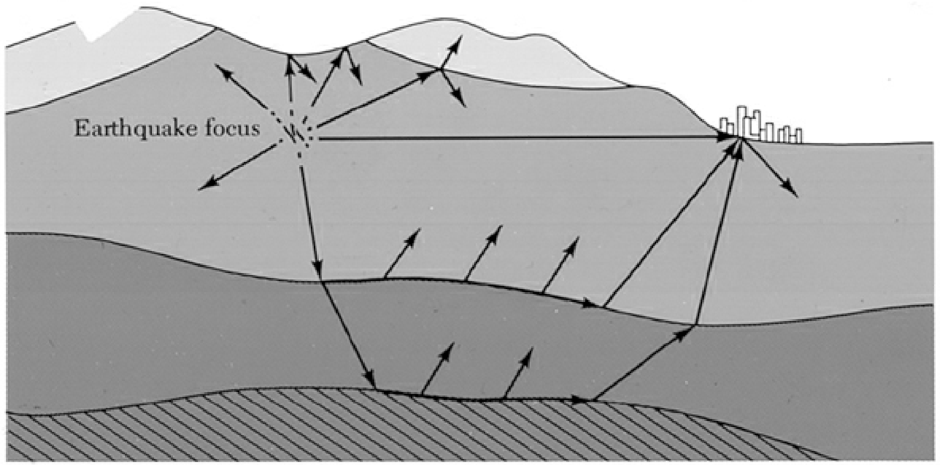
Highlighted damping

!

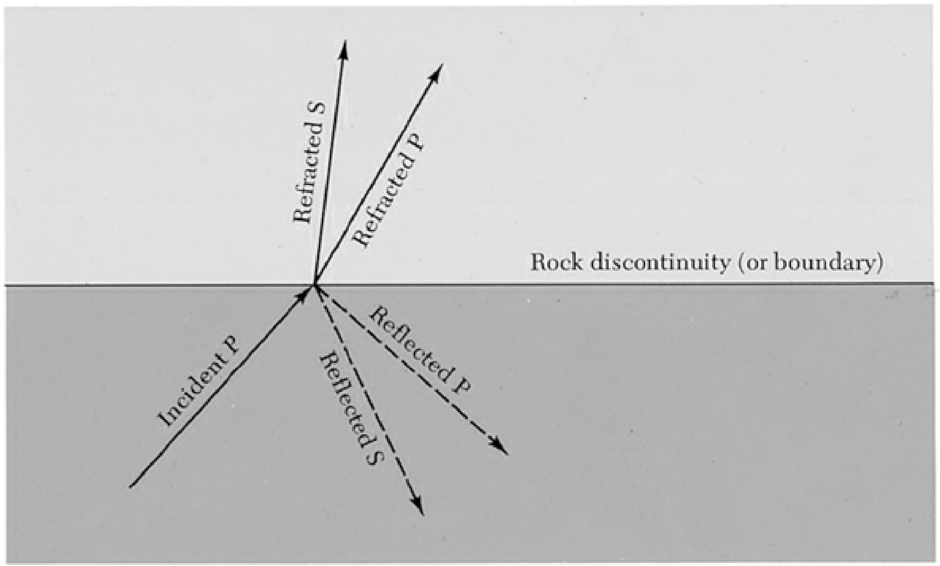
Previous formulation is valid only for sinusoidal stationary motion with a fixed frequency

Propagation of waves in elastic inhomogeneous continua

Reflection and transmission for arbitrary incidence



Paths of seismic waves which are reflected and refracted by interfaces of various geological formations within the earth crust.



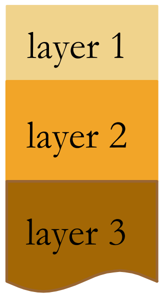
Reflection and refractions of a P seismic wave at the interface between two different rock formations.

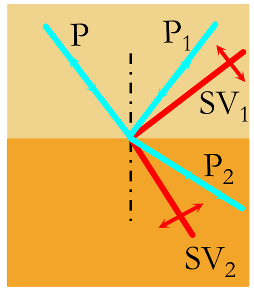
(*from* Bruce A. Bolt, Nuclear Explosions and Earthquakes, W. H. Freeman, San Francisco, 1976)

Reflection and transmission for arbitrary incidence

Body waves ➡ **influence of medium heterogeneity**

- *Discontinuous variability*

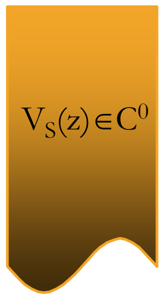


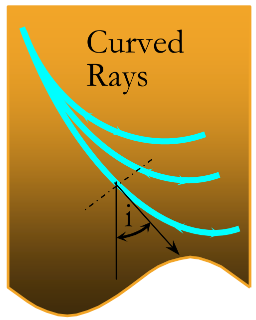


$$\frac{\sin(i)}{V(z)} = p$$

➡ Snell's law

- *Continuous variability*





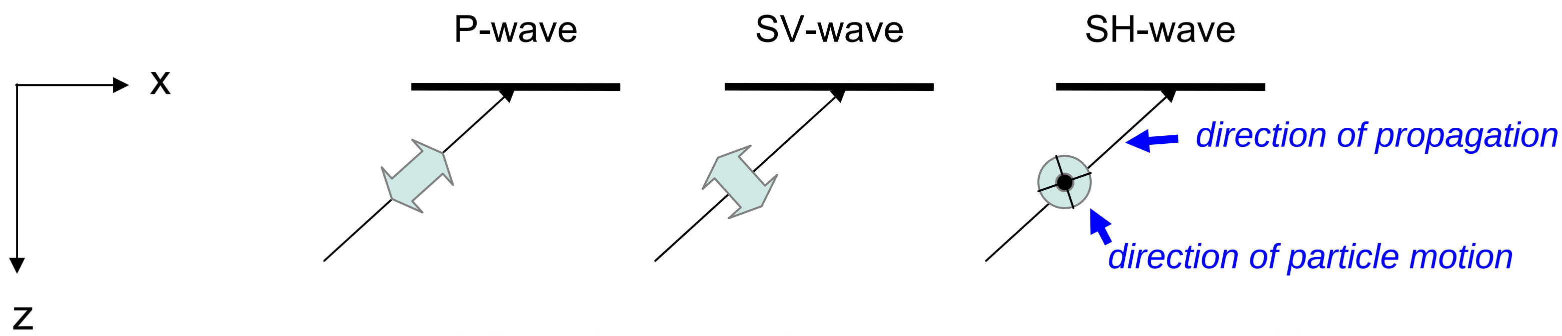
Reflection and transmission for arbitrary incidence

Body waves in elastic (isotropic) media are of two different types:

- **P-waves**
- **S-waves**

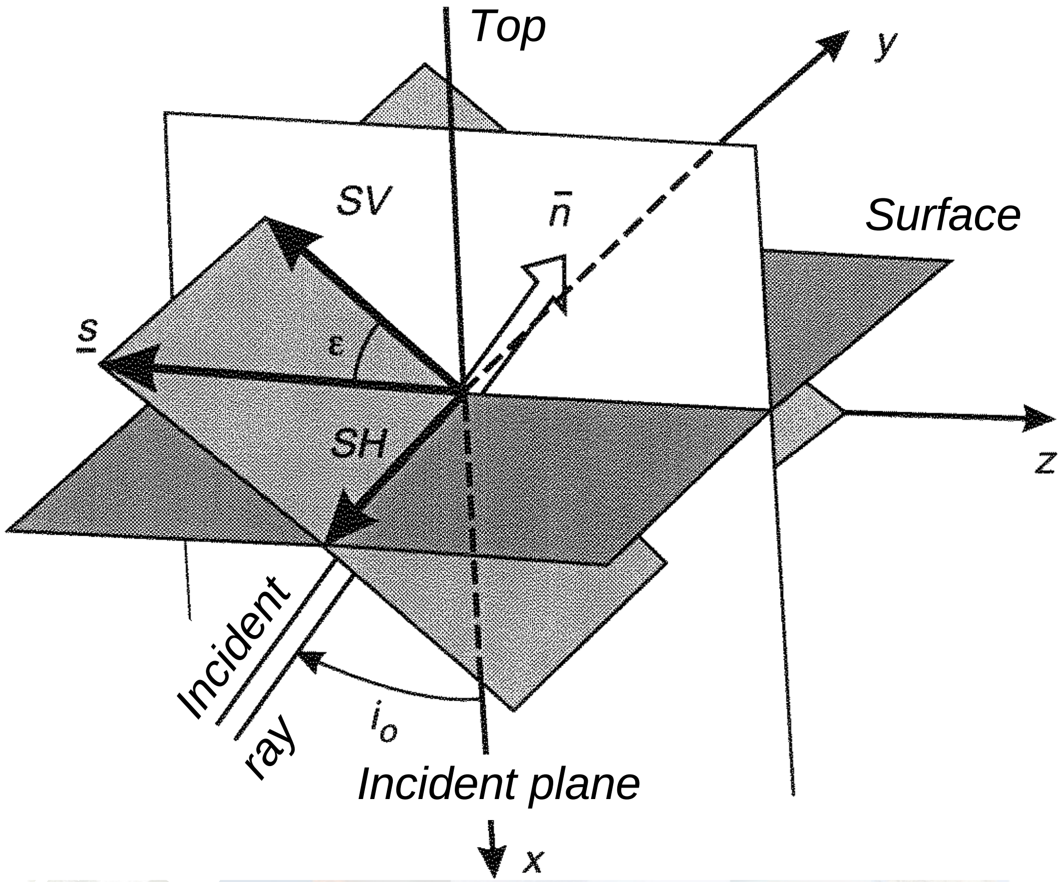
S-waves are of two different kind according to the direction of particle motion:

- *SV-wave:* displacement amplitude is oriented parallel to plane [x, z]
- *SH-wave:* displacement amplitude is oriented normal to plane [x, z]



Reflection and transmission for arbitrary incidence

Polarisation of the transversal particle motion into the SV and SH components



(*from* Faccioli, 2005)

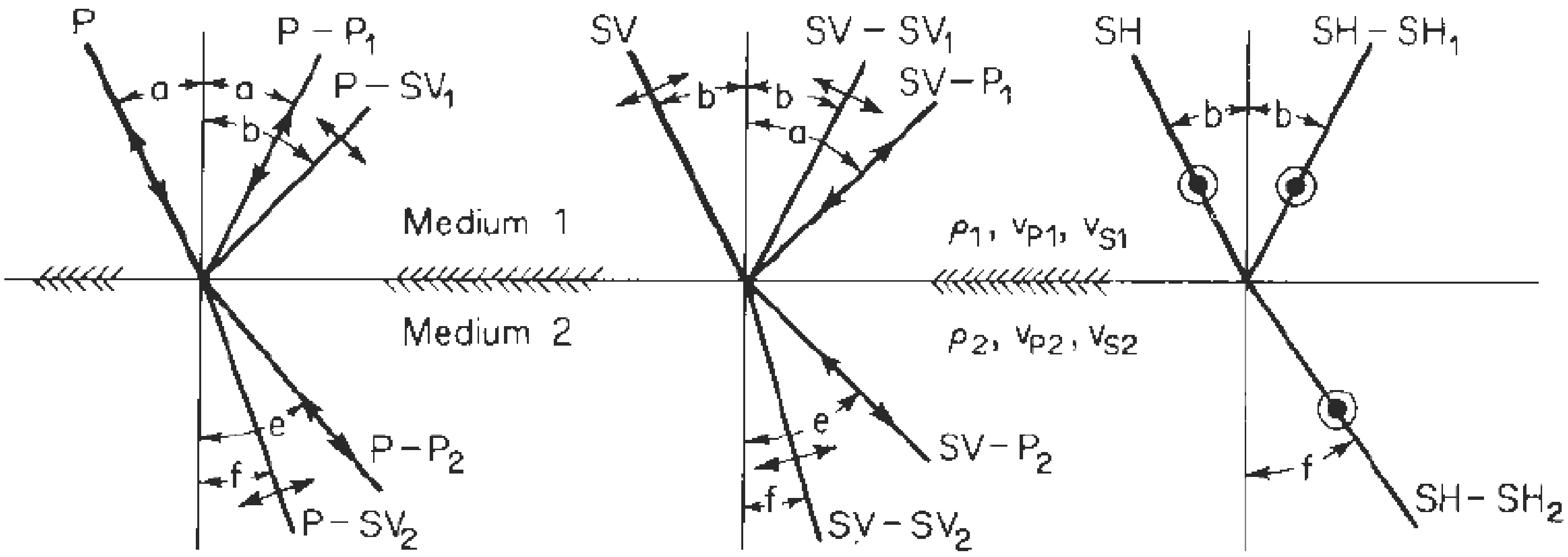
Reflection and transmission for arbitrary incidence

When body waves are impinging at an interface with an arbitrary angle of incidence, according to their polarization (orientation of particle motion) they give rise to:

- **P-waves:** generate reflected and transmitted P and SV-waves (*mode conversion*)
- **S-waves:**
  - ***SV-waves:*** generate reflected and transmitted P and SV-waves (*mode conversion*)
  - ***SH-waves:*** generate reflected and transmitted SH-waves

$$\frac{\sin a}{v_{P1}} = \frac{\sin b}{v_{S1}} = \frac{\sin e}{v_{P2}} = \frac{\sin f}{v_{S2}}$$

Snell's law



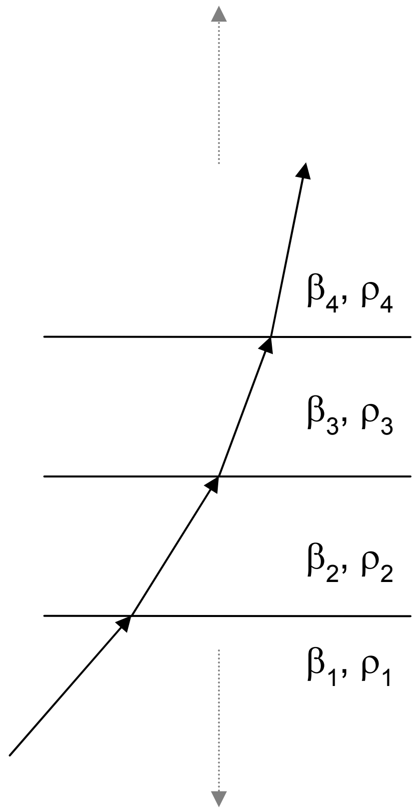
(a) Incident P-Wave. (b) Incident SV-Wave. (c) Incident SH-Wave.

(from *Richart et al.*, 1970)

Reflection and transmission for arbitrary incidence

Suppose a soil deposit where:

- layers are oriented horizontally
- stiffer layers are overlaid by softer layers (i.e. $\beta_{i+1} < \beta_i$)



- incidence angle gets smaller at each interface level !
- transmitted waves get more vertical !

This situation is valid also for seismic wave propagation in deep earth layers !

$\Rightarrow$ **normal incidence is a reasonable assumption in 1D soil modeling !**

Reflection and transmission for arbitrary incidence

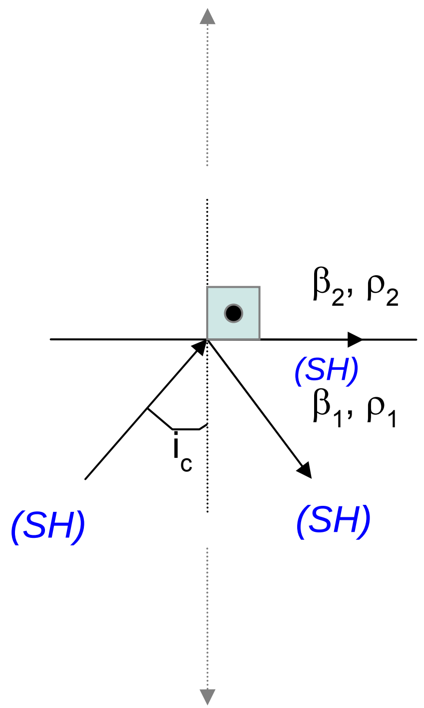
SH-WAVES

When does a wave cannot be transmitted through an interface ?

For SH-waves

$$\frac{\sin(i_c)}{\beta_1} = \frac{\sin(90^o)}{\beta_2} \rightarrow i_c = \arcsin\left(\beta_1 / \beta_2\right)$$

For incident angles greater than i_c, no transmitted SH-waves are generated !



Similar conclusions may also be drawn for 2-D P-SV propagation.

Reflection and transmission for arbitrary incidence

Influence of boundary condition on transmission and reflexion - impedance

Dr. **Dan Russel**, Kettering University, Applied Physics

http://www.kettering.edu/~drussell/Demos/reflect/reflect.html

at a fixed (hard) boundary, the displacement remains zero (Dirichlet conditions) and the reflected wave changes its polarity (undergoes a 180º phase change)

hard.gif

at a free (soft) boundary, the restoring force is zero (Neumann conditions) and the reflected wave has the same polarity (no phase change) as the incident wave

soft.gif

Permeable boundary: the incident wave is travelling from a region of low impedance towards a high impedance region

reflect1.gi

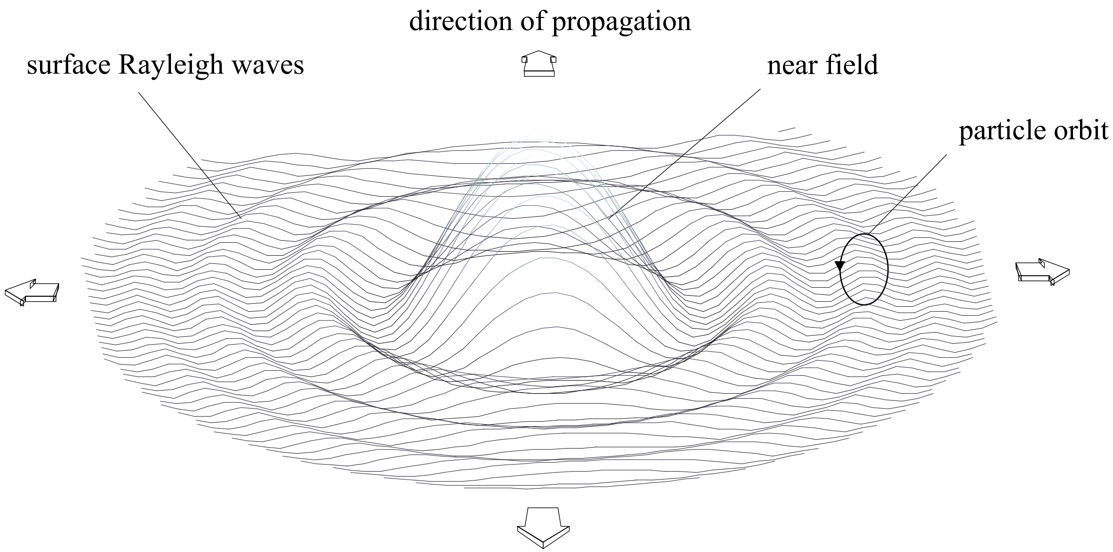
Permeable boundary: the incident wave is travelling from a high impedance region towards a low impedance region

reflect2.gi

Surface Rayleigh waves

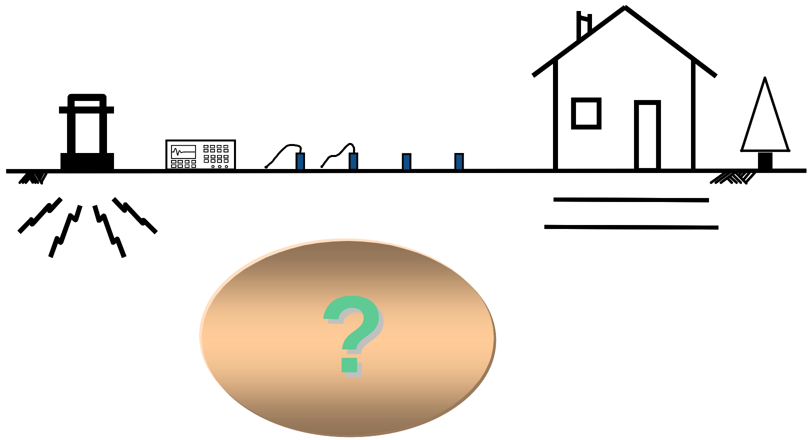
Relevance of surface waves in science and technology

- Discovered by Rayleigh in 1887, have attracted an increasing interest in:
- Solid-state physics
- Microwave engineering
- Geophysical prospecting
- Geotechnical engineering
- Non-destructive testing
- Seismological studies
- Material science
- Ultrasonic acoustics
- ….



Why surface waves are appealing ?

- They are ideal for developing non-invasive techniques for material
- characterization $\Rightarrow$ solution of parameter-identification problems:



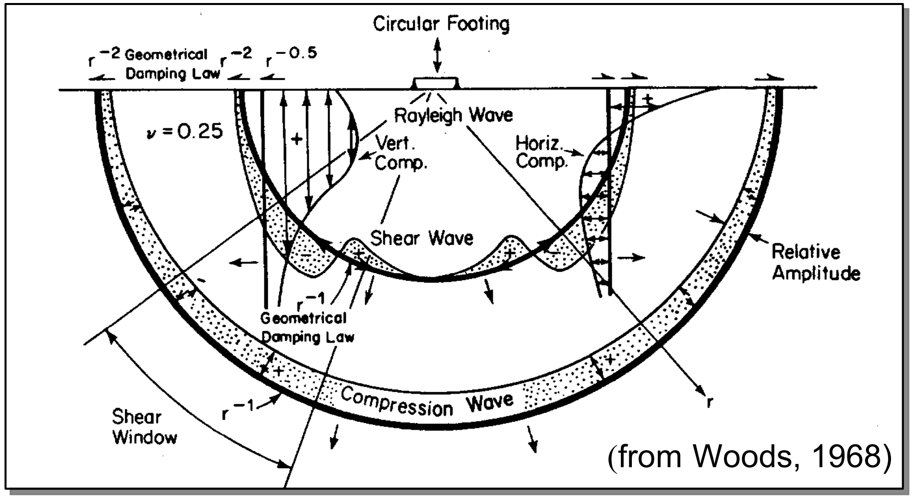
- Small scale $\Rightarrow$ ultrasonic surface waves can identify material defects
- Large scale $\Rightarrow$ seismologists use surface waves to investigate Earth’s structure
- Intermediate scale $\Rightarrow$ geophysicists use surface waves for site-characterization

Why surface waves are appealing ?

- Some properties of surface waves which make them particularly
- suitable for material characterization:

mechanical waves generated by a vertical harmonic oscillator

Wave Type	Per Cent of Total Energy
Rayleigh	67
Shear	26
Compression	7



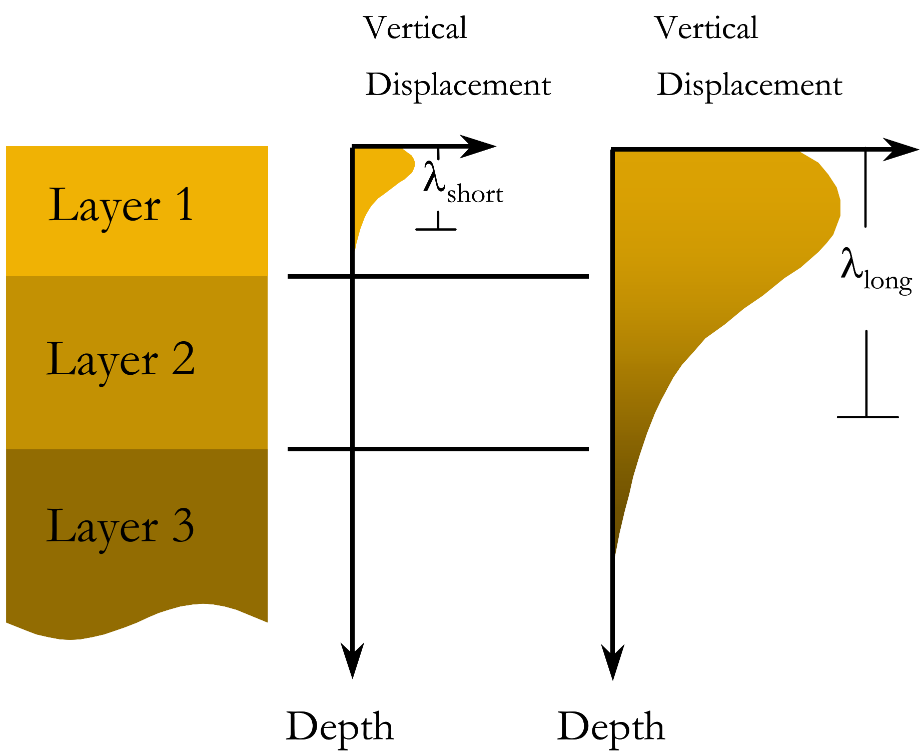
(from Woods, 1968)

- They originate from condition of vanishing stress at the boundary of a domain
- They radiation pattern is two-dimensional ⇒ lesser rate of geometric attenuation
- In direction orthogonal to propagation displacement field decays exponentially
- Most strain energy is confined within a depth of a wavelength from free-surface

Why surface waves are appealing ?

- Some properties of surface waves which make them particularly suitable for material characterization:

- In homogeneous media surface waves are non-dispersive
- In heterogeneous media surface waves are dispersive that is waves of different wavelength will travel at different speeds
- geometric dispersion can be used for material characterization



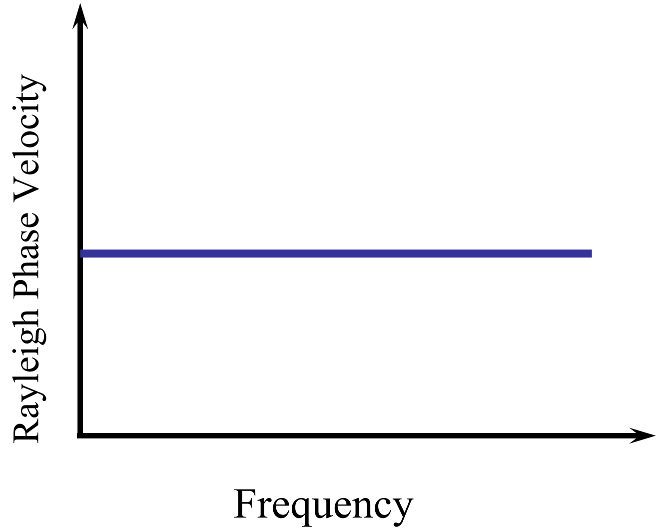
Why surface waves are appealing ?

- Some properties of surface waves which make them particularly
- suitable for material characterization:

HOMOGENEOUS CONTINUA

no geometric dispersion

homogeneous medium



dispersion relation

$$x^3 - 8 \cdot x^2 + 8 \cdot (1 + 2 \cdot r) \cdot x - 16 \cdot r = 0$$

with

$$x = \left(\frac{V_R}{\beta}\right)^2$$

and

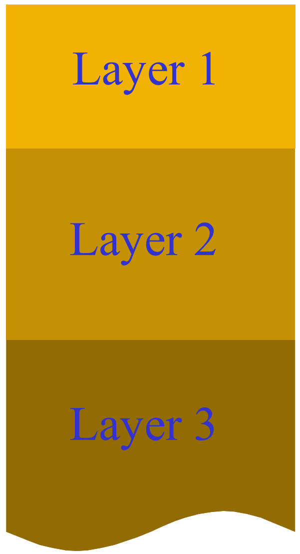
$$r := 1 - \frac{\beta^2}{\alpha^2}$$

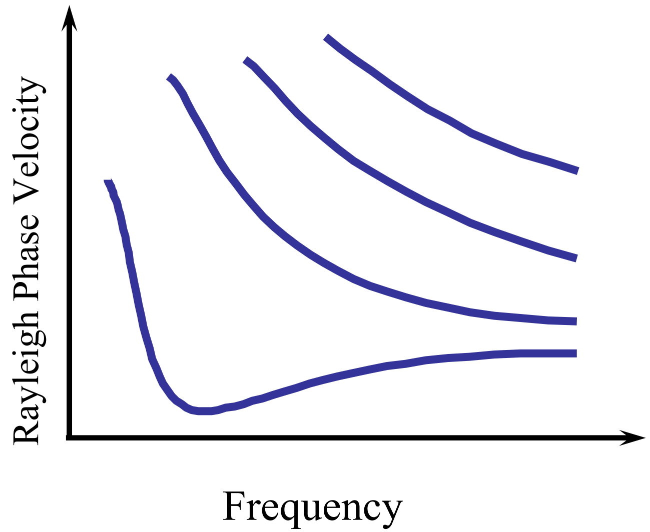
Why surface waves are appealing ?

- Some properties of surface waves which make them particularly
- suitable for material characterization:

HETEROGENEOUS CONTINUA

geometric dispersion





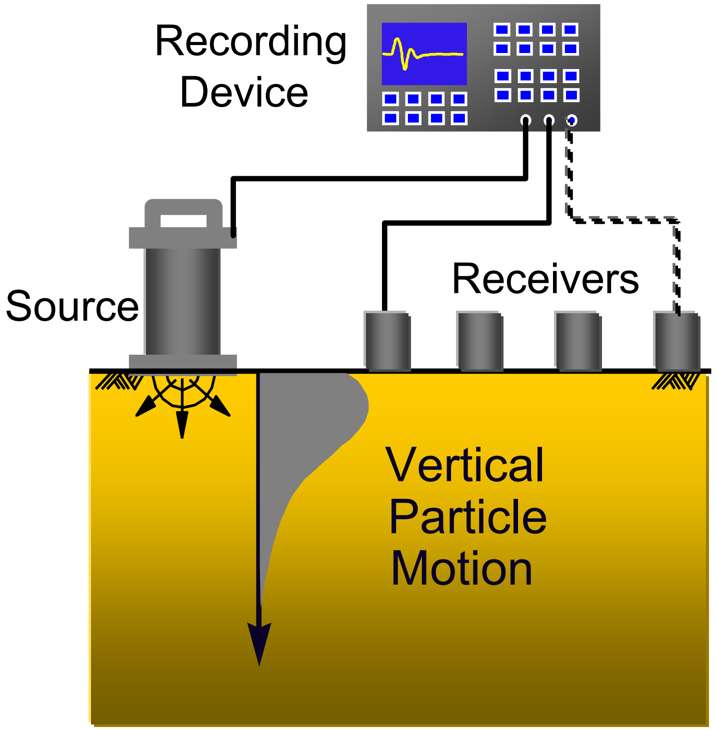
dispersion relation

$$F_R[\lambda(y), G(y), \rho(y), k_j, \omega] = 0$$

multi-mode wave propagation

Why surface waves are appealing ?

- Use these properties for development and setup of non-invasive testing
- for material and site characterization:



Acronyms: SSRM, SASW, CS-SASW, CSWS, MASW, SWM, ...

General features:

- Generation of surface waves
- Signal detection and elaboration
- Construction of experimental dispersion & attenuation curves
- Inversion of experimental curves $V_R(w)$ & $a_R(w)$ to obtain $V_S(y)$ & $D_S(y)$

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SEVENTH FRAMEWORK PROGRAMME

Rayleigh dispersion equation in homogeneous continua

PAVIA RISK CENTRE

Displacement field of Rayleigh waves

SECULAR EQUATION FOR RAYLEIGH WAVES

$$4k^2\mu\left[\left(k^2-\frac{\omega^2}{c_L^2}\right)\cdot\left(k^2-\frac{\omega^2}{c_T^2}\right)\right]^{1/2}-\left[2k^2-\frac{\omega^2}{c_T^2}\right]\cdot\left[\left(k^2-\frac{\omega^2}{c_L^2}\right)\cdot(\lambda+2\mu)-\lambda k^2\right]=0$$

substituting $k=\omega/c_R$ with $c_R=c_R(\nu)$ the propagation velocity of Rayleigh waves, function of the Poisson ratio:

⇩

$$\left(2-\frac{c_R^2}{c_T^2}\right)^2-4\left(1-\frac{c_R^2}{c_L^2}\right)^{1/2}\left(1-\frac{c_R^2}{c_T^2}\right)^{1/2}=0$$

c_R independent on the frequency

NOT DISPERSIVE waves

Displacement field of Rayleigh waves

- Rearranging, the secular equation becomes:

$$\left(\frac{c_R}{c_T}\right)^6 - 8\left(\frac{c_R}{c_T}\right)^4 + 8\left(\frac{c_R}{c_T}\right)^2 \cdot \left[1 + 2\cdot\left(1 - \frac{c_T{}^2}{c_L{}^2}\right)\right] - 16\left(1 - \frac{c_T{}^2}{c_L{}^2}\right) = 0$$

it reduces to a <u>cubic</u> equation admitting a closed form solution for c_R:

$$x^3 - 8x^2 + 8(1+2r)x - 16r = 0 \qquad \text{with: } x = \left(\frac{c_R}{c_T}\right)^2; \qquad r = 1 - \frac{c_T^2}{c_L^2}$$

⇓

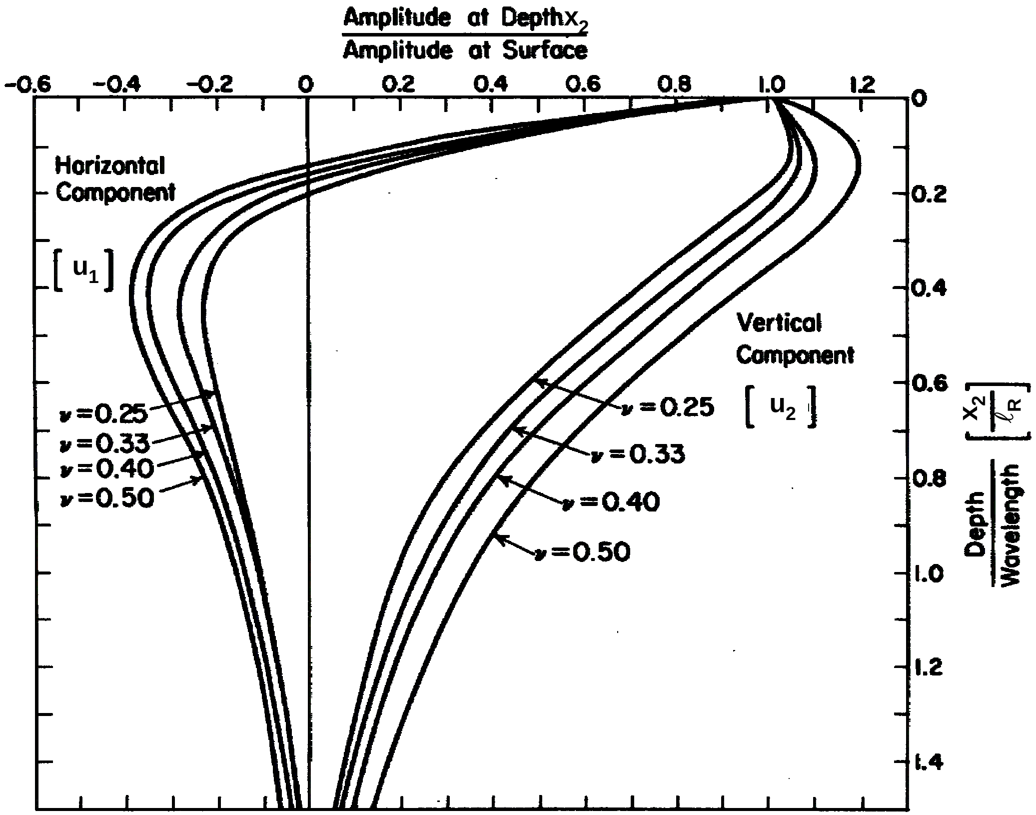
<u>one</u> admissible solution for $c_R(\nu)$, being two roots extraneous, arising from the rationalization process of squaring

- Since $0 < \nu < 0.5$

Approximate solution: $$\boxed{c_R = \frac{0.862 + 1.14\nu}{1+\nu} c_T}$$ ⇨ $$0.862 c_T < c_R < 0.955 c_T$$

c_R independent on the frequency ⇨ <u>**NOT DISPERSIVE**</u> waves

Displacement field of Rayleigh waves



$x_2 < 0.2\lambda_R$ elliptic *retrograde* motion

$x_2 > 0.2\lambda_R$ elliptic *prograde* motion

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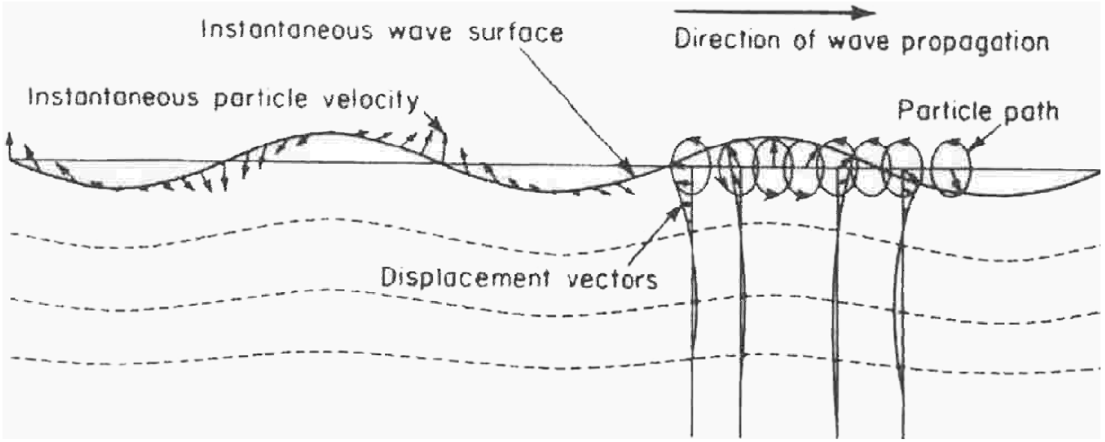
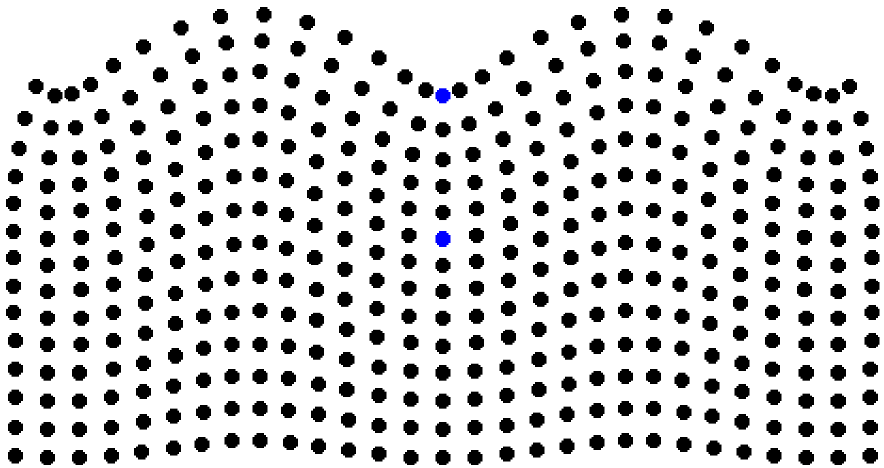
Displacement field of Rayleigh waves

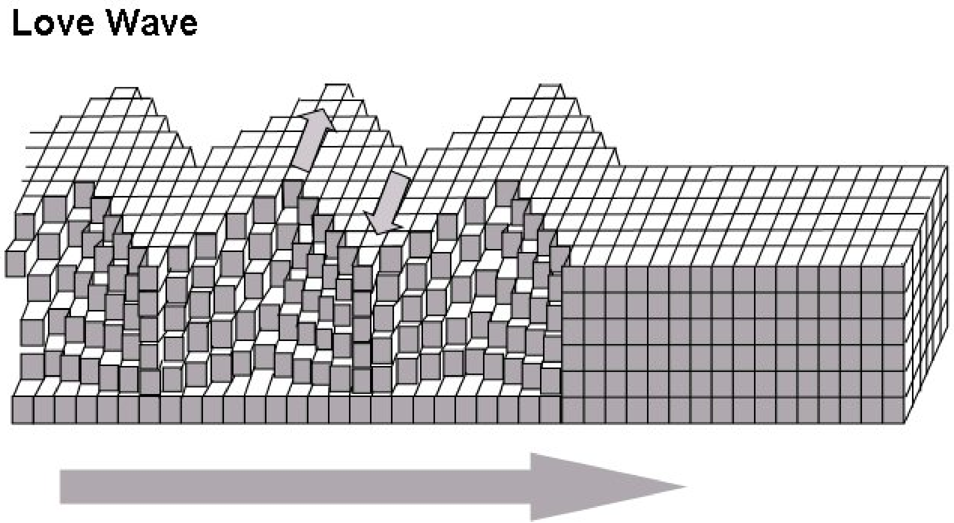




Love waves

Love waves

- They are confined at the free-surface of the earth crust ⇒ surface waves.
- They cause lateral shaking of the ground (horizontally polarized shear SH waves);
- Their velocity of propagation is slightly less that the phase velocity of SH waves;
- They exist only if the contrast of mechanical impedance of top layer with respect to half space obey certain rules.



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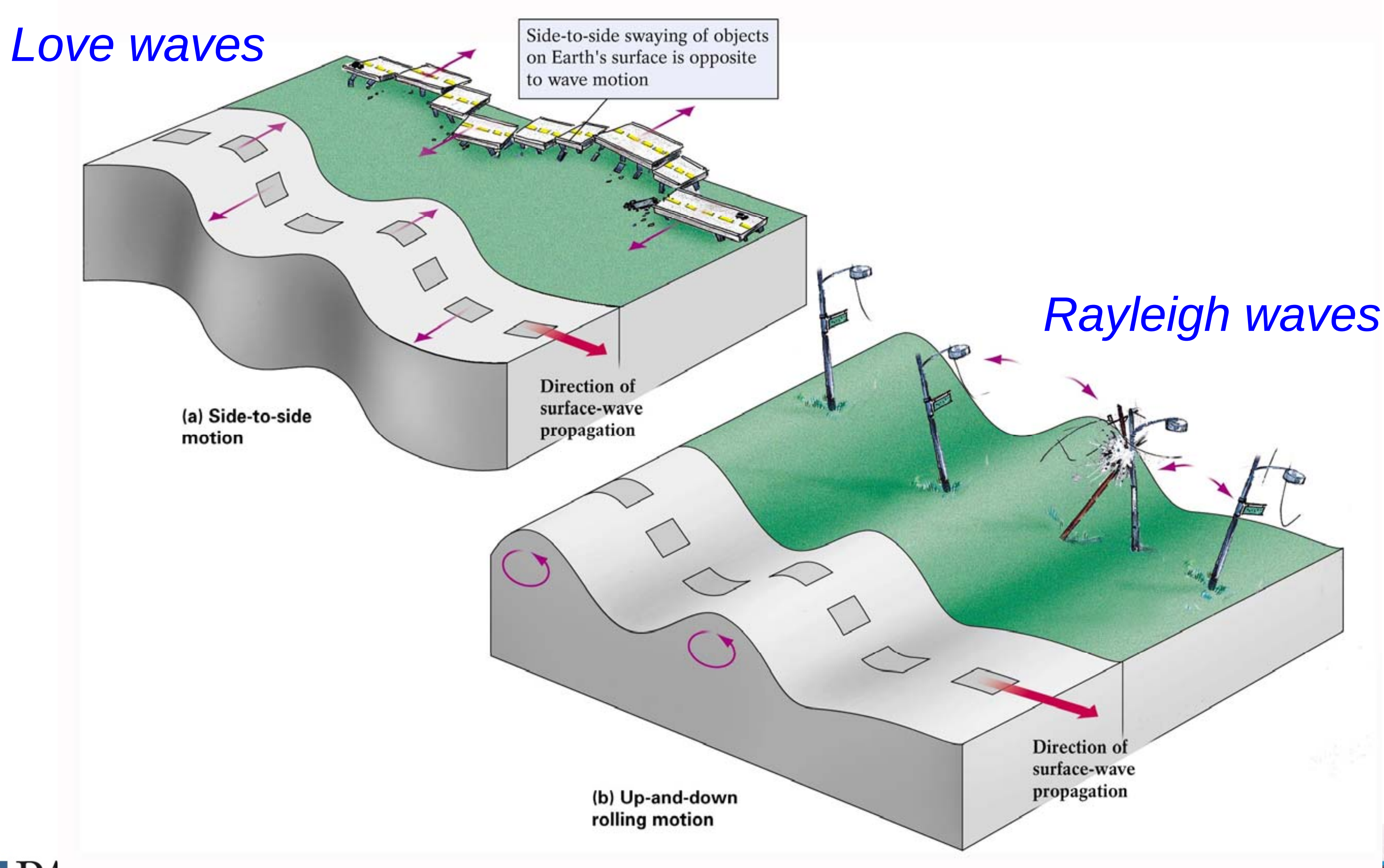
Rayleigh and Love waves



Rayleigh and Love waves

Dr. **Dan Russel**, Kettering University Applied Physics

http://www.kettering.edu/~drussell/Demos/reflect/reflect.html

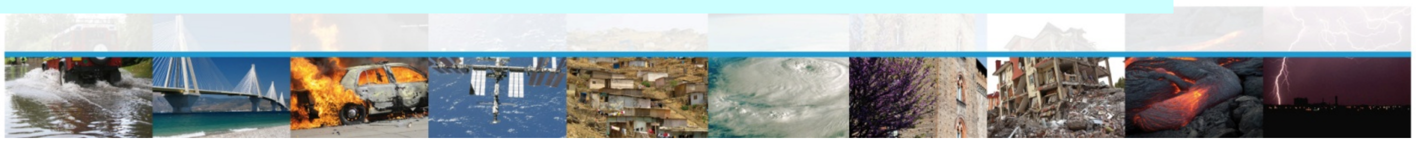
Longitudinal wave

Transversal wave

Mexican wave

Water wave

Rayleigh wave



Reflection and transmission for arbitrary incidence (more details)

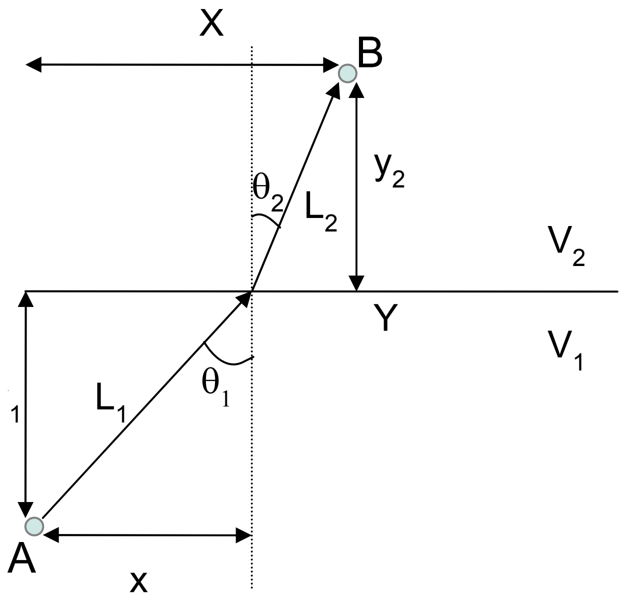
Reflection and transmission for arbitrary incidence

Fermat's principle or "*principle of least time*":

Principle in physics/optics stating that a mechanical or ELM ray takes the *shortest path* (therefore the shortest time) while travelling from point A to point B of a medium

In presence of an interface:

V_1 is different from V_2



$$L_1 = \sqrt{x^2 + y_1^2} \qquad \sin(\theta_1) = \frac{x}{\sqrt{x^2 + y_1^2}}$$

$$L_2 = \sqrt{y_2^2 + (X - x)^2} \qquad \sin(\theta_2) = \frac{X - x}{\sqrt{(X - x)^2 + y_2^2}}$$

According to Fermat's principle, total distance travelled by the ray must be *minimized*. In mathematical notation:

$$t_{total}(x) = \frac{L_1}{V_1} + \frac{L_2}{V_2} \qquad \frac{dt_{total}(x)}{dx} = \frac{1}{2} \cdot \left[\frac{x}{V_1\sqrt{x^2 + y_1^2}} - \frac{X - x}{V_2\sqrt{(X - x)^2 + y_2^2}} \right] = 0$$

$$\left[\frac{x}{V_1\sqrt{x^2 + y_1^2}} - \frac{X - x}{V_2\sqrt{(X - x)^2 + y_2^2}} \right] = \left[\frac{\sin(\theta_1)}{V_1} - \frac{\sin(\theta_2)}{V_2} \right] = 0$$

$$\frac{\sin(\theta_1)}{V_1} = \frac{\sin(\theta_2)}{V_2}$$

⇦ *Snell's law!*

Reflection and transmission for arbitrary incidence

When incident **SH-waves** are impinging at an interface, they reflect and transmit through the boundary as SH-waves with angles obeying at Snell's law

Snell's law

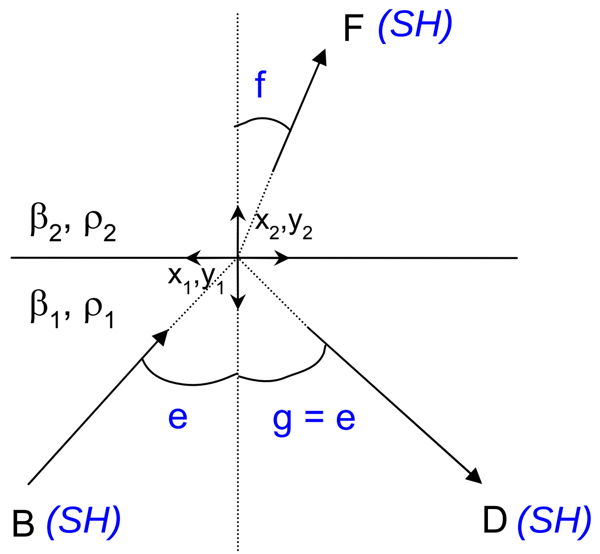
$$\frac{\sin(\theta_1)}{V_1} = \frac{\sin(\theta_2)}{V_2}$$

Number of unknowns: 2 (D, F)

Number of equations: 1 continuity of displacement
1 continuity of stress [$r b^2 d(w)/dy$] } **@ interface**

Corresponding angles are found by applying Snell's law

$$\begin{cases} w_i = B\left[\sin^2(b)\cdot e^{iw\left(t-\frac{x_1}{\beta_1 \sin(b)}\right)} + \cos^2(b)\cdot e^{iw\left(t-\frac{y_1}{\beta_1 \cos(b)}\right)}\right] \\ w_r = D\left[\sin^2(d)\cdot e^{iw\left(t-\frac{x_1}{\beta_1 \sin(d)}\right)} + \cos^2(d)\cdot e^{iw\left(t+\frac{y_1}{\beta_1 \cos(d)}\right)}\right] \\ w_t = F\left[\sin^2(f)\cdot e^{iw\left(t+\frac{x_2}{\beta_2 \sin(f)}\right)} + \cos^2(f)\cdot e^{iw\left(t+\frac{y_2}{\beta_2 \cos(f)}\right)}\right] \end{cases}$$



For incident *SH*-wave— $B + D - F = 0$

Zoeppritz equations $B - D - \frac{\rho_2}{\rho_1}\frac{v_{S2}}{v_{S1}}\frac{\cos f}{\cos b} F = 0$

Reflection and transmission for arbitrary incidence

When incident **P-waves** are impinging at an interface, 4 resulting waves are generated:

- 2 P-waves (one reflected, one transmitted)
- 2 SV-waves (one reflected, one transmitted)

Snell's law

$$\frac{\sin(\theta_1)}{V_1} = \frac{\sin(\theta_2)}{V_2}$$

Number of unknowns: 4 (C, D, E, F)

Number of equations: 2 continuity of displacement
2 continuity of stress (normal and shear stress) } **@ interface**

Corresponding angles are found by applying Snell's law:

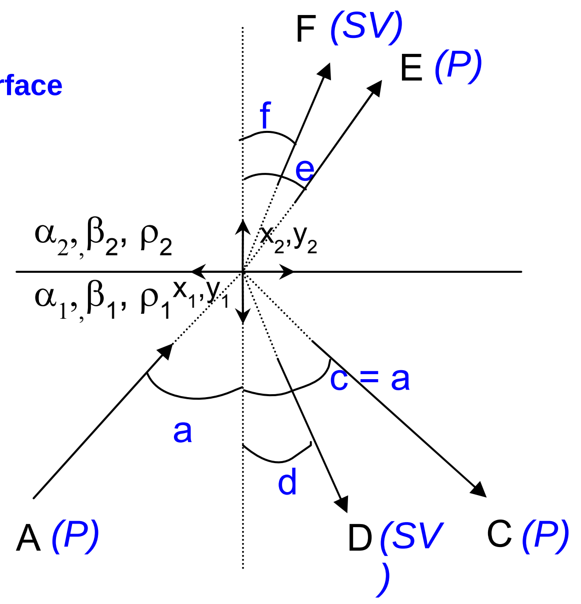
For incident *P*-wave—

$$(A - C)\sin a + D\cos b - E\sin e + F\cos f = 0$$

$$(A + C)\cos a + D\sin b - E\cos e - F\sin f = 0$$

$$-(A + C)\sin 2a + D\frac{v_{P1}}{v_{S1}}\cos 2b + E\frac{\rho_2}{\rho_1}\left(\frac{v_{S2}}{v_{S1}}\right)^2\frac{v_{P1}}{v_{P2}}\sin 2e - F\frac{\rho_2}{\rho_1}\left(\frac{v_{S2}}{v_{S1}}\right)^2\frac{v_{P1}}{v_{S2}}\cos 2f = 0$$

$$-(A - C)\cos 2b + D\frac{v_{S1}}{v_{P1}}\sin 2b + E\frac{\rho_2}{\rho_1}\frac{v_{P2}}{v_{P1}}\cos 2f + F\frac{\rho_2}{\rho_1}\frac{v_{S2}}{v_{P1}}\sin 2f = 0$$



(from *Richarts et al., 1970*)

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Reflection and transmission for arbitrary incidence

When incident **SV-waves** are impinging at an interface, 4 resulting waves are generated:

- 2 P-waves (one reflected, one transmitted)
- 2 SV-waves (one reflected, one transmitted)

Snell's law

$$\frac{\sin(\theta_1)}{V_1} = \frac{\sin(\theta_2)}{V_2}$$

Number of unknowns: 4 (C, D, E, F)

Number of equations: 2 continuity of displacement
2 continuity of stress (normal and shear stress) } **@ interface**

Corresponding angles are found by applying Snell's Law

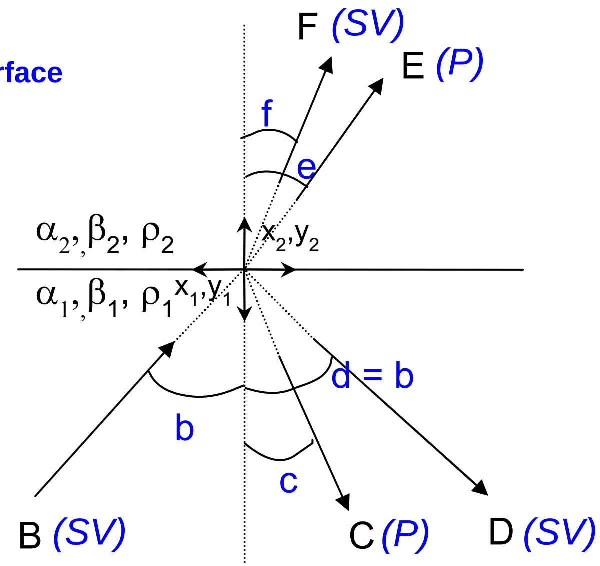
For incident SV-wave—

$$(B + D)\sin b + C\cos a - E\cos e - F\sin f = 0$$

$$(B - D)\cos b + C\sin a + E\sin e - F\cos f = 0$$

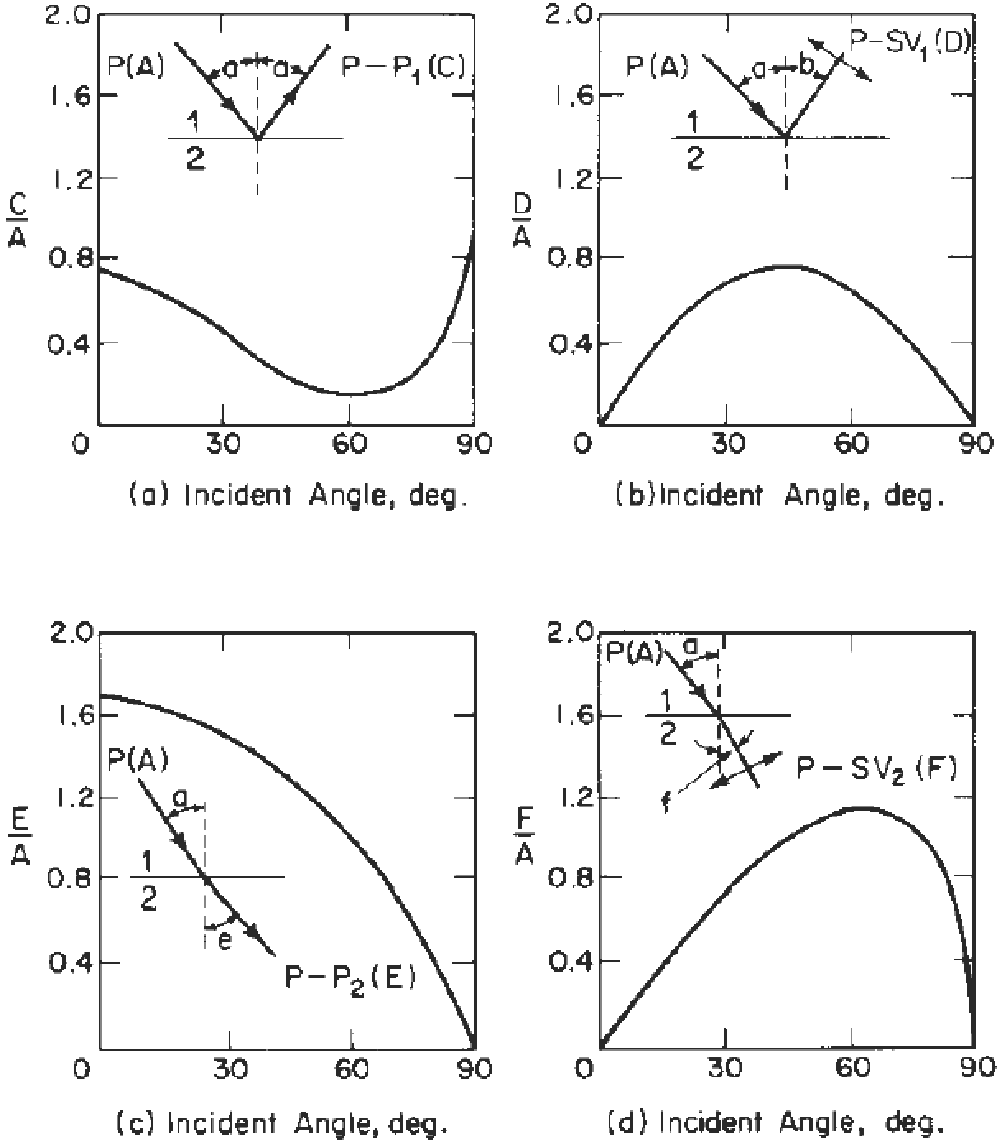
$$(B + D)\cos 2b - C\frac{v_{S1}}{v_{P1}}\sin 2a + E\frac{\rho_2}{\rho_1}\frac{v_{S2}^2}{v_{S1}v_{P2}}\sin 2e - F\frac{\rho_2}{\rho_1}\frac{v_{S2}}{v_{S1}}\cos 2f = 0$$

$$-(B - D)\sin 2b + C\frac{v_{P1}}{v_{S1}}\cos 2b + E\frac{\rho_2}{\rho_1}\frac{v_{P2}}{v_{S1}}\cos 2f + F\frac{\rho_2}{\rho_1}\frac{v_{S2}}{v_{S1}}\sin 2f = 0$$



(from *Richarts et al., 1970*)

Reflection and transmission for arbitrary incidence

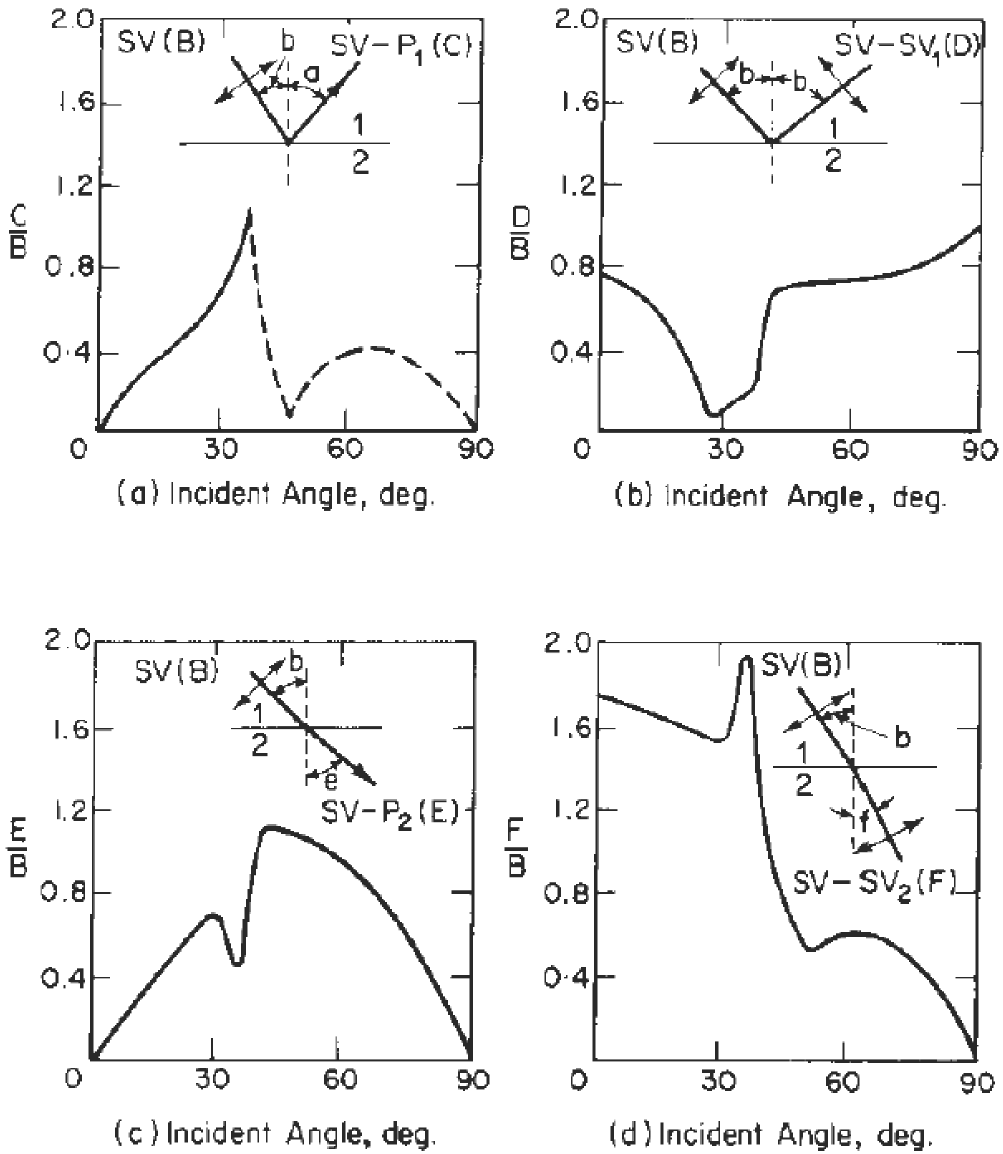


P-WAVES

	Amplitud	Angle
Incident P	A	*a*
Incident SV	B	*b*
Reflected P	C	*c*
Reflected SV	D	*d*
Transmitted P	E	*e*
Transmitted SV	F	*f*

(from *Richarts et al., 1970*)

Reflection and transmission for arbitrary incidence



SV-WAVES

	Amplitud	Angle
Incident P	A	*a*
Incident SV	B	*b*
Reflected P	C	*c*
Reflected SV	D	*d*
Transmitted P	E	*e*
Transmitted SV	F	*f*

dashed line above indicate evanescent waves !

(from *Richarts et al., 1970*)