Basics of signal processing, design of specimens, system acquisition

Analysis of signals

Dr. Simone Peloso

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Systems and signals

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Signals convey information. Systems transform signals

SIGNALS are functions that carry information, in the form of temporal and spatial patterns. In a broad definition, the concept of signal encompass virtually any data that can be represented as an organized collection of data.

SYSTEMS are functions that transform signals.

SIGNAL PROCESSING concerns primarily with signals and systems operating on signals to extract useful information. Purposes can be different:

- Describe the significant content of the signal
- Correlate different variables with causality relationships
- Find out the significant parameters of the phenomenon
- Analyse and identify the system producing the signal
- Reduce data noise
- Separate different components of the signal

…

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Example: noise filtering

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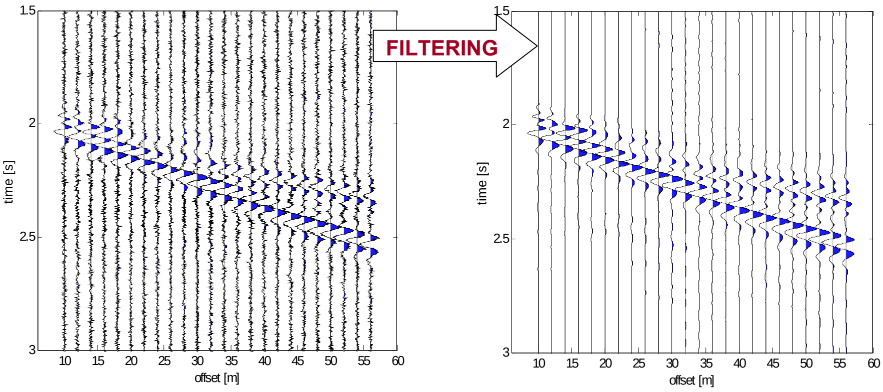
From a series of vibration signals information is found on propagating phenomena and on propagation medium



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Example: frequency domain and spectrum

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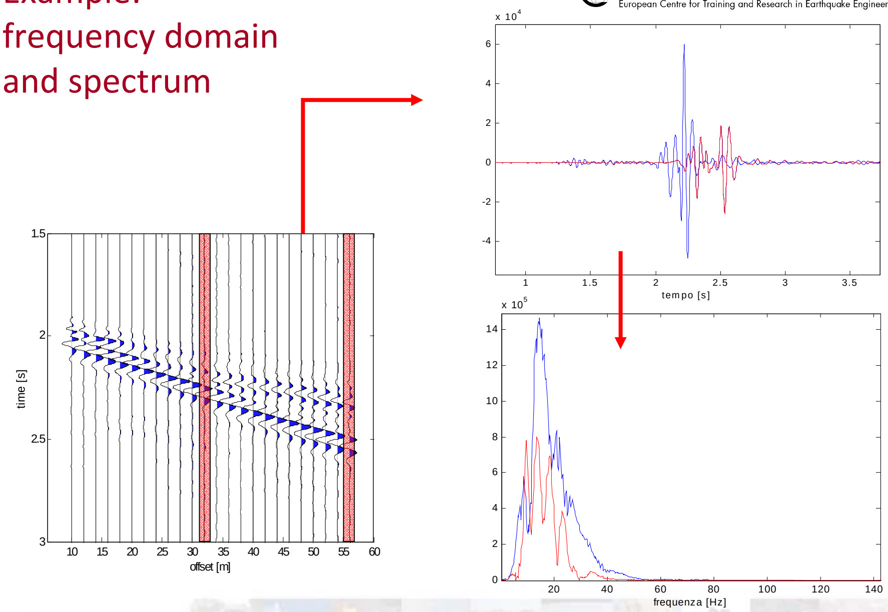

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Example: cross-correlation, time delay, synchronization

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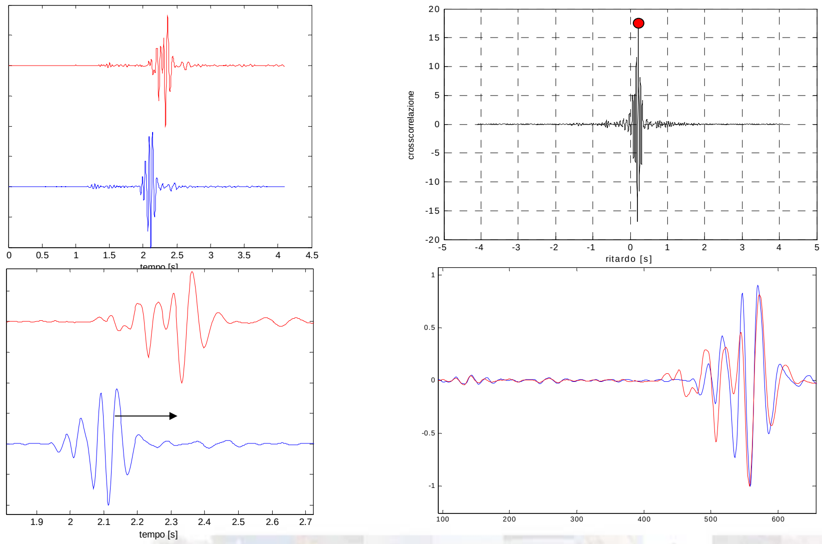

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Example: cross-spectrum between signals

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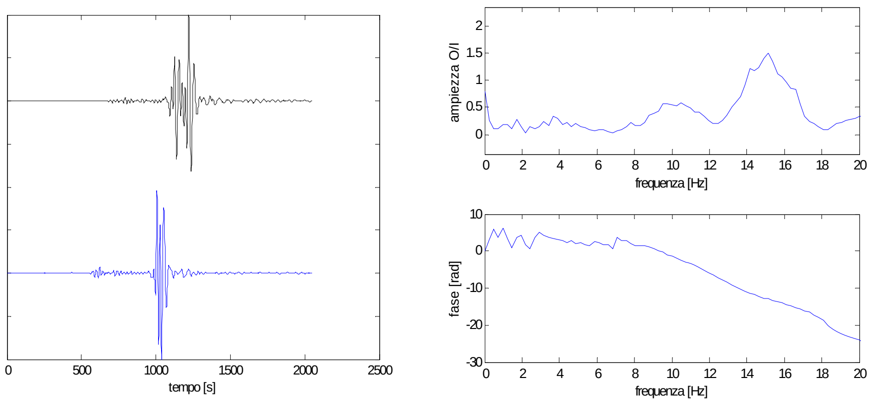

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Example: time (space) - frequency analysis

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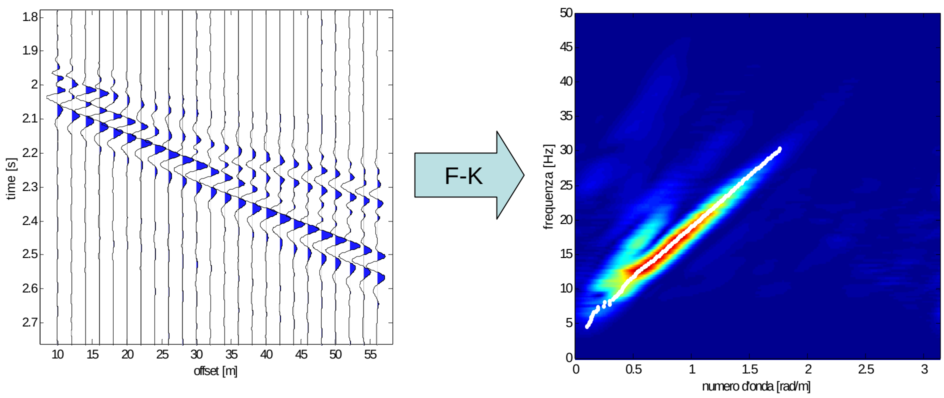

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Where to look for information?

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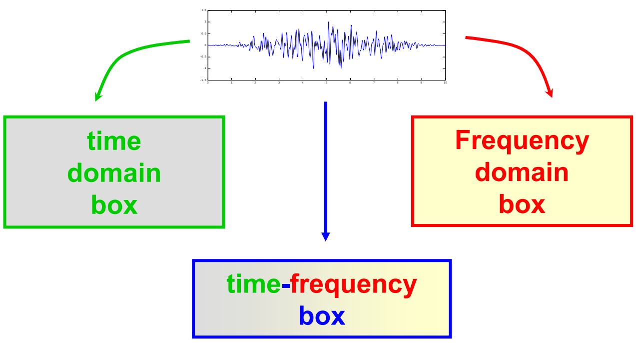
In the signal! All the information is contained in the signal



→ what changes is just the way we extract the information

→ you do not add information looking at the signal in one or in the other way

→ The domain in which we extract information is just a “dress” → the easiest way to get what we need to know

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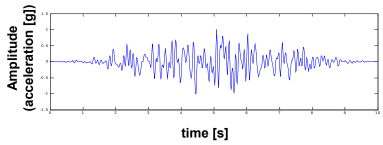
Time vs frequency domain

Signal processing can be based on the
TIME or FREQUENCY DOMAIN

A signal can be described (equivalently?) both in the time and frequency domain.



The FREQUENCY DOMAIN allows a synthetic and simple representation of the characteristics of the signal and of the system producing the signal.

The idea is that arbitrary signals can be described as sums of sinusoidal signals.

But the real justification for the frequency domain approach is that it turns out to be particularly easy to understand the effect that LTI systems (linear time invariant systems) have on signals.

Although few (if any) real-world systems are truly LTI, models can be easily constructed where the approximation is valid over some regime of operating conditions.

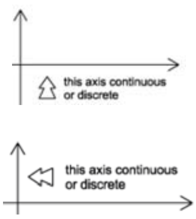
Signals

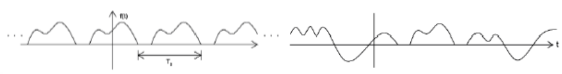
The transducer is the physical interface converting the information from its physical form (acceleration, temperature, pressure…) into a correlated electrical quantity (voltage, power…), called “SIGNAL”.

The physical event is continuous (in space or time), but is recorded in a discrete form.

Signals can be classified in:

1. continuous/discrete
2. analog/digital
3. periodic/aperiodic

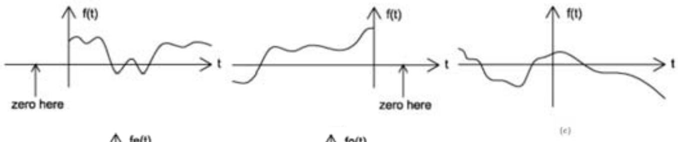




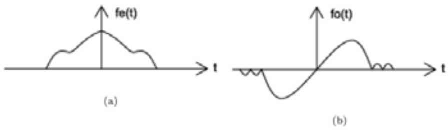
(a) (b)

Signals

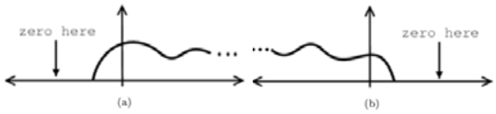
4. causal / anticausal / noncausal



5. even/odd



6. right-handed / left-handed



7. finite / infinite Length
8. deterministic/random

All signals can be decomposed into a sum of aperiodic or periodic components or into a sum of even and odd components

Deterministic / Random

Deterministic: the measured quantity can be explicitly described by mathematical relationships

Random: it is not possible to exactly predict the signal in the future

Deterministic periodic signals

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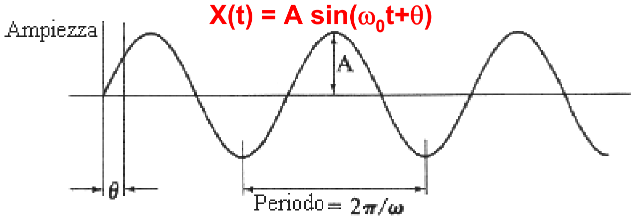
Periodic signals repeat with some period T. The fundamental period of the function, f (t), is the smallest value of T such that:

f (t) = f (T + t)

The elementary sine function is the base of the frequency analysis

$X(t) = A \sin(\omega_0 t + \theta)$



It is often convenient to consider aperiodic signals as periodic of a period corresponding to their entire duration

Frequency, amplitude, phase

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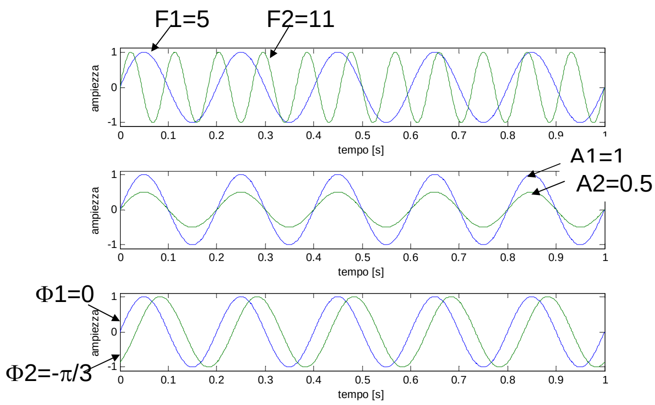
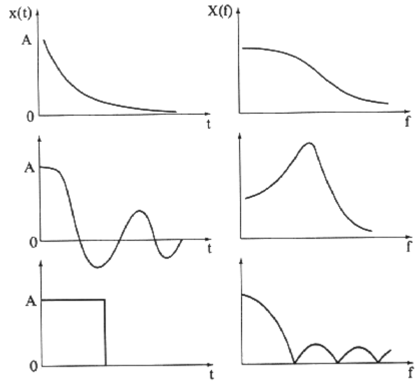
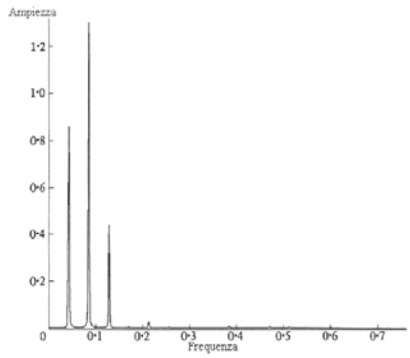


IN SPACE: the distance between two point with the same value is the wave length λ (corresponding to T in time) and $\kappa = 2\pi/\lambda$ is the wave number (corresponding to the angular frequency $\omega = 2\pi/T$ in time)

Transient signals

Transient signals are signals variable in time, reducing to zero value after a finite time interval





Random signals: ensemble averages

Random signals are constituted by an ensemble of signals representing a given phenomenon.

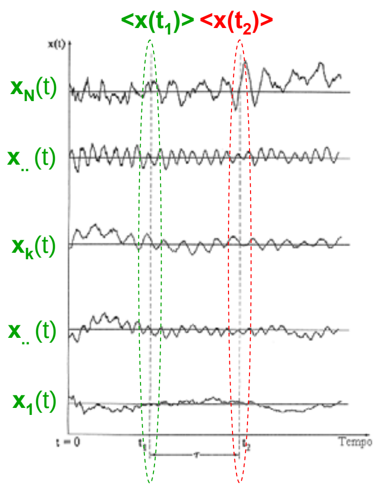
To describe random signals a statistic/probabilistic description is used, by means of the

ENSEMBLE AVERAGES (<.>)

1. Mean value
2. Autocorrelation function

MEAN VALUE: defined per each time instant as the average of all the signals

$$\langle x(t) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t)$$



Random signals: ensamble averages

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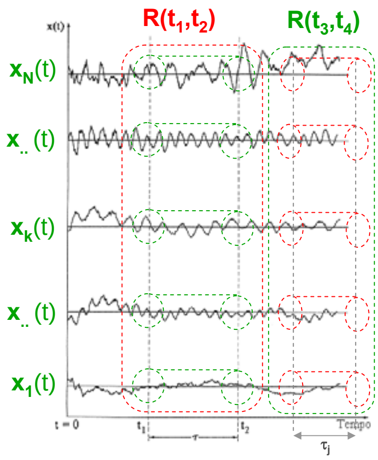
AUTOCORRELATION FUNCTION:

Between two given instants t_1 and t_2, is the mean across all the ensemble of the product of each signal at t_1 and t_2.

$$R(t_1,t_2) = \langle x(t_1)x(t_2)\rangle$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x_k(t_1)\,x_k(t_2)$$

The autocorrelation function is a representation of the dependence of actual values of the signal from past values, and of future values from actual values.



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Temporal averages

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A physical event can be captured in multiple time segments, thus forming an ensemble of signals.

Unfortunately, often only 1 time series is available, which corresponds to 1 realization of the process (e.g. a single transducer signal).

TEMPORAL AVERAGES are defined as:

$$\bar{x}(t) = \frac{1}{T}\int_0^T x(t)dt$$

$$R(t,t+h) = \frac{1}{T-h}\int_0^{T-h} x(t)x(t+h)dt$$

R(t,t) is the mean squared value of the process, which is a measure of the energy carried out by the signal:

$$\bar{x}^2(t) = \frac{1}{T}\int_0^T x^2(t)dt = R(0)$$

Temporal averages have significance for a particular class of random processes: stationary ergodic processes

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Stationary (or homogeneous) random process

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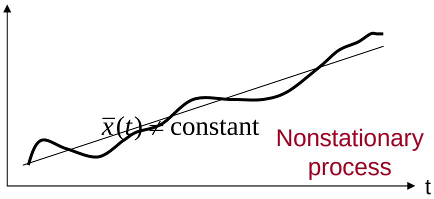
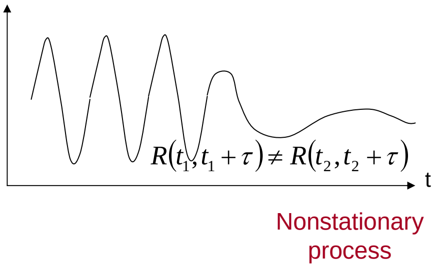
STATIONARY PROCESS: the probabilistic description of the process is independent of a shift of the origin in time.



ERGODIC PROCESS: ensemble statistics are the same as temporal statistics of any record

$\bar{x}(t)$ is constant per each time interval considered and is equal to the ensemble mean

R(τ) depends only on τ.



For stationary ergodic process ensemble and temporal averages coincides, and all the statistic quantities can be obtained from measurements of just one record, instead of averages across an ensemble of records

Stationary signals

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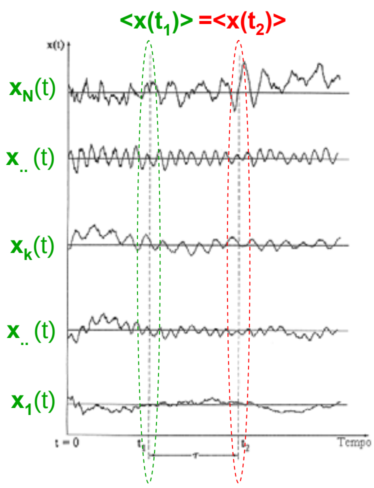
This is a particular class of signals with:

1. time-independent mean value
2. Autocorrelation function depending just on the time delay $\tau = t_2 - t_1$

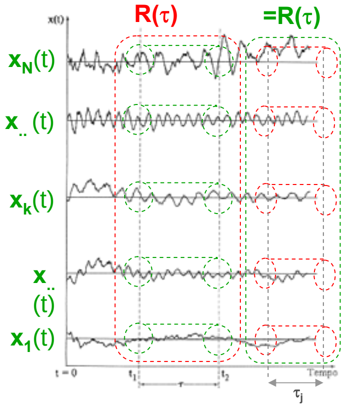
$$\langle x(t_1)\rangle = \langle x(t_2)\rangle$$

$$R(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t)\, x_k(t+\tau)$$



Stationary signals

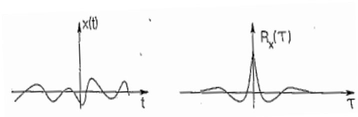
$$\left(\begin{array}{l} R(\tau = t_1 - t_2) = R(\tau = t_3 - t_4) = R(\tau) \\ \Rightarrow \quad R(0) = \text{constant} \\ \text{but} \quad R(\tau_i) \neq R(\tau_k) \\ \text{if} \quad \tau_i \neq \tau_k \end{array}\right)$$



In this case the autocorrelation function represents "how fast" the signal changes.

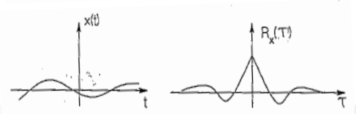
A very fast changing signal will show high values of R(τ) for small time delay τ (i.e. R(τ) strictly close to the axis origin)

1)



A very slowly changing signal will show non-zero values of R(τ) also for larger time delay τ (i.e. R(τ) more spread)

2)



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Amplitude analysis

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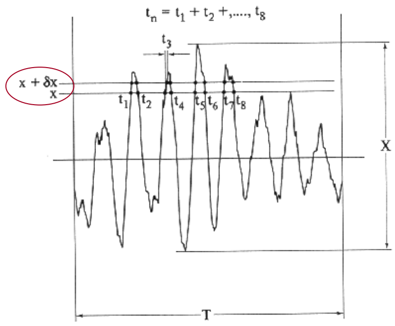
In order to understand the variable nature of the process, it has to be defined the probability distribution function of the process, which is the relative likelihood of the different outcomes of the random variable

i.e. per each value x , the total time in which the signal has values between x and x +dx :

$$t_n(x) = \sum_{p=1}^{k} t_p(x, x + dx)$$



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Amplitude analysis: probability mass function

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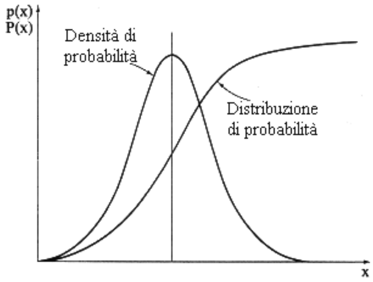
The pdf or pmf (probability density/mass function, for continuous or discrete signals, respectively) $p_x(x_i)$ defines the probability that a signal has values between x_i and x_i+dx

$$p_x(x_i) = \lim_{T\to\infty} \frac{t_n(x_i)}{T}$$

$$P_x(x) = \sum_{x_i \le x} p_x(x_i)$$

The cdf (cumulative density function) $P_x(x)$ represents the probability that a signal has values smaller than x.

Such definitions can be applied to joint data ensembles x, y, z … , defining in the same way the joint pdf and joint cdf



$$P(x,y,z,...) = \int_{-\infty}^{...} ... \int_{-\infty}^{z}\int_{-\infty}^{y}\int_{-\infty}^{x} p(x,y,z,...)dxdydz...$$

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Time domain analysis

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Means and probabilistic measures cannot capture the periodic nature of a given signal. A better estimator of the time characteristics of a signal is the

AUTOCORRELATION FUNCTION

$$R_{xx}(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t)\, x_k(t+\tau)$$

1. R(0) is the mean square value, i.e. the mean energy of the signal

2. R(τ) allows identifying internal timescales within a signal, such as repetitive patterns. Such patterns manifest as peak of R(τ).

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Identifying similarities: cross-correlation function

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Autocorrelation is a particular case of the CROSS-CORRELATION FUNCTION, which is a robust operation that allows identifying similarities between signals even in presence of noise.

$$R_{xy}(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t)\, y_k(t+\tau)$$

As it is defined, such function per each time delay τ integrates the product of the two signals shifted of τ, and it is clearly maximized when the two shifted signals are similar at most.

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Cross-correlation function

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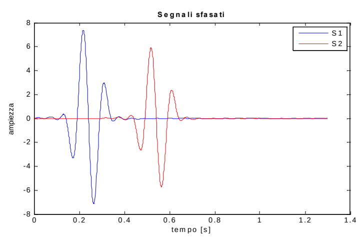
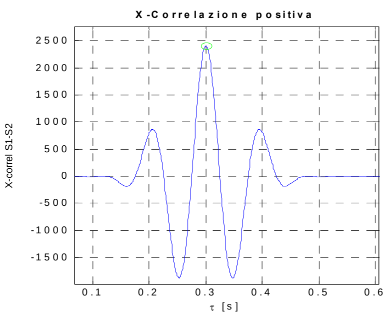
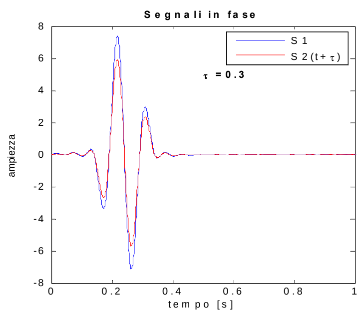


The cross-correlation allows finding the time delay by which the similitude between two signals is maximized → e.g. can be used for signal synchronization





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Cross-correlation

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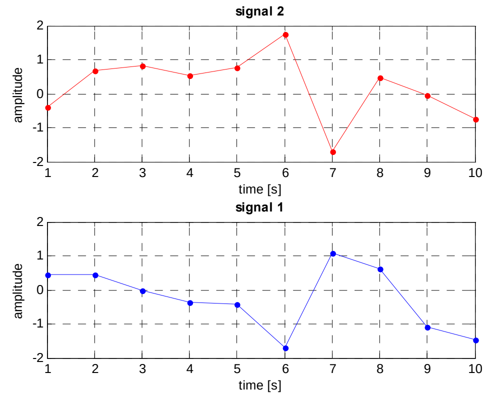
Per each the time delay τ, the X-correlation between two signals is the sum of the product of the first signal times the second shifted of τ

If the signals are of length L, the cross correlation will be long 2L-1

$$R_{xy}(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t)\, y_k(t+\tau)$$

$$R_{xy}(\tau) = \frac{1}{T}\int_0^T x(t)y(t+\tau)dt = \langle x(t)y(t+\tau)\rangle$$



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Cross-correlation computation

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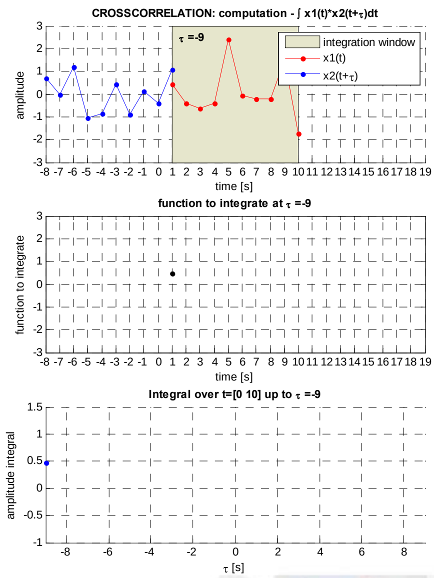

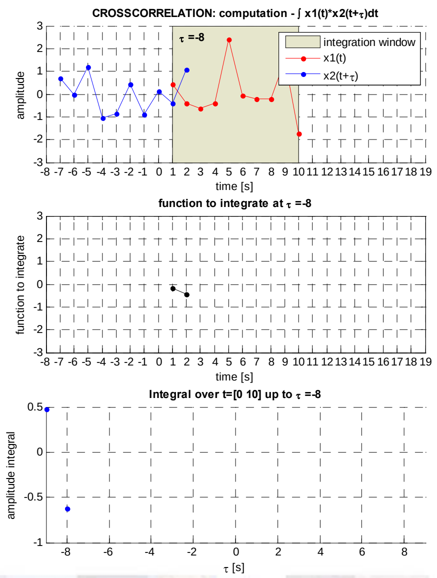

Cross-correlation computation

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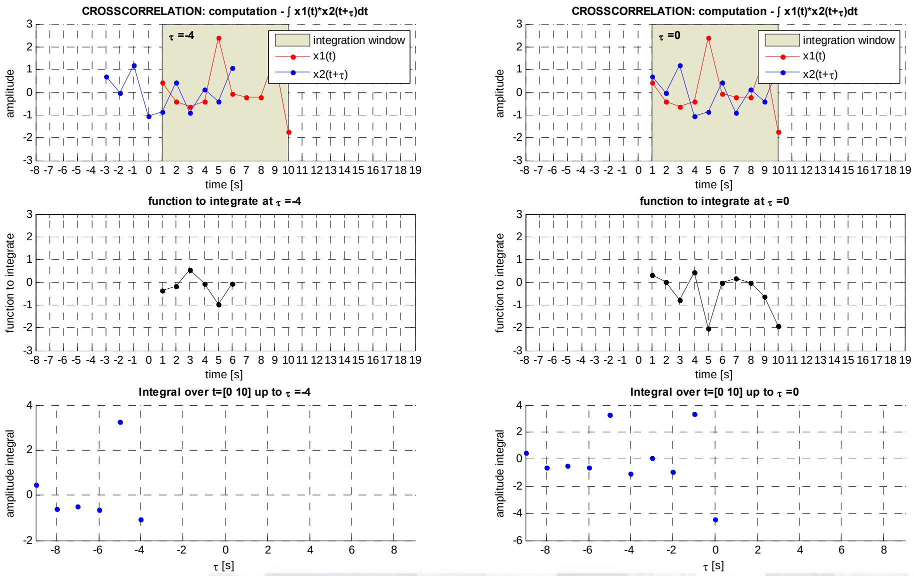

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Cross-correlation computation

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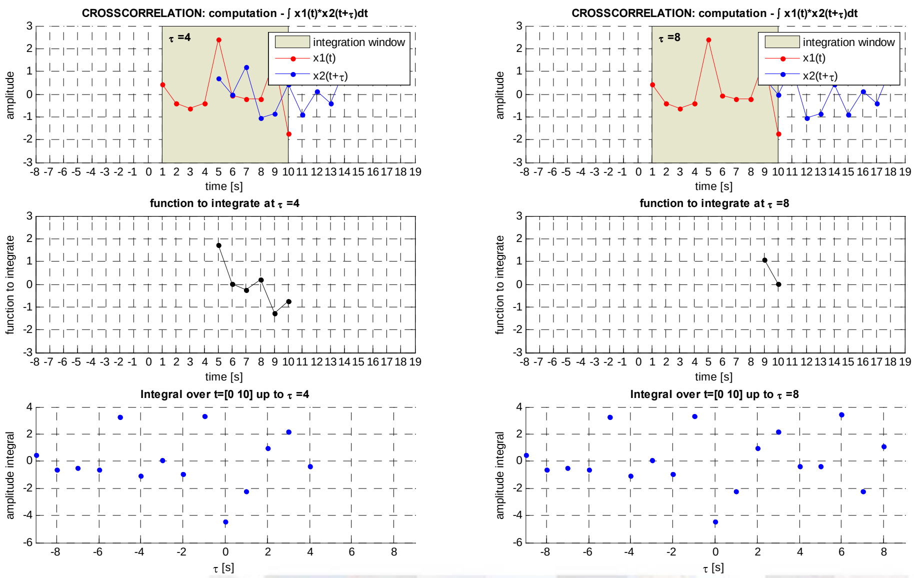

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Cross-correlation computation

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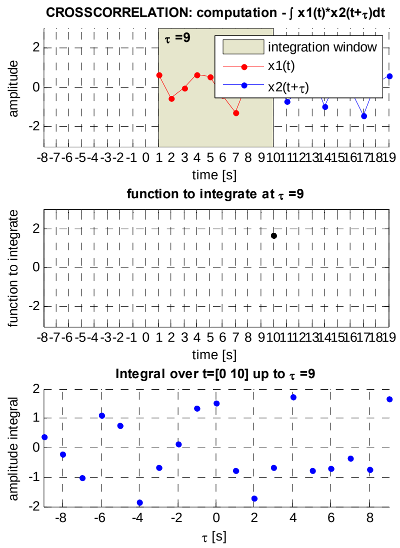

The signal length was 10 points

→ The x-correlation length is 19 points [-9:1:9]

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Cross correlation to synchronize signals

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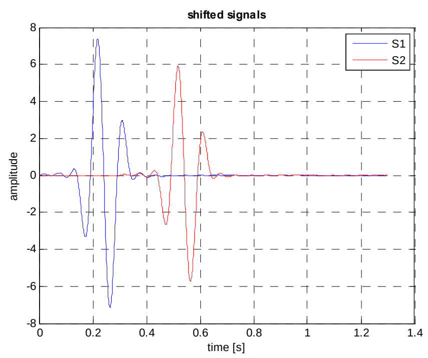
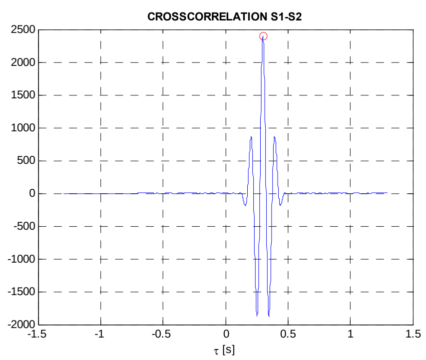
(demo_02)





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Negative cross correlation

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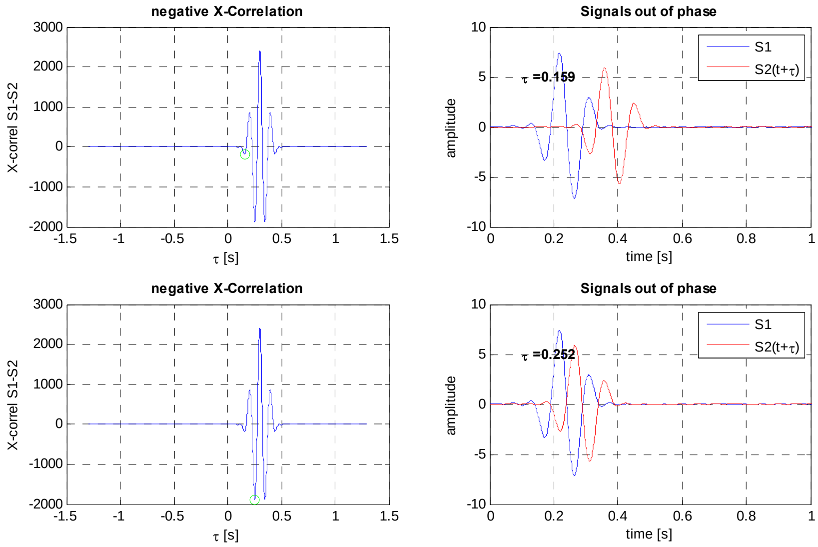

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Positive cross correlation

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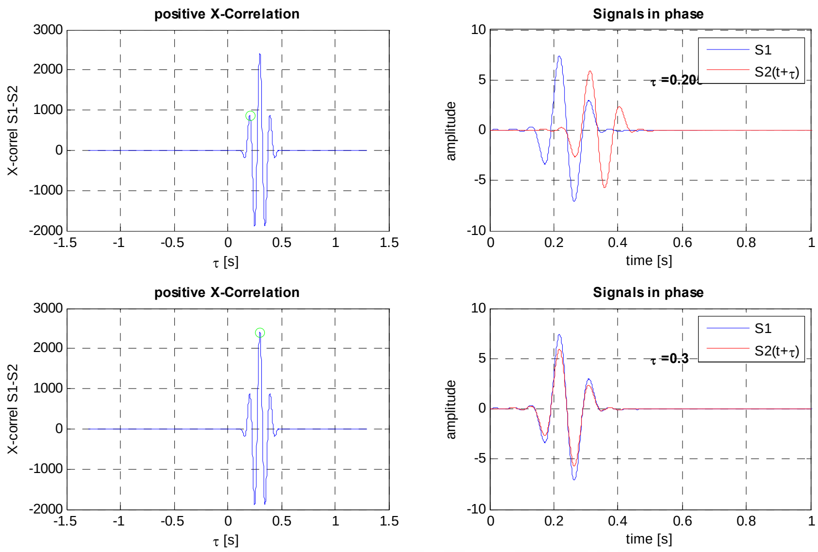

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Time domain operations: integration and differentiation

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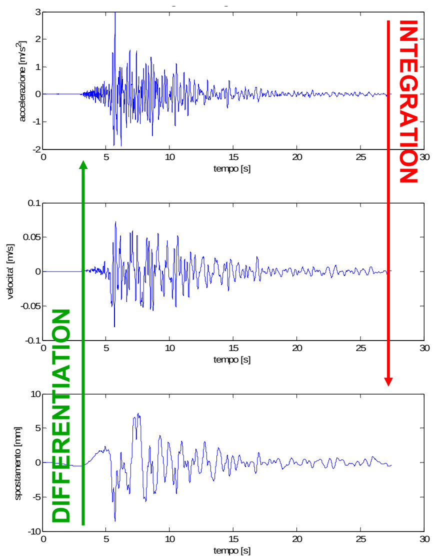
Given acceleration or displacement signals, it is theoretically possible to get displacement and acceleration, respectively by differentiating or integrating the acquired signal.

However such operation can result in undesired outcomes:

1. Numerical integration of a sampled signal enhance the low frequency content
2. Numerical differentiation enhance the high frequency content



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Application: the TVC controller

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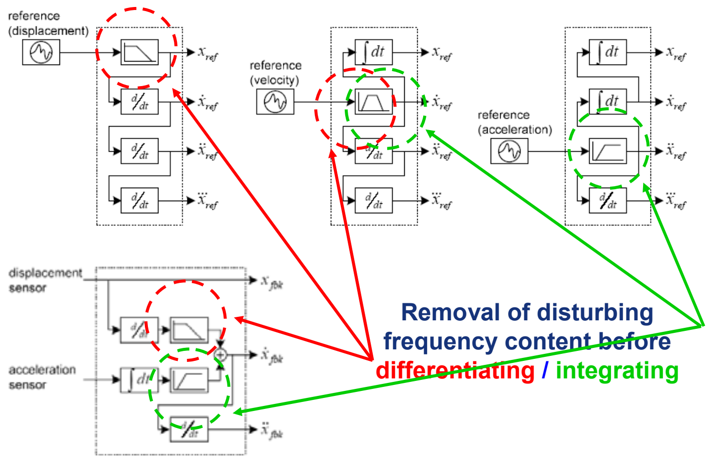
In order to avoid such shortcomings, often the integration/differentiation of signals is associated with other operations:

TVC reference generator

TVC feedback generator



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Noise

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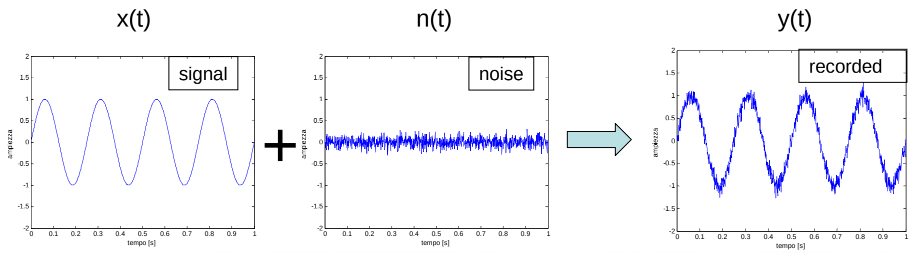
Generally, the signal is affected by the presence of noise which can be related to physical, mechanical, electrical phenomena.

The NOISE is any random signal interacting with the original signal and affecting its clearness.

Given the signal amplitude Vs and the noise amplitude Vn, the signal-to-noise-ratio (SNR) is defined as:

$$SNR = \left(\frac{V_S}{V_N}\right)^2 \quad \text{or} \quad SNR[dB] = 20\,log_{10}\left(\frac{V_S}{V_N}\right)$$



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Signal detrending and spike removal

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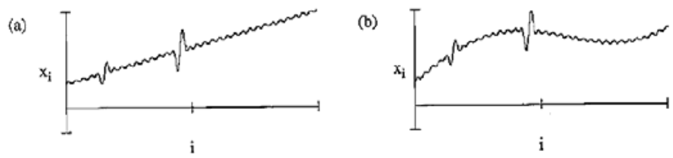
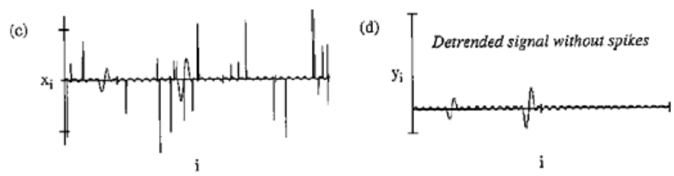
Low-frequency noise can be removed by signal de-trending in the time domain, by least squares fitting a low frequency function to the noisy signal and subtracting the trend from the measurement.



Spikes are impulses randomly distributed along the signal, which can be 'clipped', or removed and replaced by locally compatible signal values



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Noise control in the time domain: signal stacking

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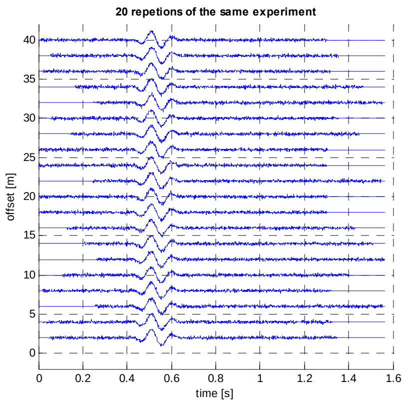
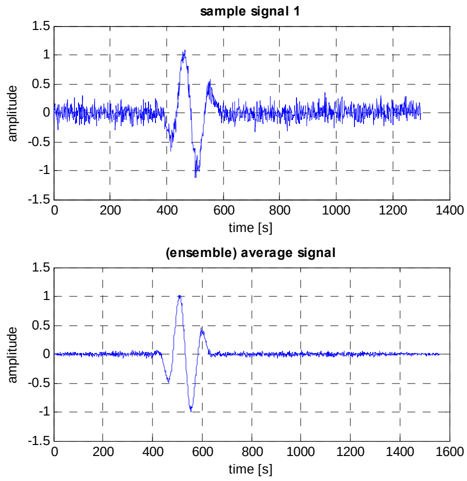
Signal stacking improves SNR and resolution. It consist in measuring the signal multiple times and averaging across the ensemble.

Considering M measurements:

$$SNR \propto \sqrt{M}$$





Xcorrelation and signal stacking to eliminate noise

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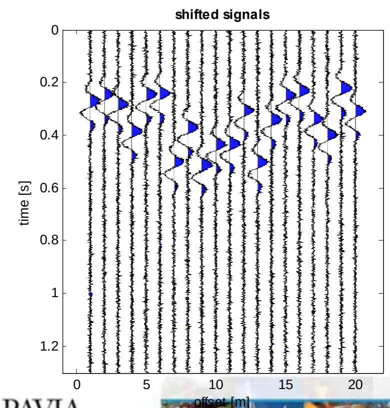
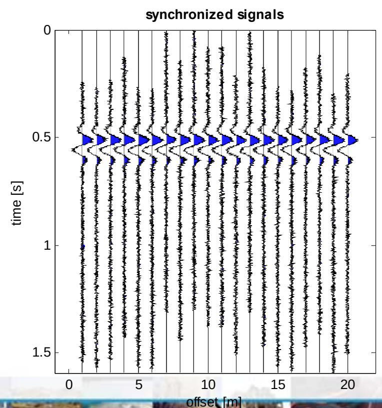
(demo_03) Consider 20 repetitions of the same experiment:

1. Synchronize measurements (according to delay corresponding to the maximum cross-correlation)





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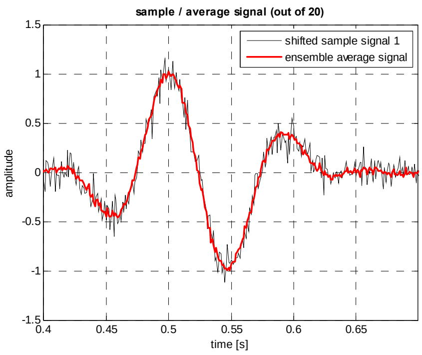
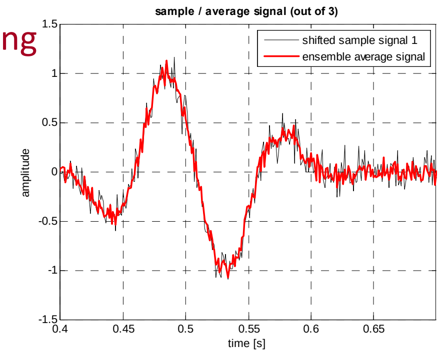
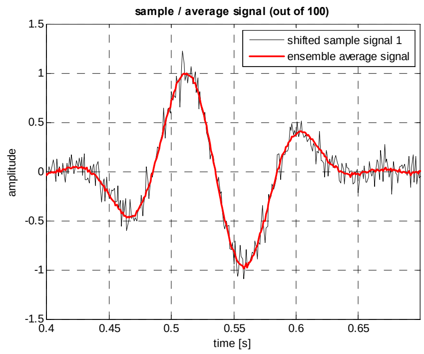
Xcorrelation and signal stacking to eliminate noise

2. Average the signal across the ensemble







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Frequency domain analysis: the Fourier Series

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Discrete time signals can be decomposed into a linear combination of sine functions:

$$x(t_i) = \sum_{n=-\infty}^{+\infty} a_n \cos\left(n\frac{2\pi}{T}t_i\right) + b_n \sin\left(n\frac{2\pi}{T}t_i\right)$$

1. T: fundamental period
2. $(a_n^2+b_n^2)^{1/2}$: amplitudes of different harmonics
3. $b_n / a_n = \mathrm{tg}\,\theta$: phases of different harmonics
   (i.e. delays of higher harmonics from fundamental
4. x(t)=x(t+nT)

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The ‘transform’ concept

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$$\underline{x} = [x_0, x_1, x_2, ...] \quad \text{in } t - x \text{ coordinates}$$

$$P(t)_{LSM\underline{x}} = a + bt + ct^2 + dt^3 + ...$$

$$\underline{P} = [a, b, c, d, ...] \quad \text{polynomial transform of } x$$

$$\Rightarrow$$

$$X(t_i) = \sum_{n=-\infty}^{+\infty} a_n \cos\left(n\frac{2\pi}{T}t_i\right) + b_n \sin\left(n\frac{2\pi}{T}t_i\right)$$

$$\underline{X} = [a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3, ...] \quad \text{Fourier transform of } x$$

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Discrete Fourier Transform

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Discrete time signals can be decomposed into a series of sine functions:

1. T: fundamental period
2. X_n: complex coefficient, representing amplitudes and phases of different harmonics
3. x(t)=x(t+nT)

A discrete signal is thus decomposed in the sum of a finite number of sine functions of proper frequency, amplitude and phase.

$$x(t_k) = \sum_{n=-\infty}^{+\infty} X_n \, exp\left(j\frac{2\pi n}{T}t_k\right)$$

(discrete time)

$$t_k = k\Delta t \quad T = N\Delta t$$

$$x_k = \sum_{n=-\infty}^{+\infty} X_n \, exp\left(j\frac{2\pi n}{N}k\right)$$

(nyquist criterion)

$$x_k = \sum_{n=-N/2}^{N/2-1} X_n \, exp\left(j\frac{2\pi n}{N}k\right)$$

$$x_k = \sum_{n=0}^{N-1} X_n \, exp\left(j\frac{2\pi n}{N}k\right)$$

normalization for energy content preservation in $t \rightarrow$
$\rightarrow f \rightarrow t$ transformations

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n \, exp\left(j\frac{2\pi n}{N}k\right)$$

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Fourier Decomposition examples

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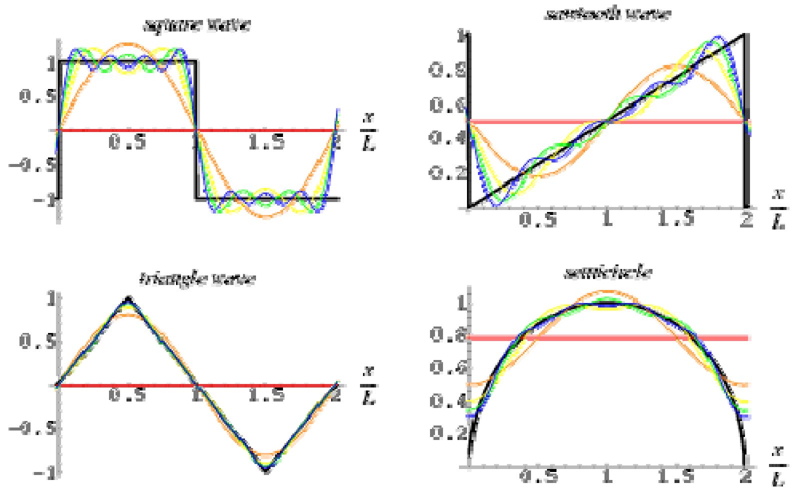

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Fourier Decomposition examples

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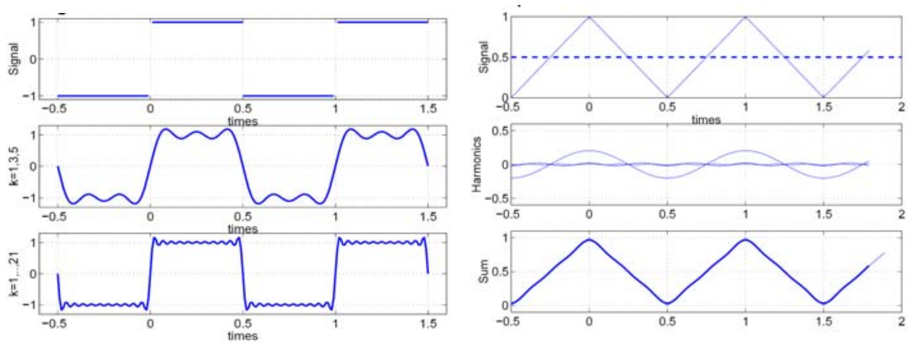

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Frequency representation of simple signals

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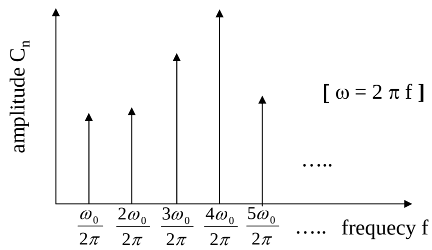
1. Each harmonic is a multiple of the fundamental harmonic
2. With independent amplitude
3. And independent phase

$$x(t_i) = \sum_{n=0}^{+\infty} a_n \cos\left(\frac{2\pi n}{T} t_i\right) + b_n \sin\left(\frac{2\pi n}{T} t_i\right)$$

$$= C_0 + \sum_{n=0}^{+\infty} C_n \sin\left(\frac{2\pi n}{T} t_i + \theta_n\right)$$

The spectral content is a number of impulses corresponding to each frequency
(both for amplitude and phases)



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Periodic Signal

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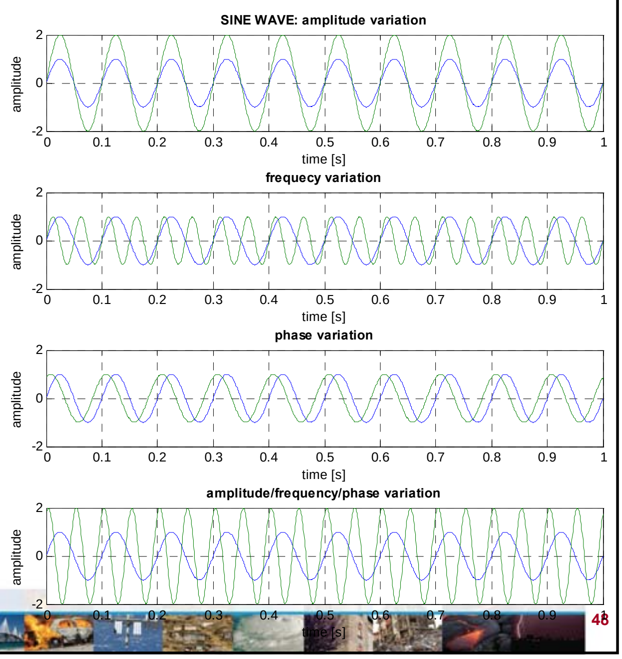
(demo_04)

Characteristic quantities:
Amplitude
Frequency
phase



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Periodic Signal

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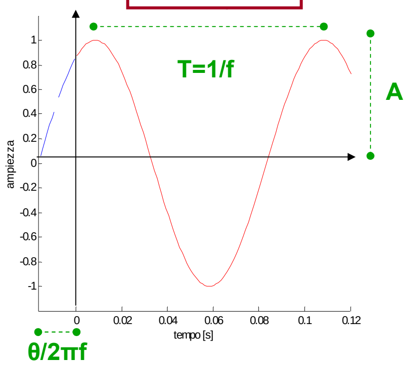
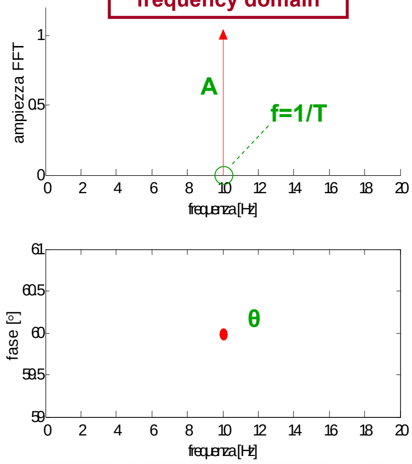
Periodic signals are stationary/ergodic. Their representation in the time domain is EXACTLY DUAL to the frequency domain representation

time domain



frequency domain



SAME INFORMATION contained in the two representations

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Fourier transform

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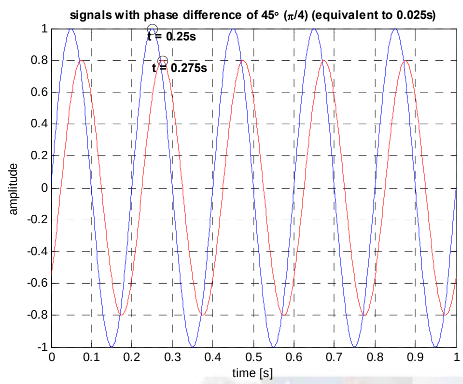
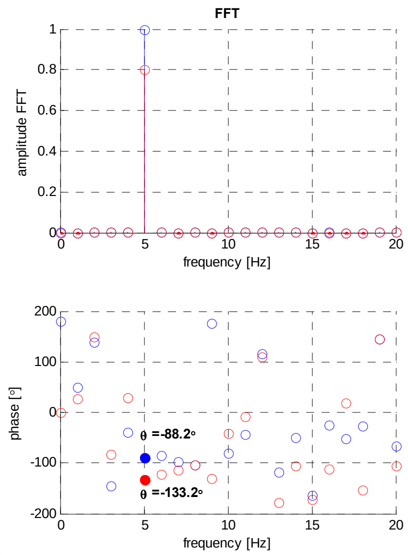
demo_05

Signal f = 5 Hz

Signal difference of phase: 45°

$\Delta t = \pi/4 \, / \, 2\pi f = 0.025s$





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Fourier Transform Coefficients

Fourier coefficients could be obtained by least square fitting the signal x to the Fourier series.

A better alternative calls upon the orthogonality of the harmonics to identify how much the signal x resemble a given sinusoid of frequency $\omega_n = 2\pi n / N\Delta t$:

i.e. each coefficient is found as the zero-shift value of the X-correlation function of the signal x with the given sine:

$$X_n = \sum_{k=0}^{N-1} x_k \, exp\left(-j\frac{2\pi n}{N}k\right)$$

NOTE: the summation index is k!!!

The coefficient $Re(X_0)$ capture the static component of the signal, while $Im(X_0)$ is assumed zero.

Fourier Transform Coefficients

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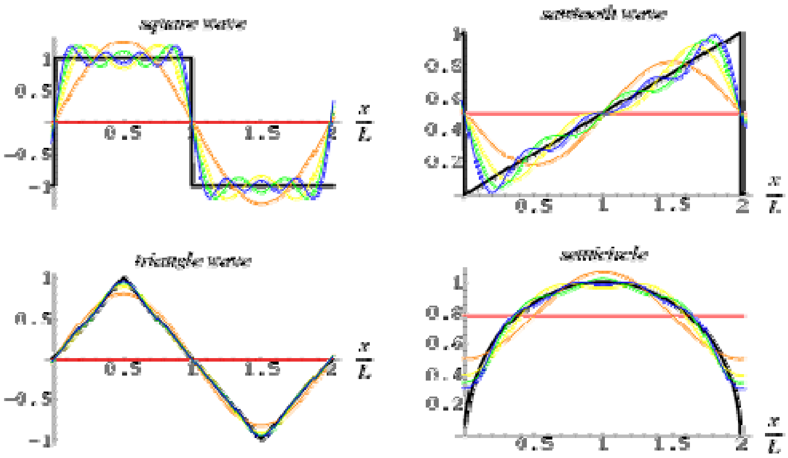
The coefficient $Re(X_0)$ capture the static component of the signal: i.e. the temporal mean of the signal (red line). The $Im(X_0)$ is assumed zero, i.e. that value is kept constant, it is not going up and down like a sine function.



Fourier Transform Coefficients

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N points in time correspond to N points in frequency

Since $X_n = Re(X_n) + i\,Im(X_n)$
It appears that N X_n would imply 2N coefficients

Remembering that

$$\exp\left(j\frac{2\pi n}{N}k\right) = \cos\left(\frac{2\pi n}{N}k\right) + j\sin\left(\frac{2\pi n}{N}k\right)$$

→ Due to periodicity and symmetry property of the Fourier transform, coefficients reduce to N.

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Fourier Transform Properties

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(1) Linearity

$x(t) \stackrel{\text{Fourier}}{\Longleftrightarrow} X(f)$

$y(t) \stackrel{\text{Fourier}}{\Longleftrightarrow} Y(f)$

⇒ $a\,x(t)+b\,y(t) \stackrel{\text{Fourier}}{\Longleftrightarrow} a\,X(f)+b\,Y(f)$

(2) Duality

$h(t) \stackrel{\text{Fourier}}{\Longleftrightarrow} H(f)$ ⇒ $H(t) \stackrel{\text{Fourier}}{\Longleftrightarrow} h(-f)$

(3) Scale factor in the time domain

$h(t) \stackrel{\text{Fourier}}{\Longleftrightarrow} H(f)$ ⇒ $h(kt) \stackrel{\text{Fourier}}{\Longleftrightarrow} \frac{1}{|k|}H(f/k)$

(4) Scale factor in the frequency domain

$h(t) \stackrel{\text{Fourier}}{\Longleftrightarrow} H(f)$ ⇒ $\frac{1}{k}h(t/k) \stackrel{\text{Fourier}}{\Longleftrightarrow} H(kf)$

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Fourier Transform Properties (2)

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(5) Translation in the time domain

$h(t) \Leftrightarrow^{Fourier} H(f) \Rightarrow h(t \pm t_0) \Leftrightarrow^{Fourier} H(f) \cdot e^{\pm j 2 \pi f t_0}$

(6) Translation in the frequency domain

$h(t) \Leftrightarrow^{Fourier} H(f) \Rightarrow h(t) \cdot e^{\mp j 2 \pi f t_0} \Leftrightarrow^{Fourier} H(f \pm f_0)$

(7) Even Function

$h(t)$ ***real & even*** $\Leftrightarrow^{Fourier}$ $H(f)$ ***real & even***

(8) Odd function

$h(t)$ ***real & odd*** $\Leftrightarrow^{Fourier}$ $H(f)$ ***imaginary & odd***

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Fourier Transform Properties (3)

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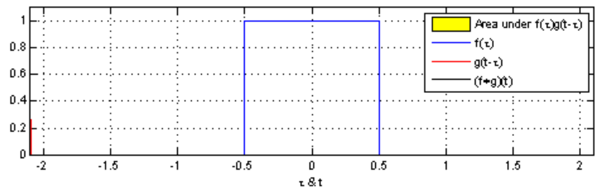
(9) Convolution Theorem

$h(t) * g(t) \Leftrightarrow^{Fourier} H(f)G(f)$

$h(t)g(t) \Leftrightarrow^{Fourier} H(f) * G(f)$

Def: Convolution

$$[f * g](t) = \int_0^t f(\tau) g(t-\tau) d\tau \qquad f * g = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau$$



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Transform Pairs

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Definition in continuous domain

$$x(t) = \int_{-\infty}^{+\infty} X(f)\exp(j2\pi ft)df \qquad X(f) = \int_{-\infty}^{+\infty} x(t)\exp(-j2\pi ft)dt$$

for stationary systems:

time domain ⟷ **Frequency domain**

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Fourier transform: linear combination of signals

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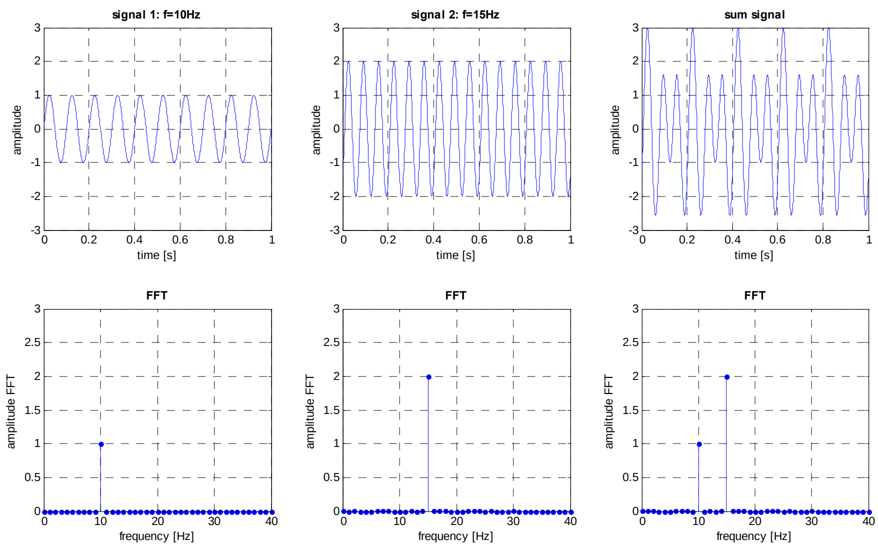
demo_06-07



Fourier transform: linear combination of signals

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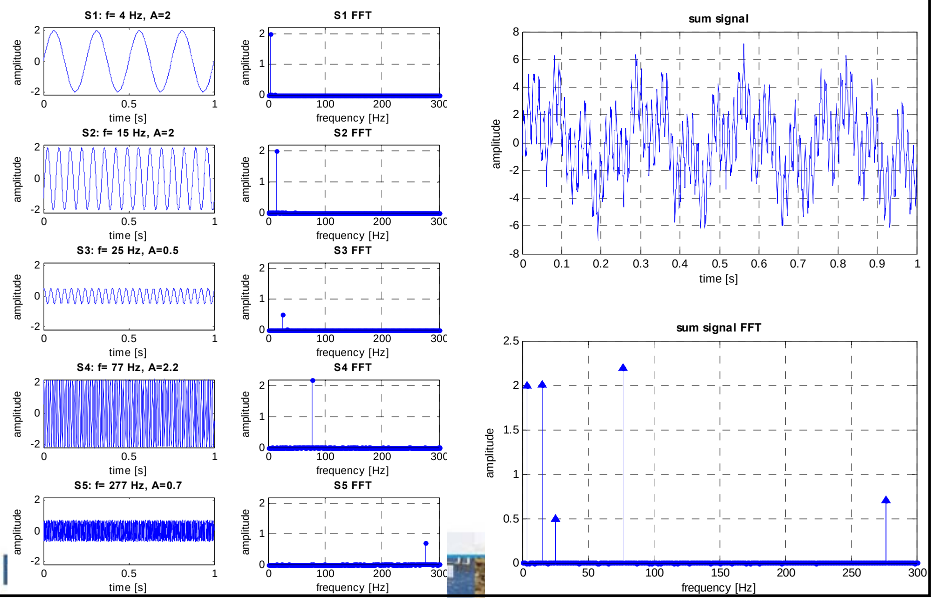

Nonstationary Signals

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The Fourier transform do loose any timing information

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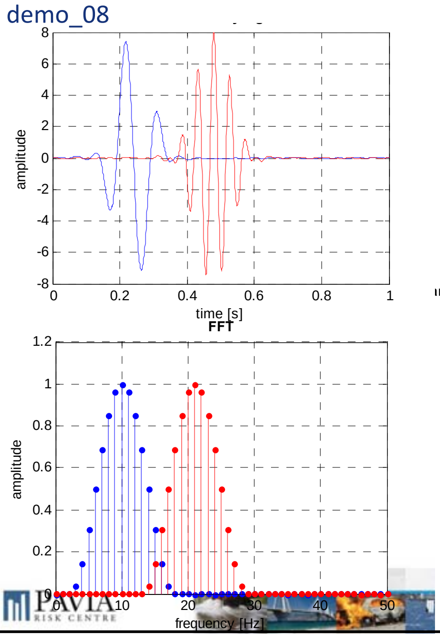
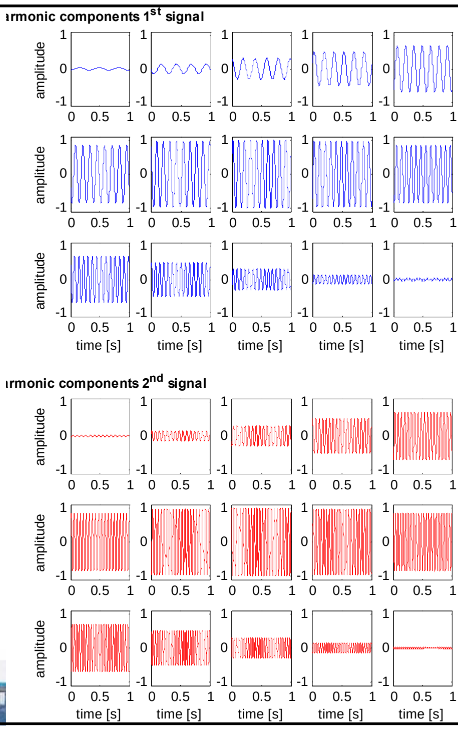
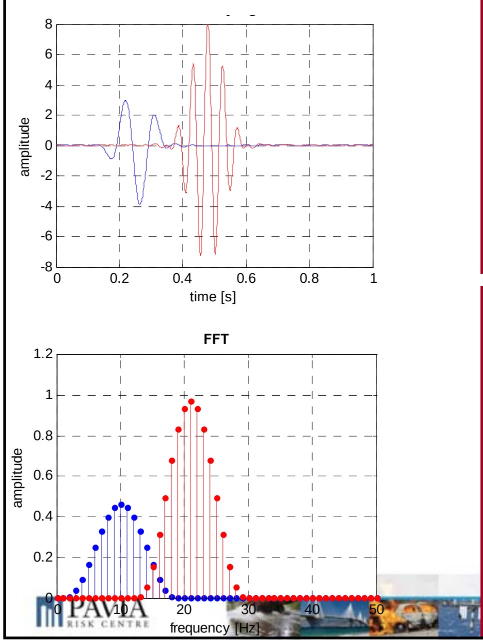
Stationary signals

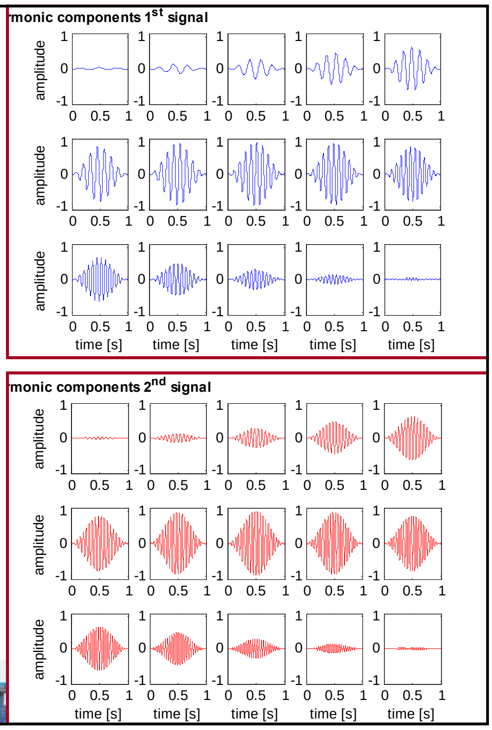
demo_08





Nonstationary signals





Nonstationary signal

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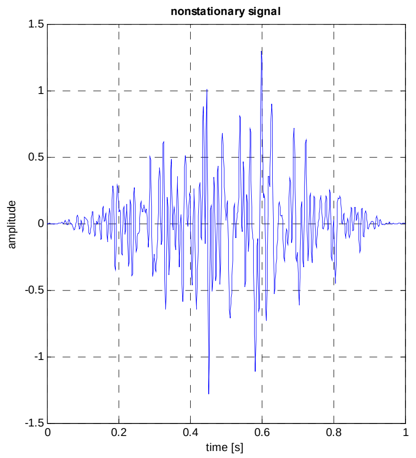
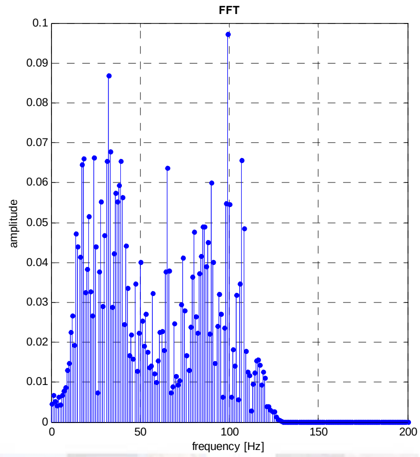
demo_09





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Implications of digitalization: aliasing

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It is intuitive that in a sampled signal frequencies higher than the sample frequency (SR, sampling rate, i.e. number of sample per second) cannot be captured.

In particular it is not possible to detect frequencies higher than SR/2, called NYQUIST FREQUENCY f_{nyq}

The spectrum of a sampled signal represents the original continuous signal in the band of frequencies lower than f_{nyq}

Beyond f_{nyq} an overlapping of shifted replicas of the transform is found.
The phenomenon is called ALIASING

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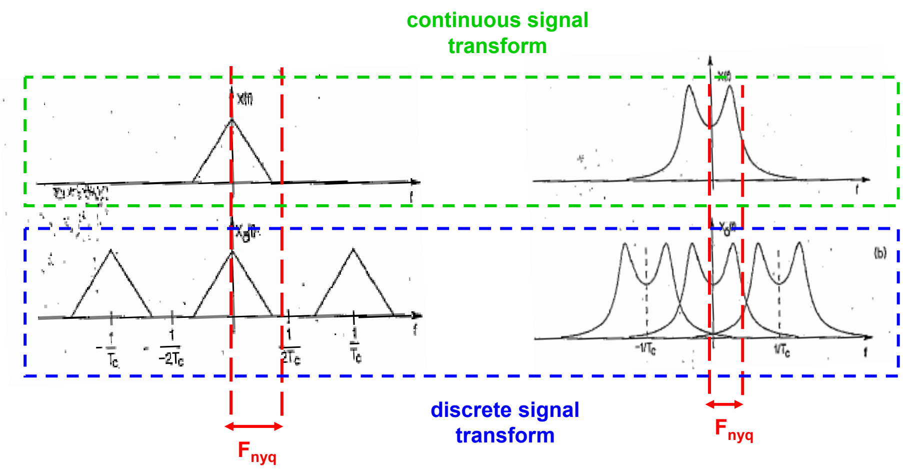
Aliasing



Sampling theorem

A signal with limited frequency bandwidth (i.e. with zero spectrum out of the bandwidth B) is fully described by the corresponding discrete signal only if the SR is higher than 2B

SR>2B

→ the aliasing is unavoidable for signals with unlimited frequency bandwidth, but if the frequency range of interest is known, it can be prevented with a proper sampling rate

SR=2B

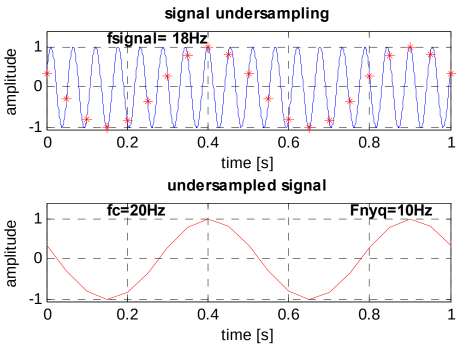
On a periodic signal, the frequency is captured, but not necessarily amplitude peaks

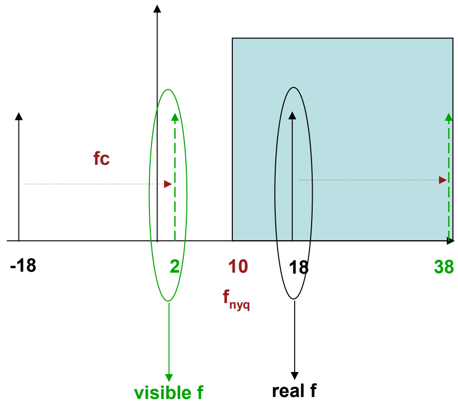
SR>2B

On a periodic signal, the frequency is captured, and also necessarily amplitude peaks

Signal undersampling

In the visible frequency bandwidth ($f<f_{nyq}$) the spectrum of an undersampled signal is not the actual spectrum but it is one of the shifted "alias", at a frequency lower than the actual.



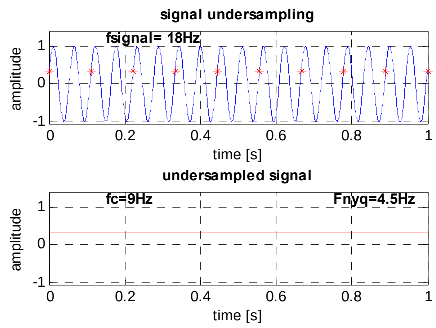


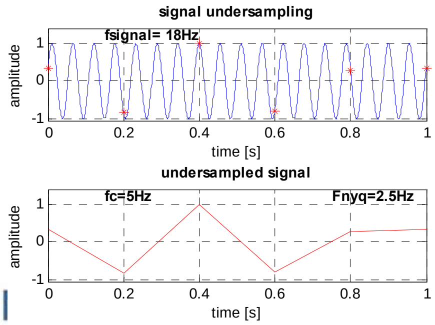
Aliasing

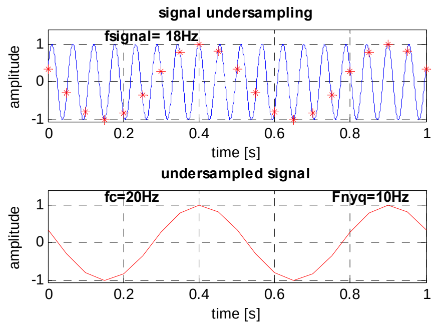
demo_10

The signal f is 18 hz. Which is a good sampling frequency?

For the samplig theorem, to detect 18 Hz, SR ≥ 36 … but is it enough?







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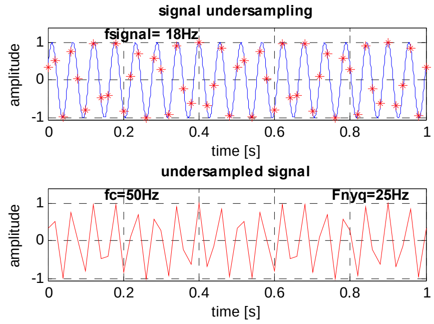
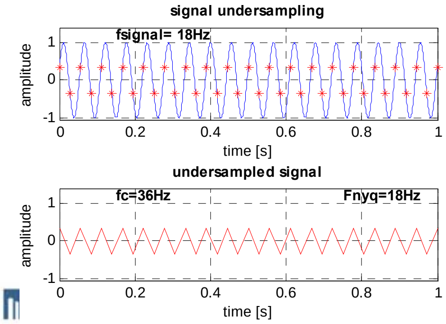
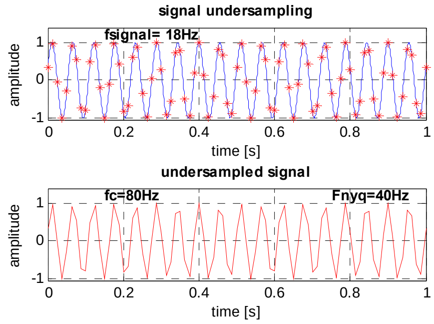
At SR=36 the signal periodicity is kept, but no the actual amplitude!

At SR>36 the periodicity and amplitude of the signal is captured

But still the signal is poorly reproduced







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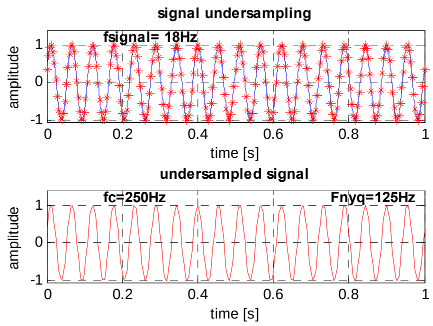
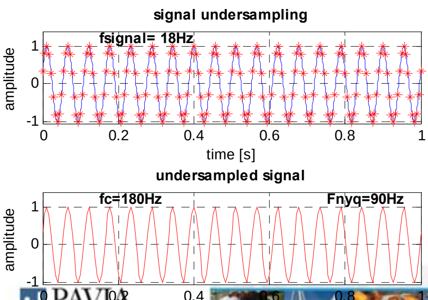
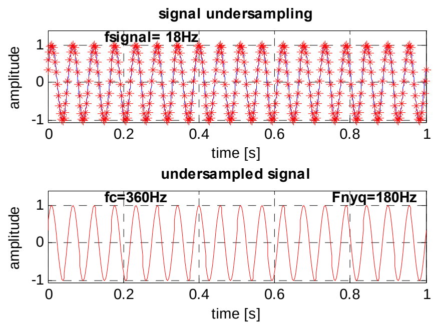
The higher is SR, the better is the signal reproduction

→ For a good signal, SR should be significantly higher than the scale of interest!!







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Aliasing effect: lower frequency alias

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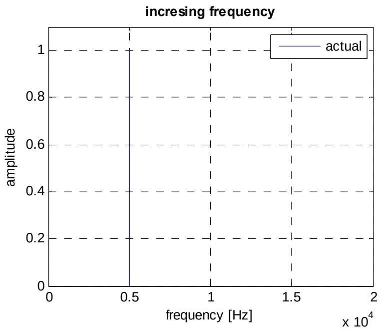
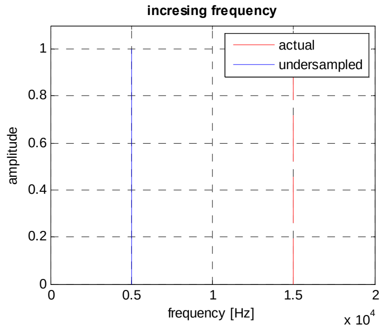
demo_11

The signal has frequency increasing with time from 0 to 20000 hz.

What if the sampling frequency is 10000 Hz?





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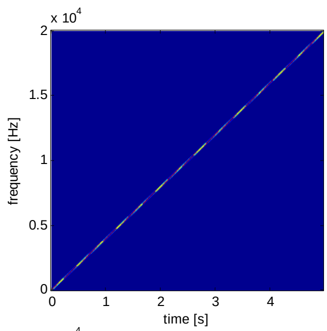
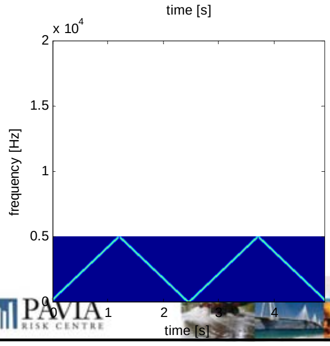


Sampling at 40000 Hz it is possible to see the frequency increasing



Sampling at 10000 Hz, after 5000 Hz (f_{nyq}), fake frequencies are seen, due to the undersampling of the signal (aliasing effect)

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The Decibel Scale

In signal processing is common practice to describe signals amplitudes in the logaritmic or decibel (DB) scale, which is a relative scale, referred to a reference amplitude.

The DB scale can be referred to amplitude or power, with analogous meanings.

Given a reference amplitude A_0 or power $P_0=cA_0^2$, the amplitude A_1 in DB is:

$$r_{dB} = 20\log_{10}\left(\frac{A_1}{A_0}\right) = 10\log_{10}\left(\frac{P_1}{P_0}\right)$$

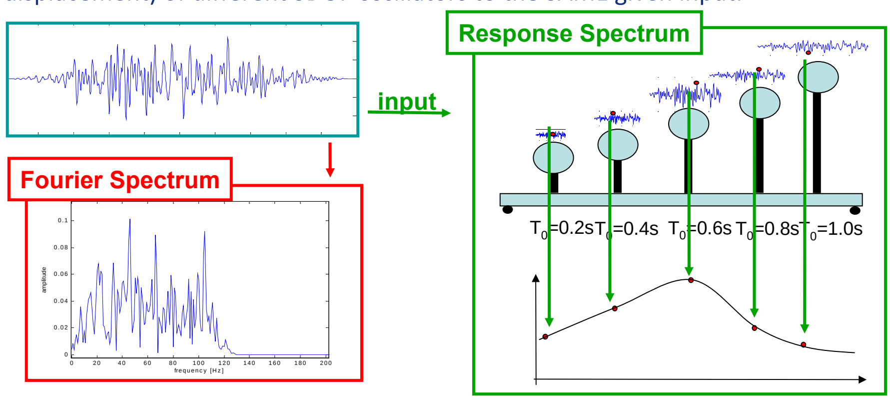
$$\frac{A_1}{A_0} = 10^{\frac{r_{dB}}{20}}$$

- 6dB → A_1 is twice A_0
- 9.5dB → A_1 is 3 times A_0
- 20dB → A_1 is 10 times A_0
- 40dB → A_1 is 100 times A_0
- 60dB → A_1 is 1000 times A_0
- 80dB → A_1 is 10000 times A_0

Fourier Spectrum vs Response Spectrum

THEY ARE DIFFERENT CONCEPTS!!!

The response spectrum is a structural engineering concept, representing the envelope of the response maximum values (acceleration / velocity / displacement) of different SDOF oscillators to the SAME given input.



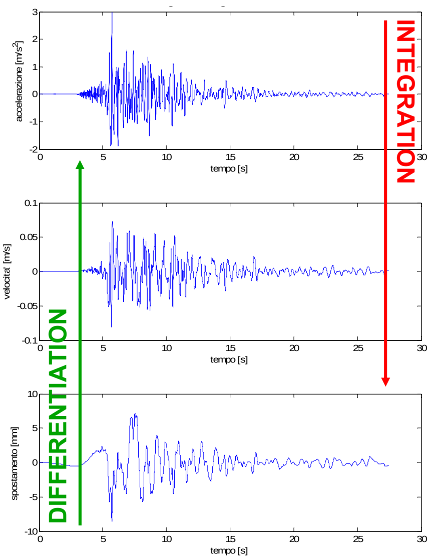
Frequency domain operations: integration and differentiation

Such operation can result in undesired outcomes, i.e.:

1. Numerical integration of a sampled signal enhance the low frequency content
2. Numerical differentiation enhance the high frequency content

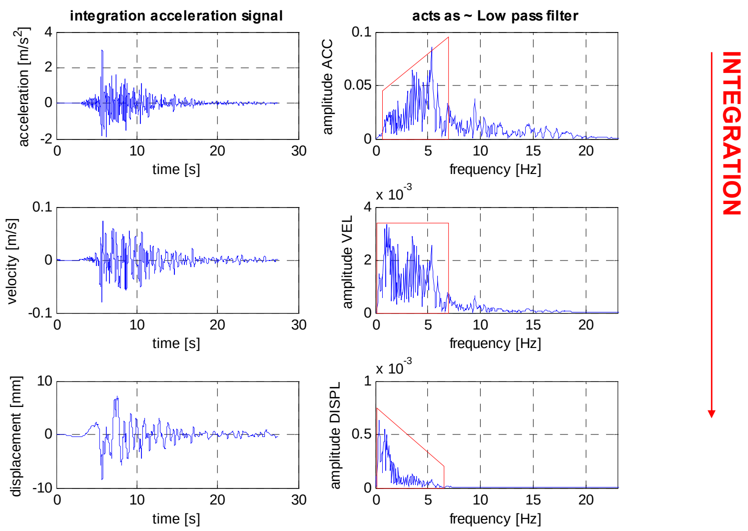
It is easy to see the effect in the frequency domain, given the differentiation and integration properties of the Fourier Transform:

$$x(t) \rightarrow X(f)$$
$$\dot{x}(t) \rightarrow \div fX(f)$$
$$\int x(t)dt \rightarrow \div X(f)/f$$



Signal integration

demo_12



Signal differentiation

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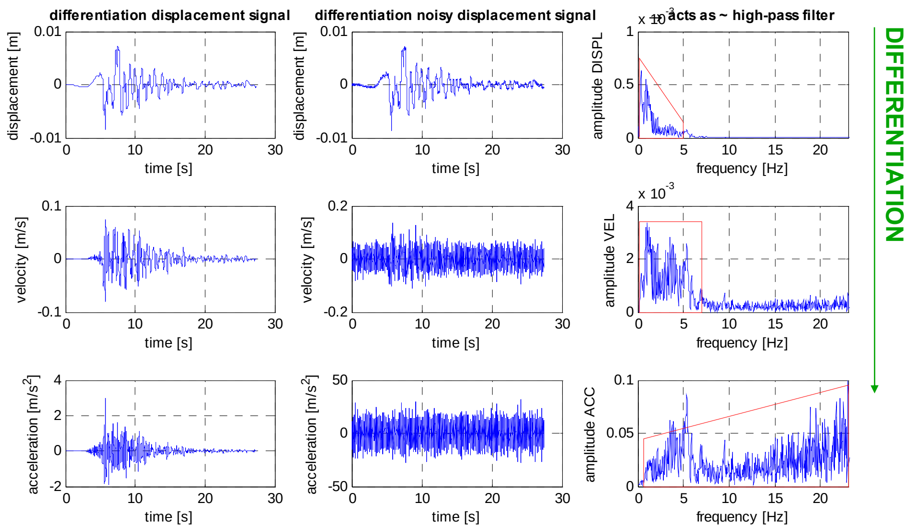

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Frequency vs time domain analysis

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Time domain | **Frequency domain**

problem → **transform problem**

Time-consuming & difficult analysis | **Fast & easy analysis**

solution ← **transform solution**

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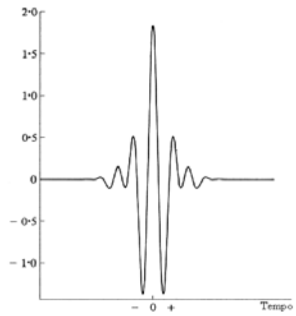
Discrete Fourier Transform (DFT) calculation

Due to the finite duration of the signal, the "FINITE" DFT is related to the "INFINITE" DFT by means of a weigthed integral with the sinc function:

$$X_n = \int_{-\infty}^{+\infty} \exp\{2\pi j(fT-n)\} \operatorname{sinc}(fT-n) X(f) df$$

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



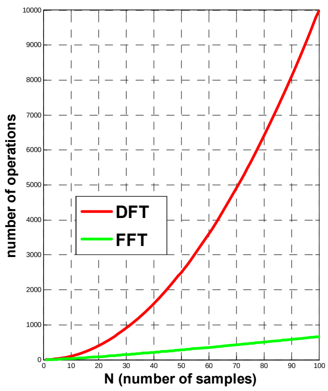
The signal power is no more concentrated on the X_n component, but distributed among all the frequencies n/T, by means of the sinc function

The Fast Fourier Transform (FFT)

1. The sinc function is highly oscillating and converges slowly, introducing fictitious structures in the spectrum
2. Using a weighted average on the spectrum, a better shape is obtained with faster convergence
3. The operation of "windowing" is needed to smooth the spectrum, e.g. using the Hanning window as weight function.

The Cooley & Tukey (1965) algorithm, known as FAST FOURIER TRANSFORM, is way faster than the direct calculation of the DFT. Considering N sampled values:

1. The FFT algorithm uses N $\log_2 N$ operations
2. The DFT algorithm uses N^2 operations



To remember

For a given N number of samples:

N time values x_k → N freq numbers X_n

$T=N*\Delta t$

Δt (time resolution)

$t_k=k*\Delta t$

$SR=1/\Delta t$ (sampling rate)

$f_{min}=1/T$

$f_n = n*f_{min} = n/(N*\Delta t)$

$f_{max} = f_{nyq} = 1/(2\Delta t)$

$\Delta f = f_{max}/(N/2) = 1/(N\,\Delta t) = SR/N$ (frequency resolution)

→ $N = 1/(\Delta f*\Delta t)$

The resolution in frequency is inversely proportional to resolution in time

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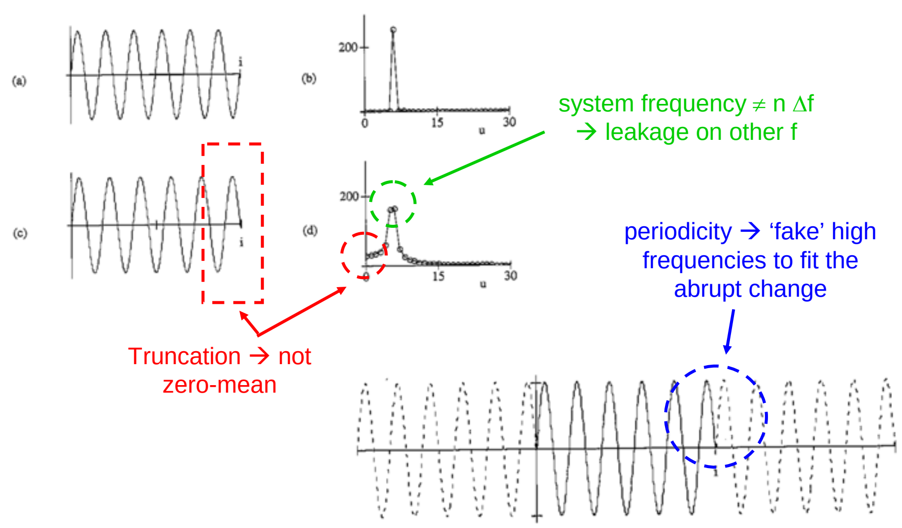
Truncation, leakage and windowing



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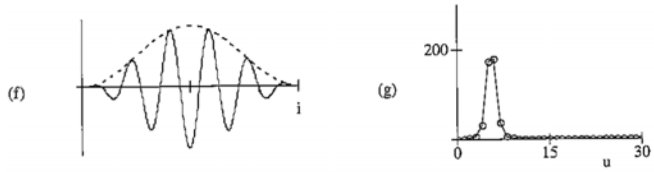
Truncation, leakage and windowing

Leakage: energy “leaks” to frequencies other than the real components → the spectrum does not appear as it should be

1. Frequency not a multiple of the Δf (not solvable)
2. Periodicity assumption → ‘fake’ high frequencies introduced to fit the abrupt change due to the truncation
3. Truncation gives not zero mean (static component)

→ Windowing can solve issue 2 and 3, even if reduces the energy content of the original signal



Padding

To increase the frequency resolution (i.e. decrease Δf), several techniques of signal “extension” may be used. Remember $N = 1/(\Delta f^* \Delta t)$

- ZERO PADDING consists in appending zeros to the signal.
- CONSTANT PADDING consists in appending the last value to the signal
- LINEAR PADDING consists in extending the signal maintaining constant the first derivative at the signal end
- PERIODIC PADDING consists in repeating the signal

The following observations apply:

1. Signal padding DOES NOT add information
2. The real effect of padding is create harmonic components better fitting the signal
3. Zero and periodic padding may create discontinuities
4. Negative effects of padding are reduced if signal is first detrended and windowed
5. Padding the signal at the beginning will change the signal phases with a frequency dependent phase shift

Padding

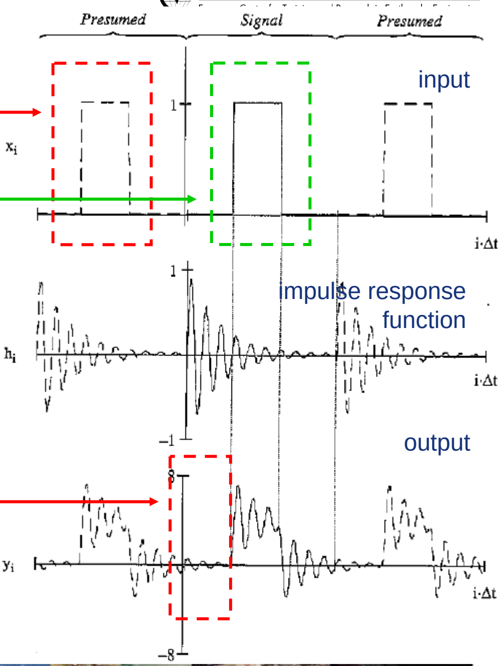
6. When the main frequency f* of the system under study is known, the padding should be done in such a way that f* is a multiple of the frequency resolution. (also the sampling rate!!!)
7. Since the DFT presumes the signal period of period T, the padding increase such a period, preventing "circular convolution effects" in the frequency domain computations (the "output before input" effect).
8. In case of random signals , the extention must preserve stationarity conditions (so padding should not be applied using random signals like noise)

→ Enhanced frequency resolution with harmonics better fitting the signal lead to more accurate system identification

(Circularity effect)

DFT periodicity assumption

actual input



If the output signal is computed by implementing the convolution of the input and the impulse response function with frequency domain operations the system responds before the input is applied!!!

Non-causal output response

Plots

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1. The frequencies associated with the peak FFT response are the oscillator resonant frequencies

2. The (power) spectrum in the log scale allows identifying low amplitude oscillation modes

3. Low damping is denoted as multiple oscillations in the time domain, as narrow peaks in the frequency domain (imagine the SDOF impulse response function at different damping values)

Spectral Analysis

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The spectral analysis can be performed as:

1. Amplitude analysis: determine each harmonic amplitude → amplitude spectrum

2. Power analysis: determine the energy content associated to each harmonic → power spectrum

Power Spectral Density

The square values of the FFT indicate the energy content associated to the different harmonics.

The power spectral density is defined as: $PSD_n(f) = |X_n(f)|^2$

The sum of the square values of a signal is an indicator of the energy carried on by the signal.

The Parseval theorem (energy conservation) states that:

$$\sum_{k=0}^{N-1} x_k^2 = \frac{1}{N}\sum_{n=0}^{N-1} |X_n|^2$$

$$\int |X(f)|^2 df = \int |x(t)|^2 dt$$

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Cross-correlation and cross-spectrum

The comparison between two signals can be done:

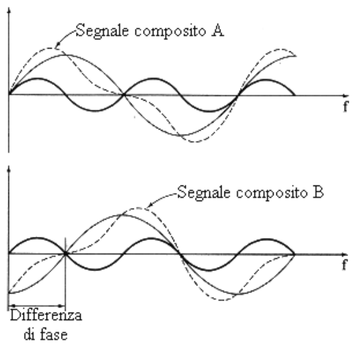
1. In the time domain, by means of the cross-correlation function

$$R_{xy}(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x_k(t)\, y_k(t+\tau)$$

2. In the frequency domain, by means of the CROSS-SPECTRUM, defined as:

$$S_{xy,n} = Y_n \overline{X}_n = FFT(R_{xy}(\tau))$$

$$S_{xx,n} = PSD_n = X_n \overline{X}_n = |X_n|^2$$



The X-spectrum gives information about: the coherency of the signals harmonics, but also about the phase relation of the different harmonics

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Cross spectrum

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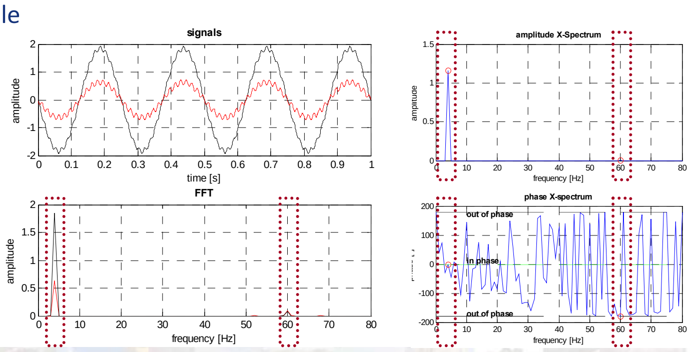
The X-spectrum of two signals represents the series of harmonics having per amplitude the amplitude product and per phase the phase difference.

→ the X-spectrum phase indicates the delay of the common harmonics of one signal wrt the other, i.e. if for a given frequency two point are moving in phase or out of phase.

Let's see an example



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Cross-spectrum: information on the relationship between two signals

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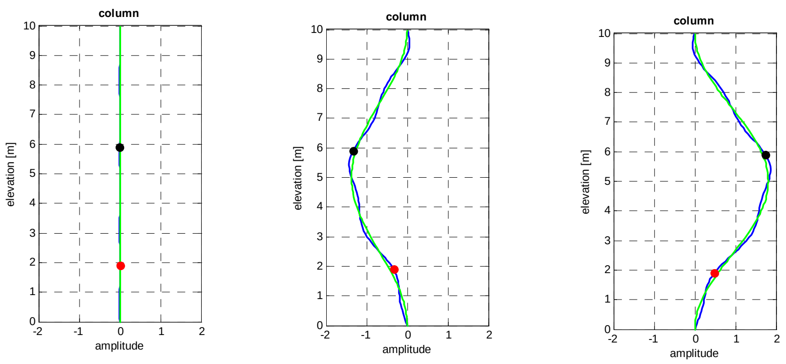
demo_13

Oscillating column: monitoring the displacement of two points along the column



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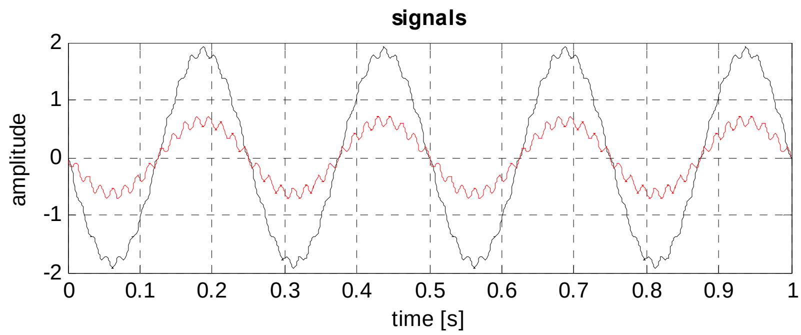
Information on the relationship between two signals: time domain

Time domain analysis: both the points are moving according to a wide sine wave (low frequency) in phase, with a small high frequency oscillation out of phase



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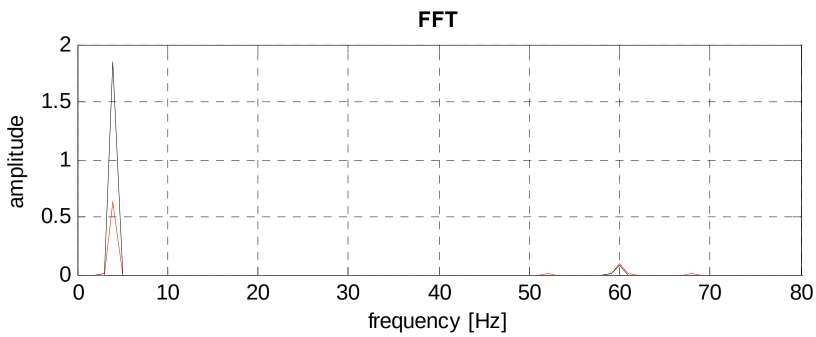
Information on the relationship between two signals: frequency domain

The same information can be extracted from the FFT: unfortunately signals are not so easy to analyse.



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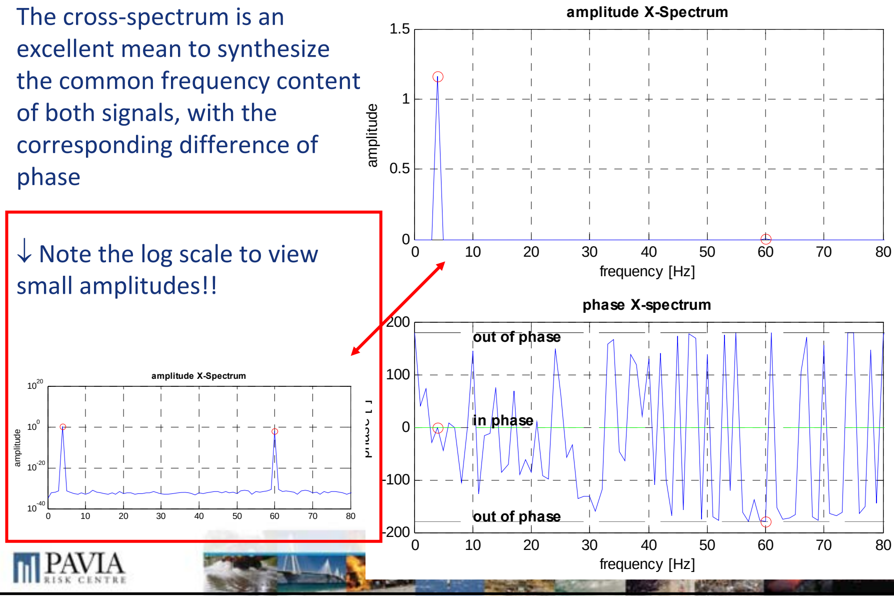
Information on the relationship between two signals: frequency domain

The cross-spectrum is an excellent mean to synthesize the common frequency content of both signals, with the corresponding difference of phase

↓ Note the log scale to view small amplitudes!!



Linear Time-Invariant Systems (LTI) Theory

It investigates the response of a linear and time-invariant system to an arbitrary input signal.

1. Linearity means that the relationship between the input and the output of the system is a linear map

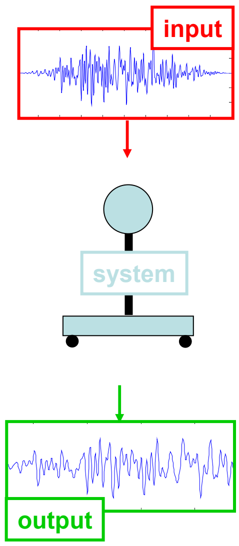
2. Time invariance means that the output does not depend on the particular time the input is applied.

LTI system theory is good at describing many important systems.

LTI systems are considered "easy" to analyze, compared to the time-varying and/or nonlinear case.

Any system that can be modeled as a linear homogeneous differential equation with constant coefficients is an LTI system.

Examples of such systems are ideal spring–mass–damper systems.



LTI Theory

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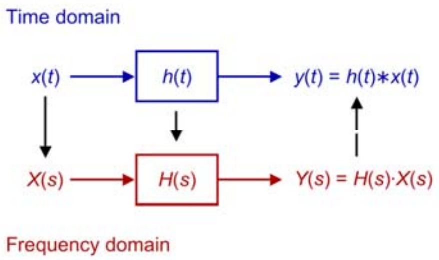
In the time domain, LTI system can be characterized entirely by a single function called the system's IMPULSE RESPONSE h(t).

The output of the system is simply the convolution of the input x(t) to the system with the system's impulse response: y(t)=h(t)*x(t)

In the frequency domain,
LTI system can be characterized
by the system's
TRANSFER FUNCTION H(f),
which is the transform of the
system's impulse response.



→ The frequency domain analysis
is faster and easier!!!

Convolution

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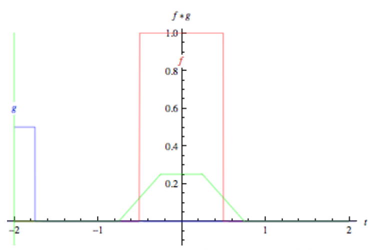
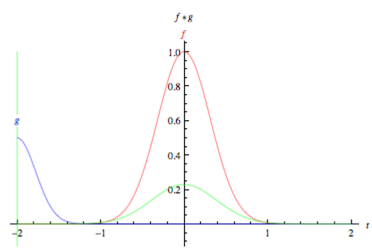
As a result of the properties of these transforms, the output of the system in the frequency domain is the product of the transfer function H(f) and the transform of the input X(f): Y(f) = H(f) X(f)

In other words, convolution in the time domain is equivalent to multiplication in the frequency domain.

The convolution operator is an integral that expresses the amount of overlap of one function g as it is shifted over another function f. It therefore "blends" one function with another.





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$$[f * g](t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

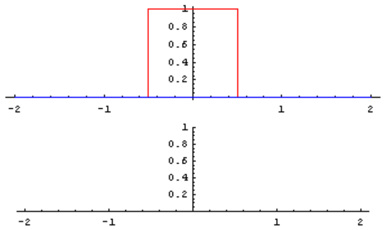
$$f * g = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau$$

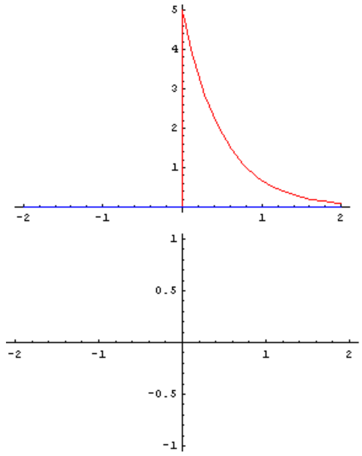
properties

$$f * g = g * f$$

$$f * (g * r) = (f * g) * r$$

$$f * (g + r) = f * g + f * r$$





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Transfer function computation: phase unwrapping

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The determination of h(t) in the time domain is hampered by the mathematical nature of the impulse signal.

The most effective way to determine H(f) is to apply a broadband input signal x and to process the data y in the frequency domain, according to the point by point ratio of the output-input DFT:

$$H_n = \frac{Y_n}{X_n}$$

The amplitude analysis is straightforward.

The phase analysis requires one more step

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Transfer function computation: phase unwrapping

Consider a system causing a linear phase shift.

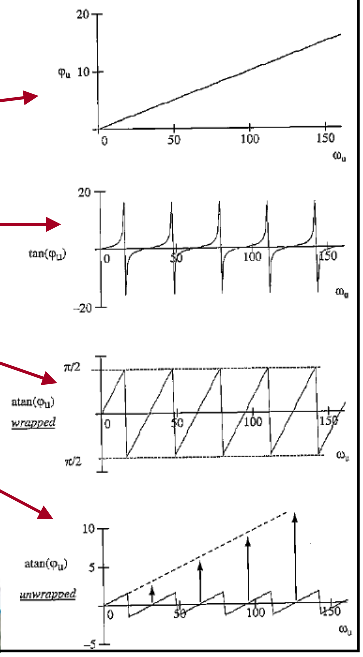
1. The direct computation is of $\tan(\theta_n)=\mathrm{Im}(H_n)/\mathrm{Re}(H_n)$

2. The phase, computed as arctan (θ_n) appears clipped between $-\pi/2$ and $\pi/2$

The phase unwrapping means shifting every segment up everywhere the phase jumped from $\pi/2$ to $-\pi/2$

Sometime local jumps may be related to the system nature.

→ Phase unwrapping must be guided by the physical insight into the system



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SDOF linear oscillator

The equation of motion of a SDOF oscillator can be written and solved in the time domain convolving the impulse response function h(t) (Duhamel's integral).

In the frequency domain the solution is way faster using the system transfer function H(f), which is the DFT of h(t)

$$\ddot{q}(t)+2\xi\omega_0\dot{q}(t)+\omega_0^2 q(t)=\frac{1}{m}f(t)$$

$$Q(\omega)=H(\omega)F(\omega)$$

$$H(\omega)=\frac{1}{m}\frac{1}{\omega_0^2-\omega^2+2j\xi\omega_0\omega}$$

$$\omega=2\pi f$$

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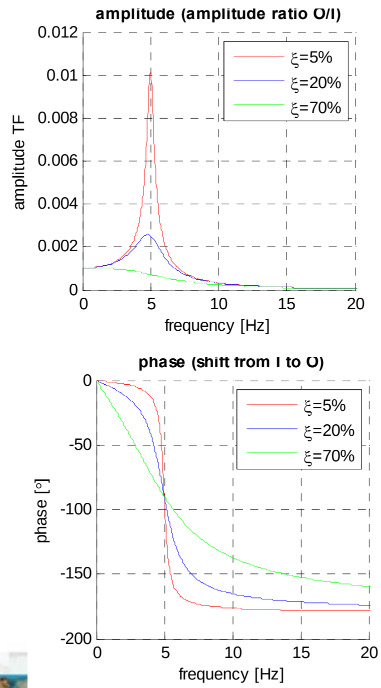
SDOF linear oscillator

The H(f) amplitude represents the output/input ratio of the different harmonics

$$|H(\omega)| = \frac{1}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\xi^2\omega_0^2\omega^2}}$$

The H(f) phase represents the phase variation of the different harmonics

$$\theta(H(\omega)) = \arctan\left(\frac{-2\xi\omega_0\omega}{\omega_0^2-\omega^2}\right)$$



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LTI system: transfer function

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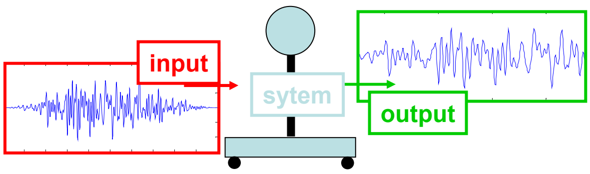
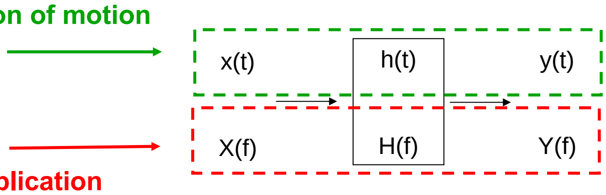


Integration equation of motion
→ convolution



Frequency domain
→ transform multiplication
→ Anti-trasform to get the response in the time domain

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LTI system: transfer function of a SDOF

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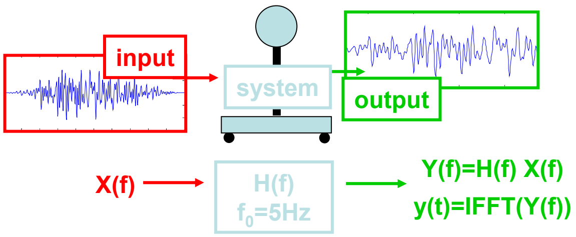
demo_14
Acceleration response transfer function



$$\ddot{q}(t)+2\xi\omega_0\dot{q}(t)+\omega_0^2 q(t)=\frac{1}{m}x(t)$$

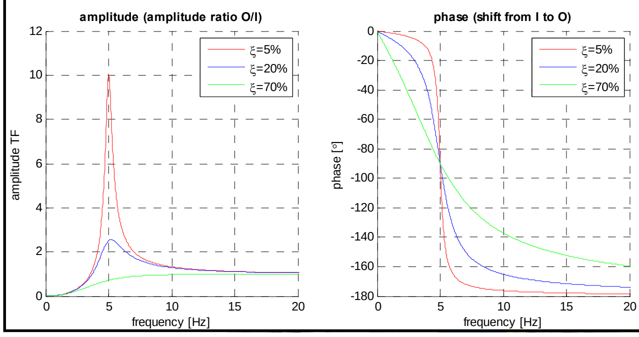
$$f(t)=F_0\exp(-j\omega t)$$

$$q(t)=H(\omega)f(t)$$

$$H(\omega)=\frac{1}{m}\frac{1}{\omega_0^2-\omega^2+2j\xi\omega_0\omega}$$

$$\ddot{q}(t)=\omega^2 q(t)$$

$$H_{acc}(\omega)=\frac{1}{m}\frac{\omega^2}{\omega_0^2-\omega^2+2j\xi\omega_0\omega}$$



LTI system: transfer function of a SDOF

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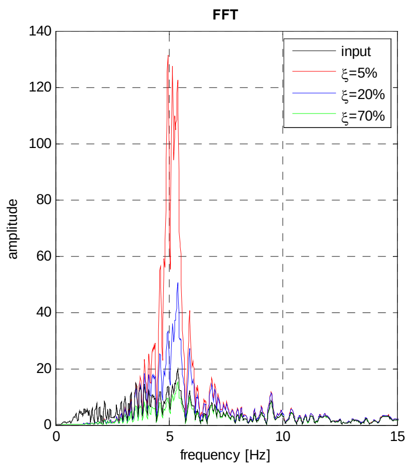

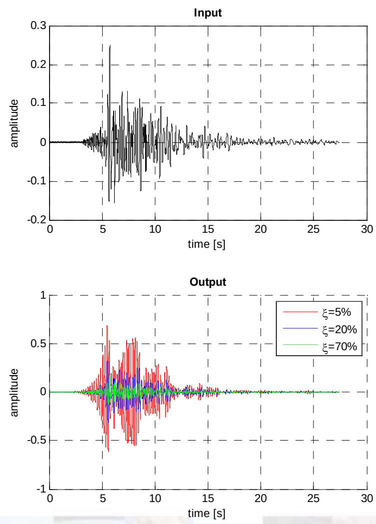

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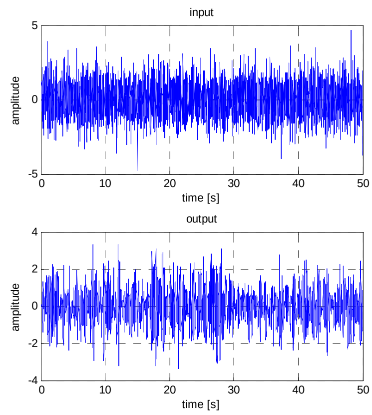
Computation of the transfer function across multiple windows: the Welch periodogram

In order to get a better TF of recorded signals, it is sometime useful to calculate the TF on reduced length signal windows, then averaging the TF.

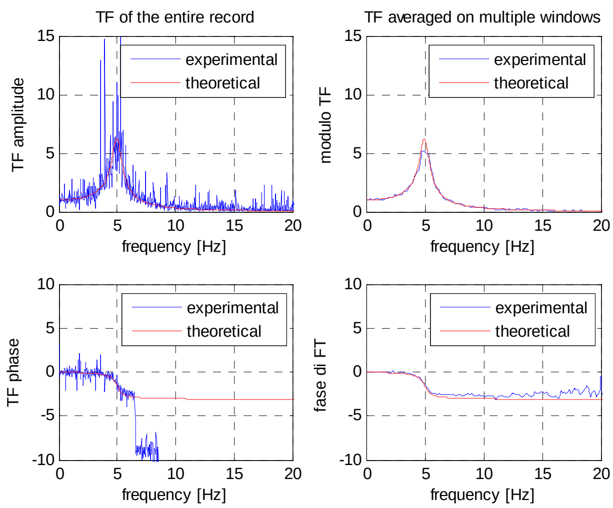
Small windows smooth the TF, but worsen the frequency resolution.

Lets consider the TF of a linear SDOF

demo_15



Computation of the transfer function across multiple windows: the Welch periodogram



LTI: transfer function applied to the response spectrum evaluation

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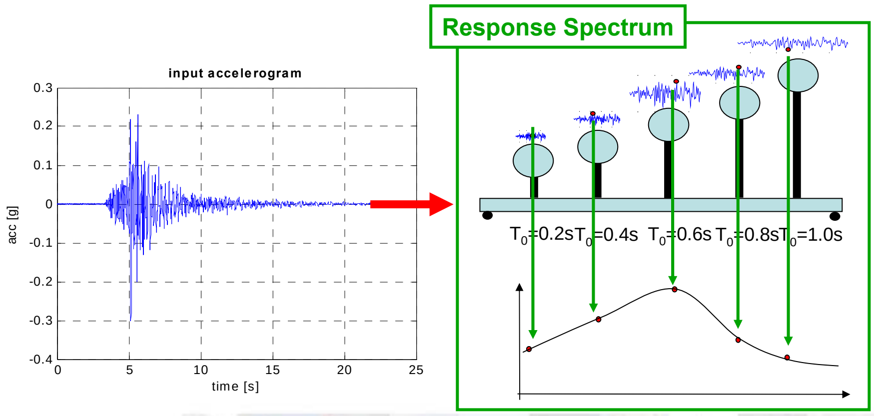
demo_16

The definition of elastic response spectrum of an accelerogram is the maximum response of elastic SDOF of different T_0



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Response spectrum evaluation

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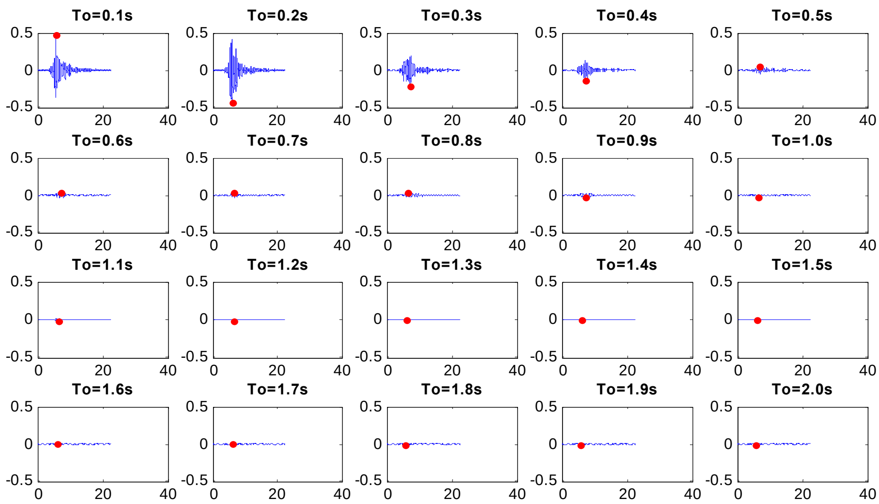

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Response spectrum evaluation

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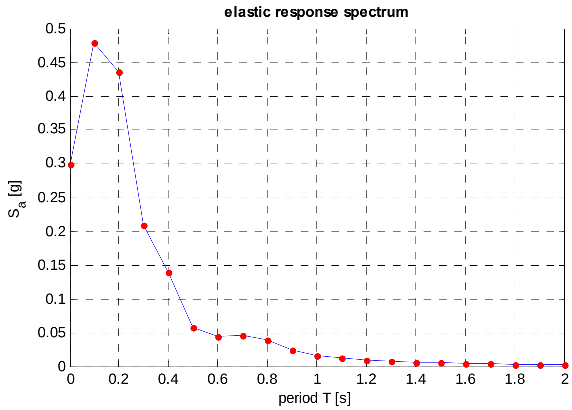

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Signal Coherence

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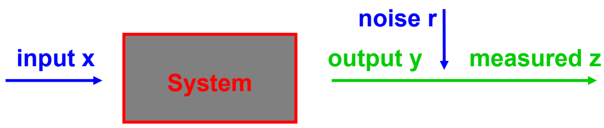

The coherence represents the variation of the transfer function between two signals. It measures the energy in the measured output caused by the input. It is a measure of the SNR as a function of the frequency.

It is determined using average spectra for an ensemble of signals:

$$\gamma_n = \sqrt{\frac{(S_{xz,n})^2_{avr}}{(S_{xx,n})_{avr}(S_{zz,n})_{avr}}} \dots \cong \sqrt{\frac{(S_{yy,n})_{avr}}{(S_{zz,n})_{avr}}}$$

DEF. The transfer function between two signals represents the linear system (or filter) that transforms the first signal in the second. In the Frequency domain it is the ratio between the FFT of the two signals (see the filter definition)

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Signal Coherency and SNR

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Coherence is a valuable diagnostic tool!!

Coherence less than 1 may indicate:

1. Noise in the output
2. Unaccounted inputs in the system
3. Nonlinear system behaviour
4. Lack of frequency resolution and leakage: a local drop in coherency close to a resonant peak suggest that the system resonant frequency is not a multiple of the frequency resolution.

The SNR can be defined as:

$$SNR_n = \frac{\left(S_{yy,n}\right)_{avr}}{\left(S_{rr,n}\right)_{avr}} = \frac{\gamma_n^2}{1-\gamma_n^2}$$

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Signal and noise: coherence analysis

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The coerehnce measures the energy in the measured output caused by the input. It is a measure of the SNR as a function of the frequency. It is determined using average spectra for an ensemble of signals.

Two cases:

1. noise out of the range of frequency of the signal
2. noise in of the range of frequency of the signal

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Noise out of the range of frequency of the signal

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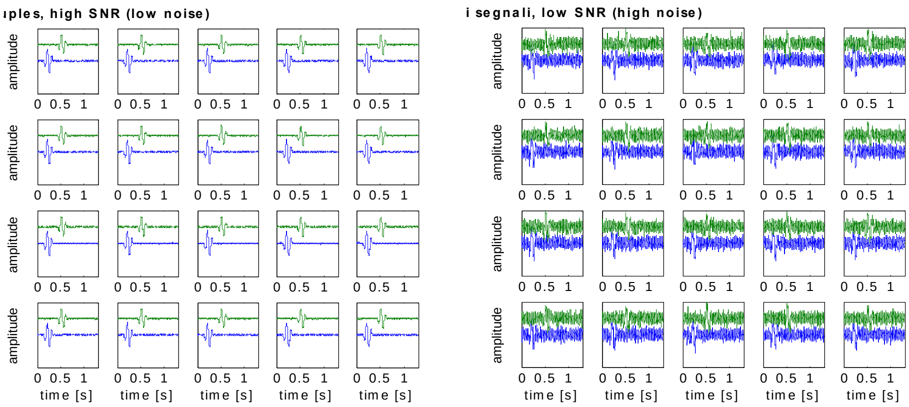
Let's analyse two systems

High SNR

Low SNR



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Noise out of the range of frequency of the signal

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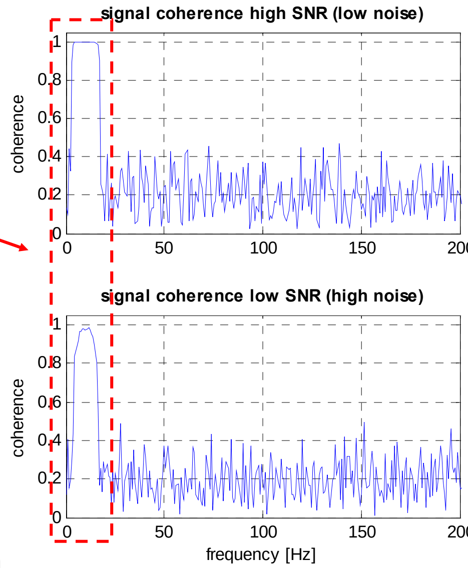
The noise does not influence the signal coherency in the range of frequencies where the signal is strong

→ This means that where the coherence is close to 1 the output is representative of the system and the noise does not disturb even if the SNR is low



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Noise in the range of frequency of the signal

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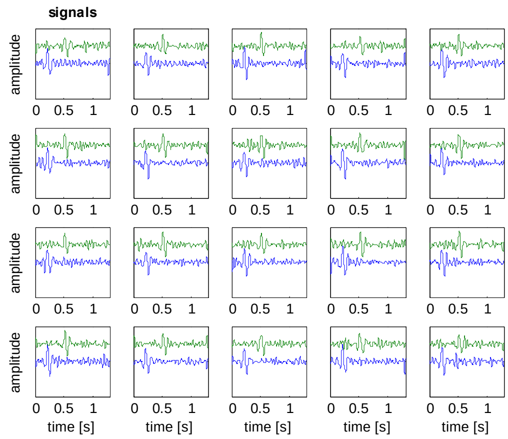
The SNR is low, and the noise is within the range of frequency of the system

demo_18



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Noise in the range of frequency of the signal

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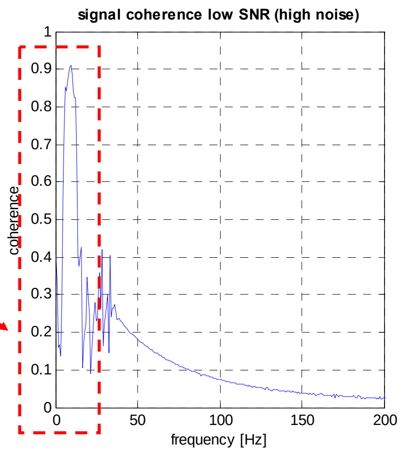
The coherence is close (but lower) than 1 in a smaller range of frequencies, where the output can still be considered representative of the system, but the noise is now disturbing in the range of frequencies of the system.



$$\gamma_n = \sqrt{\frac{(S_{xz,n})^2_{avr}}{(S_{xx,n})_{avr}(S_{zz,n})_{avr}}} \ldots \cong \sqrt{\frac{(S_{yy,n})_{avr}}{(S_{zz,n})_{avr}}}$$

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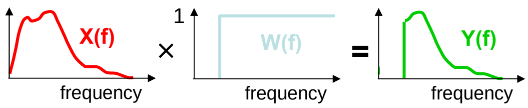
Filters

A filter in the frequency domain is a window W that passes selected frequency components X_n, rejecting the others. It is a point by point multiplication

$$Y_n = X_n W_n$$



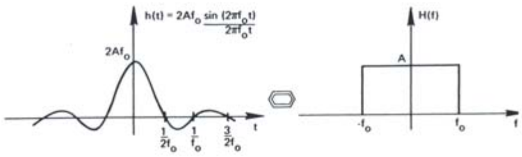
The transformed Y is converted back to the time domain into y(t).

Filters can alter the amplitude spectrum, the phase spectrum or both, depending on the filter coefficients W_n.

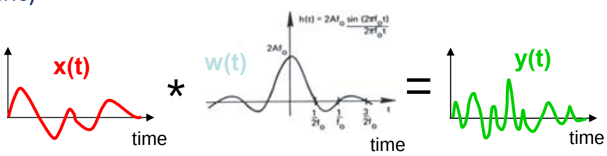
Filters

A filter is as a matter of fact a LTI system, and can be implemented in the time domain through its function h(t).

Unfortunately, the anti-transform of a box window is a sinc function:



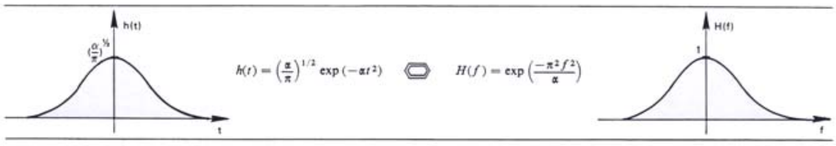
Which would imply to convolve the original signal with a very highly oscillating function (sinc)



Filters

In order to avoid such oscillations in the filtered signal, smoother windows are used to filter, e.g.:

1. Hamming
2. Hanning
3. Butterworth
4. Blackmann-Harris
5. etc…



Filter types

According to the rejected ranges of frequencies, filters may be:

1. Low pass
2. High pass
3. band-pass
4. band-reject (e.g. notch)
5. all-pass (effects only on the phases)

The transition region “pass” to “reject” may be more or less gradual.
The “cutoff” frequency corresponds to a reduction in the signal magnitude of -3dB, i.e. about of the 70%.

Another classification is:

1. IIR filters (Infinite Impulse Response Filter): cause a nonlinear phase distortion
2. FIR filters (Finite Impulse Response Filter): cause a linear phase distortion

Zero-phase filter implementation

In order to eliminate phase distortion in the filtering process, special algorithms are implemented.

E.g. the Matlab© "filtfilt" function uses the information in the signal at points before and after the current point, in essence "looking into the future," to eliminate phase distortion.

The function filtfilt performs zero-phase digital filtering by processing the input data in both the forward and reverse directions. After filtering the data in the forward direction, filtfilt reverses the filtered sequence and runs it back through the filter.

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Digital Filter Transfer Function

The transfer function is a frequency domain representation of a digital filter, expressing the filter as a ratio of two polynomials. It is the principal discrete-time model for filter implementation:

$$Y(z) = H(z)\,X(z) = \frac{b_1 + b_2 z^{-1} + \ldots + b_{n_b+1} z^{-n_b}}{a_1 + a_2 z^{-1} + \ldots + a_{n_a+1} z^{-n_a}}\,X(z)$$

The constants b_i and a_i are the filter coefficients, and the ORDER OF THE FILTER is the maximum of n_a and n_b.

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Digital Filter Transfer Function

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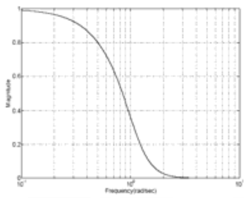
When constructing a filter of given transfer function, some filter requirements traditionally include passband ripple (Rp, in decibels), stopband attenuation (Rs, in decibels), and transition width (Ws-Wp, in Hertz).

This means find filter coefficients fitting specified requirements

→ NOTE: when constructing a filter very strict requirements, ALWAYS check the filter response, in order to avoid unexpected filter response due to non-convergence of parameters



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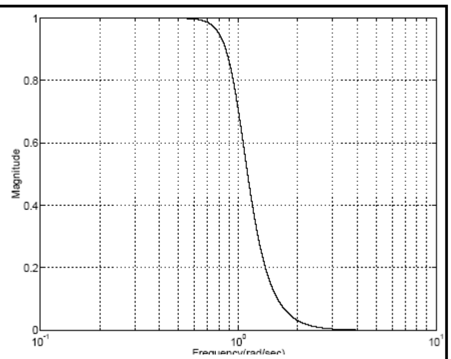
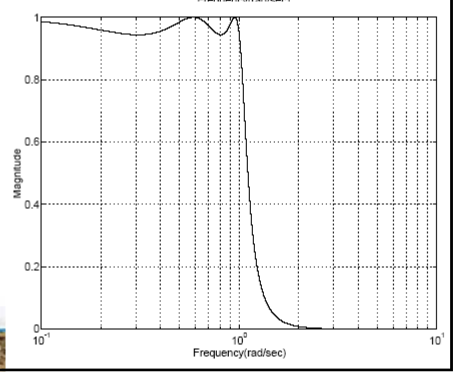
Classical Infinite Impulse Response (IIR) filter Types



BUTTERWORTH FILTER

provides the best Taylor Series approximation to the ideal lowpass filter. Response is monotonic overall, decreasing smoothly.



CHEBYSHEV TYPE I FILTER

minimizes the absolute difference between the ideal and actual frequency response over the entire passband by incorporating an equal ripple of Rp dB in the passband. Stopband response is maximally flat.

The transition from passband to stopband is more rapid than for the Butterworth.

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Classical IIR filter Types

CHEBYSHEV TYPE II FILTER

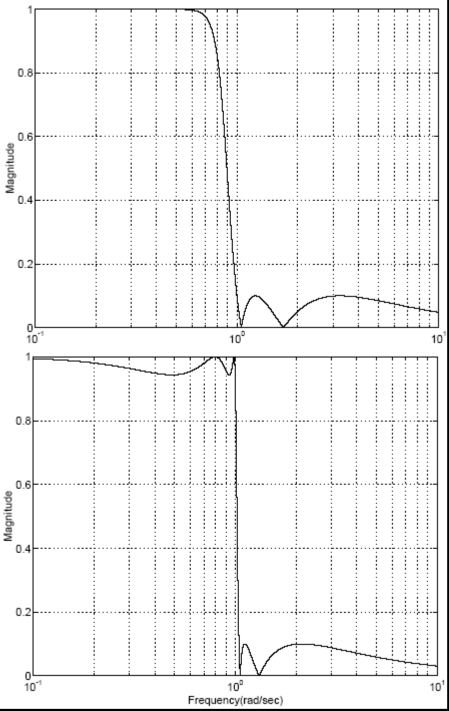
minimizes the absolute difference between the ideal and actual frequency response over the entire stopband, by incorporating an equal ripple of Rs dB in the stopband. Passband response is maximally flat.

The stopband does not approach zero as quickly as the type I filter. The absence of ripple in the passband, however, is often an important advantage.

ELLIPTIC FILTERS

are equiripple in both the passband and stopband. They generally meet filter requirements with the lowest order of many other filter types.

Given a filter order n, passband ripple Rp in decibels, and stopband ripple Rs in decibels, elliptic filters minimize transition width.



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Finite Impulse Response Filter (FIR) Filters

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They have both advantages and disadvantages compared to IIR filters.

FIR filters have the following primary advantages:

1. They can have exactly linear phase.
2. They are always stable.
3. The design methods are generally linear.
4. They can be realized efficiently in hardware.
5. The filter startup transients have finite duration.

The primary disadvantage of FIR filters is that they often require a much higher filter order than IIR filters to achieve a given level of performance. Correspondingly, the delay of these filters is often much greater than for an equal performance IIR filter.

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Filtering to eliminate noise

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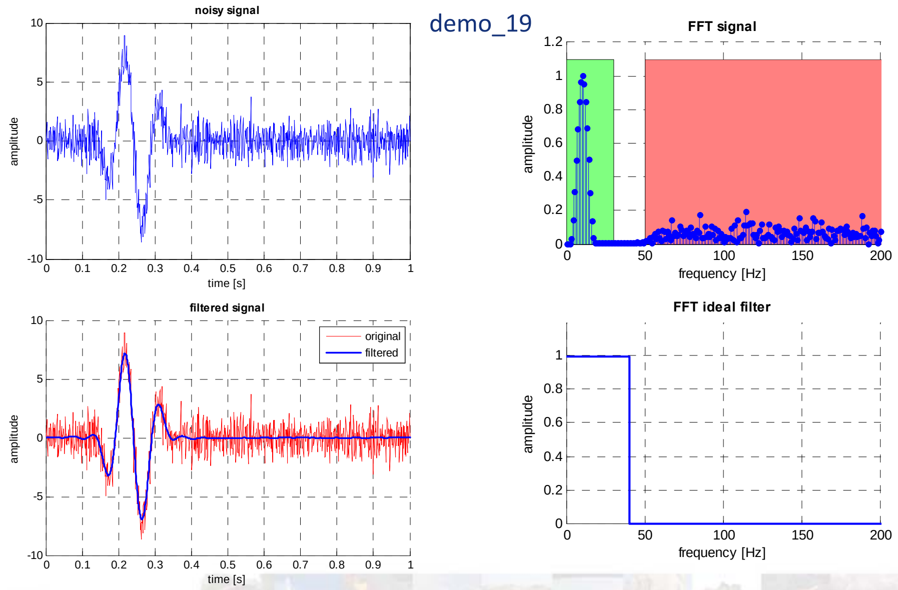

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Filter implementation

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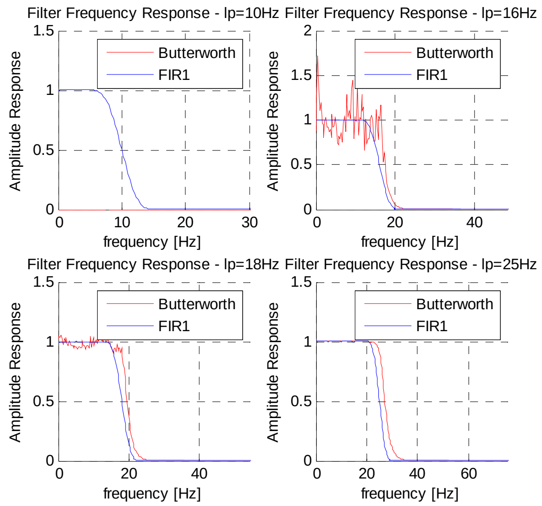
demo_20

NOTE: when constructing a filter very strict requirements, ALWAYS check the filter response, in order to avoid unexpected filter response due to non-convergence of parameters

Compare FIR1 and Butterworth
Filter response for different
Lowpass frequencies

→ FIR filter are always stable



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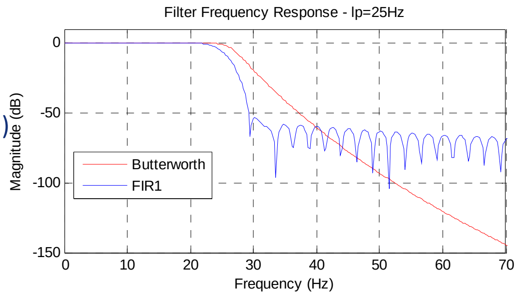
Often the filter response is visualized in the DB scale:

0 dB → response=1 (passband)

-100 dB → response 10^{-5} (stopband)

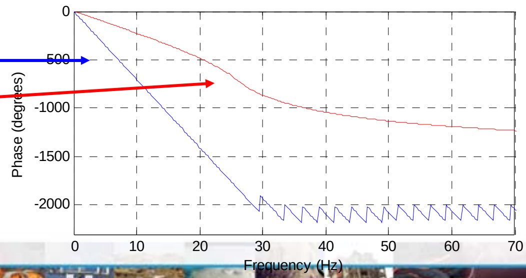


Note

FIR1 linear phase

Butterworth nonlinear phase

FIR1 has linear phase shift in the pass band



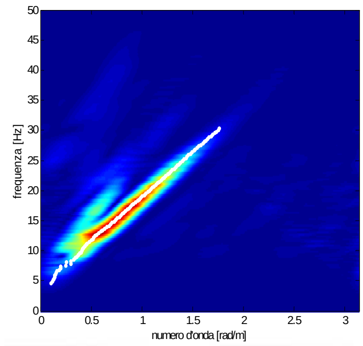
Time Variation and Nonlinearity

The Discrete Fourier Transform (DFT) does not give any timing-related information → drawback for Nonlinear-time variant systems, where time-varying frequency/amplitude content, and/or abrupt changes in the system response can be encountered.

Techniques for the analysis of nonstationary signals include:

1. Short-Time Fourier Transform
2. band-pass filters
3. Wavelet analysis

Such techniques convert the 1D array in time-domain into a 2D array in the time-frequency space in order to capture the time-varying nature of the signal



Time Variation and Nonlinearity

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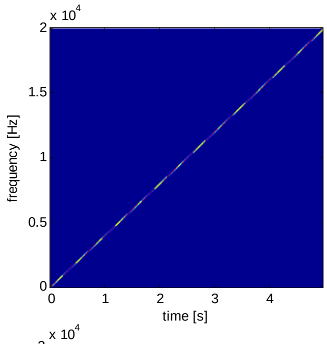
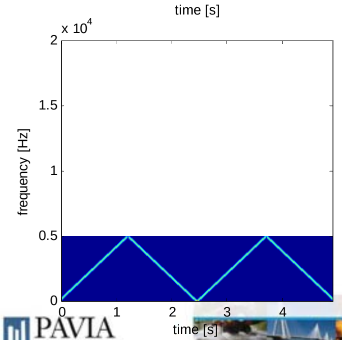


Sampling at 40000 Hz it is possible to see the frequency increasing



Sampling at 10000 Hz, after 5000 Hz (f_{nyq}), fake frequencies are seen, due to the undersampling of the signal (aliasing effect)

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STFT (Gabor, 1946)

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Drawbacks in the global DFT (computed along the entire NL-TV signal) can be decreased by extracting time windows of the original signal and analysing each window in the frequency domain.

the k-th DFT of the k-th windowed signal defines a matrix of

SHORT-TIME FOURIER TRANSFORMS (STFT) of the original signal

a 3D plot is obtained in which time-frequency and DFTs are plotted on the three axes, also called SPECTROGRAM

$$STFT_x(\tau, f) = \int_{-\infty}^{+\infty} x(t)\, g^*(t-\tau)\exp(-j2\pi f t)\, dt$$

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STFT (Gabor, 1946)

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The STFT:

1. Map the signal in a 2D space time-frequency
2. Gives information on the evolution of the signal frequency content
3. Care should be done in selecting the time window length and overlapping

Window length: $M \Delta t$

Window separation $q \Delta t$

Frequency resolution $\Delta f_{res} >\cong 1/M \Delta t$

Longer windows enhance the frequency resolution but comprise longer time segments, worsening the time resolution and leading to inaccuracy.

Time resolution $\Delta t_{res} > M \Delta t/2$

A good trade off between time and frequency resolution is such that:

→ $\Delta f_{res} \Delta t_{res} > 1/2$

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Time-frequency analysis

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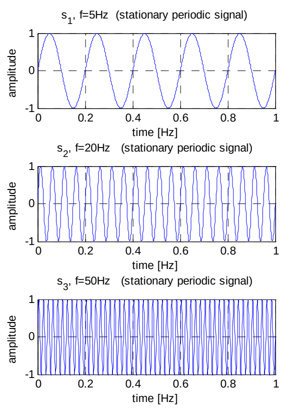
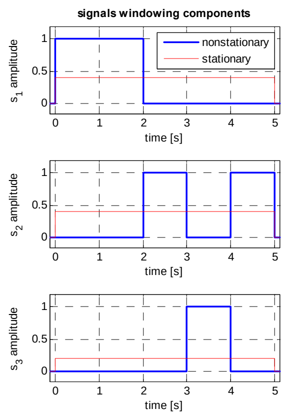
demo_21

When the frequency content of a signal varies with time (nonstationary signals, nonlinear systems, etc…), another type of representation is needed.

Lets consider 3 harmonic components





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Time-frequency analysis

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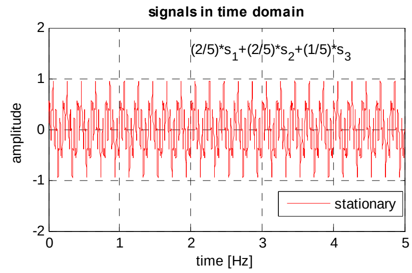
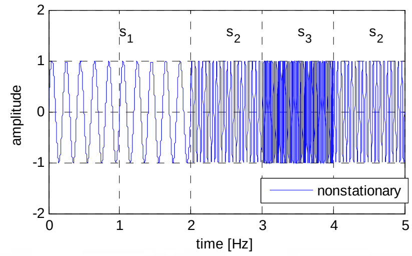
1. Stationary signal: frequency content constant in time



2. Nonstationary signal: frequency content varying with time



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Time-frequency analysis

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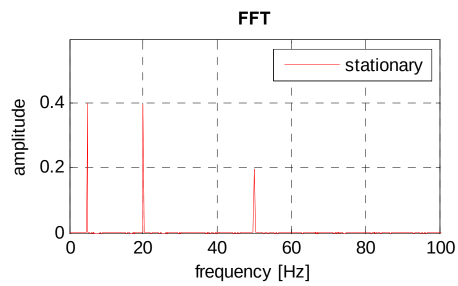
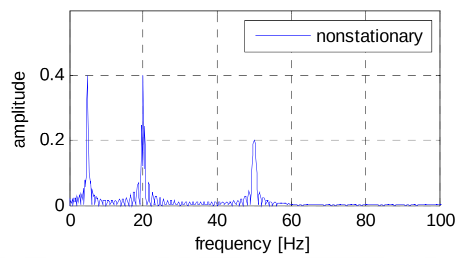
→ FFT with equal frequency content

The frequency analysis does not give any timing information!!!





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Time-frequency analysis: STFT

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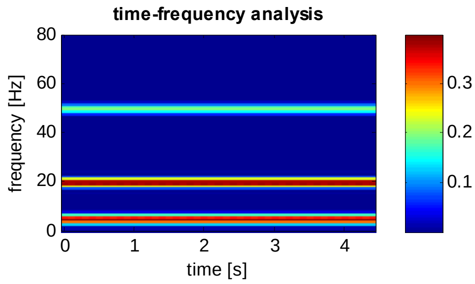
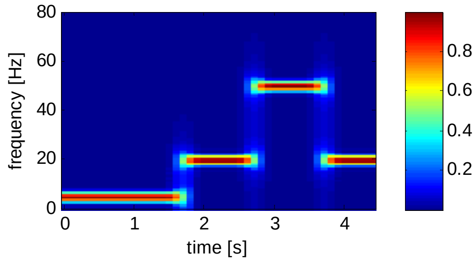
The STFT is able to detect the variation of the frequency content with time

1. Stationary signal, frequency content constant in time



2. Nonstationary signal, frequency content varying with time



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